Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 21, 2003, 375–386

AN ESSENTIAL MAP THEORY FOR $\mathcal{U}_c^\kappa$ AND PK MAPS

Ravi P. Agarwal — Donal O'Regan

Abstract. This paper presents a continuation theory for $\mathcal{U}_c^\kappa$ maps. The analysis is elementary and relies on properties of retractions and fixed point theory for self maps. Also we present a separate theory for a certain subclass of $\mathcal{U}_c^\kappa$ maps, namely the PK maps.

1. Introduction

In this paper we present a new essential map approach for $\mathcal{U}_c^\kappa$ maps. In particular we obtain new results for maps which are either

(a) approximable, or
(b) admissible (strongly) in the sense of Górniewicz.

The maps considered will also satisfy various compactness criteria described in Section 2. Our analysis is elementary and combines properties of the Minkowski functional with fixed point theory for self maps. In [1] we introduced the notion of an essential map for a particular subclass of $\mathcal{U}_c^\kappa$ maps, namely the DKT maps. In this paper we extend this notion (the proofs here are different) to enable us to discuss a more general class of maps, namely the PK maps. The theory and results in this paper complement and extend previously known results in the literature (see [1], [2], [6], [8], [10], [11] and the references therein).

For the remainder of this section we present some definitions and some known facts. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and E_2,

2000 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Continuation theory, essential maps.

respectively. We will look at maps $F: X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F: X \to K(Y)$ is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F: X \to K(Y)$ is *acyclic* if F is upper semicontinuous with acyclic values. $F: X \to K(Y)$ is said to be an *O'Neill map* if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

Given two open neighbourhoods U and V of the origins in E_1 and E_2, respectively, a (U,V)-approximate continuous selection (see [6]) of $F: X \to K(Y)$ is a continuous function $s: X \to Y$ satisfying

$$s(x) \in (F[(x+U) \cap X] + V) \cap Y \quad \text{for every } x \in X.$$

We say $F: X \to K(Y)$ is *approximable* if it is a closed map and if its restriction $F|_K$ to any compact subset K of X admits a (U,V)-approximate continuous selection for every open neighbourhood U and V of the origins in E_1 and E_2, respectively.

For our next definition let X and Y be metric spaces. A continuous single valued map $p: Y \to X$ is called a *Vietoris map* if the following two conditions are satisfied:

(a) for each $x \in X$, the set $p^{-1}(x)$ is acyclic,
(b) p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Definition 1.1. A multifunction $\phi: X \to K(Y)$ is *admissible* (strongly) in the sense of Górniewicz, if $\phi: X \to K(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p: Z \to X$ and $q: Z \to Y$ such that

(a) p is a Vietoris map, and
(b) $\phi(x) = q(p^{-1}(x))$ for any $x \in X$.

Remark 1.1. It should be noted (see [8, p. 179]) that ϕ upper semicontinuous is redundant in Definition 1.1.

Suppose X and Y are Hausdorff topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F: X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. A class $\mathcal{U}$ of maps is defined by the following properties:

(a) $\mathcal{U}$ contains the class $\mathcal{C}$ of single valued continuous functions,
(b) each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued, and
(c) for any polytope P, $F \in \mathcal{U}_c(P,P)$ has a fixed point, where the intermediate spaces of composites are suitably chosen for each $\mathcal{U}$.

Definition 1.2. $F \in \mathcal{U}_c^\kappa(X,Y)$ if for any compact subset K of X, there is a $G \in \mathcal{U}_c(K,Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Examples of $\mathcal{U}_c^\kappa$ maps are the Kakutani maps, the acyclic maps, the O'Neill maps, and the maps admissible in the sense of Górniewicz.

For a subset K of a topological space X, we denote by $\mathrm{Cov}_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $\mathrm{Cov}(K) = \mathrm{Cov}_X(K)$). Given two maps $F, G: X \to 2^Y$ and $\alpha \in \mathrm{Cov}(Y)$, F and G are said to be α-close, if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$.

By a space we mean a Hausdorff topological space. In what follows Q denotes a class of topological spaces. A space Y is an *extension space* for Q (written $Y \in \mathrm{ES}(Q)$) if for any pair (X,K) in Q with $K \subseteq X$ closed, any continuous function $f_0: K \to Y$ extends to a continuous function $f: X \to Y$.

A space Y is an *approximate extension space* for Q (and we write $Y \in \mathrm{AES}(Q)$) if for any $\alpha \in \mathrm{Cov}(Y)$ and any pair (X,K) in Q with $K \subseteq X$ closed, and any continuous function $f_0: K \to Y$, there exists a continuous function $f: X \to Y$ such that $f|_K$ is α-close to f_0.

Definition 1.3. Let V be a subset of a Hausdorff topological space E. Then we say V is *Schauder admissible* if for every compact subset K of V and every covering $\alpha \in \mathrm{Cov}_V(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha: K \to V$ such that

(a) π_α and $i: K \to V$ are α-close,
(b) $\pi_\alpha(K)$ is contained in a subset $C \subseteq V$ with $C \in \mathrm{AES}(\text{compact})$.

If $V \in \mathrm{AES}(\text{compact})$ then V is trivially Schauder admissible. If V is an open convex subset of a Hausdorff locally convex topological space E, then it is well known that V is Schauder admissible.

The following fixed point result was established in [5].

Theorem 1.1. *Let V be a Schauder admissible subset of a Hausdorff topological space E and $F \in \mathcal{U}_c^\kappa(V,V)$ a compact map. Then F has a fixed point.*

A nonempty subset X of a Hausdorff topological vector space E is said to be *admissible* if for every compact subset K of X and every neighborhood V of 0, there exists a continuous map $h: K \to X$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E. X is said to be *q-admissible* if any nonempty compact, convex subset Ω of X is admissible. X is said to be *q-Schauder admissible* if any nonempty compact, convex subset Ω of X is Schauder admissible.

The following fixed point result was established in [4].

Theorem 1.2. *Let Ω be a q-Schauder admissible, closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F \in \mathcal{U}_c^\kappa(\Omega,\Omega)$ with the following property:*

$$A \subseteq \Omega,\ A = \overline{\mathrm{co}}(\{x_0\} \cup F(A)) \text{ implies } A \text{ is compact.} \tag{1.1}$$

Then F has a fixed point in Ω.

Let (E,d) be a pseudometric space. For $S \subseteq E$, let $B(S,\varepsilon) = \{x \in E : d(x,S) \le \varepsilon\}$, $\varepsilon > 0$, where $d(x,S) = \inf_{y\in Y} d(x,y)$. The measure of noncompactness (see [7]) of the set $M \subseteq E$ is defined by $\alpha(M) = \inf Q(M)$ where

$$Q(M) = \{\varepsilon > 0 : M \subseteq B(A,\varepsilon) \text{ for some finite subset } A \text{ of } E\}.$$

Let E be a locally convex Hausdorff topological vector space, and let P be a defining system of seminorms on E. Suppose $F\colon S \to 2^E$; here $S \subseteq E$. The map F is said to be a countably P-concentrative mapping (see [7]) if $F(S)$ is bounded, and for $p \in P$ for each countably bounded subset X of S we have $\alpha_p(F(X)) \le \alpha_p(X)$, and for $p \in P$ for each countably bounded non-p-precompact subset X of S (i.e. X is not precompact in the pseudonormed space (E,p)) we have $\alpha_p(F(X)) < \alpha_p(X)$; here $\alpha_p(\,\cdot\,)$ denotes the measure of noncompactness in the pseudonormed space (E,p). In this paper when we consider countably P-concentrative maps it is worth remarking here that in fact the results hold if the maps are countably condensing in the sense of [12, pp. 353, 356].

The following fixed point result was established in [9].

Theorem 1.3. *Let Ω be a nonempty, closed, convex subset of a Fréchet space E (P is a defining system of seminorms). Suppose $F \in \mathcal{U}_c^\kappa(\Omega,\Omega)$ is a countably P-concentrative mapping. Then F has a fixed point in Ω.*

The following fixed point results were established in [3].

Theorem 1.4. *Let Ω be a q-Schauder admissible closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F \in \mathcal{U}_c^\kappa(\Omega,\Omega)$ with the following conditions holding:*

$$\begin{cases} \text{for any relatively compact, convex subset } A \text{ of } \Omega \\ \text{with } \mathrm{co}(F(A)) \subseteq A \text{ we have } F(\overline{A}) \subseteq \overline{\mathrm{co}(F(A))} \end{cases} \tag{1.2}$$

and

$$A \subseteq \Omega,\ A = \mathrm{co}(\{x_0\} \cup F(A)) \text{ implies } \overline{A} \text{ is compact.} \tag{1.3}$$

Then F has a fixed point in Ω.

Theorem 1.5. *Let Ω be a q-Schauder admissible closed, convex subset of a Hausdorff topological vector space E with $x_0 \in \Omega$. Suppose $F \in \mathcal{U}_c^\kappa(\Omega, \Omega)$ maps compact sets into relatively compact sets and assume* (1.2) *holds. In addition suppose the following conditions are satisfied:*

$$(1.4) \qquad \begin{cases} A \subseteq \Omega,\ A = \mathrm{co}(\{x_0\} \cup F(A)) \text{ with } \overline{A} = \overline{C} \\ \text{and } C \subseteq A \text{ countable, implies } \overline{A} \text{ is compact,} \end{cases}$$

$$(1.5) \qquad \begin{cases} \text{for any relatively compact subset } A \text{ of } \Omega \\ \text{there exists a countable set } B \subseteq A \text{ with } \overline{B} = \overline{A}, \end{cases}$$

and

$$(1.6) \qquad \text{if } A \text{ is a compact subset of } \Omega \text{ then } \overline{\mathrm{co}}(A) \text{ is compact.}$$

Then F has a fixed point in Ω.

Remark 1.2. If F is a Kakutani map then (1.2) is not needed in Theorems 1.4 and 1.5 (see [3]).

Remark 1.3. If $F: \Omega \to 2^\Omega$ is lower semicontinuous then (1.2) holds (see [3]).

Finally let Z and W be subsets of Hausdorff topological vector spaces Y_1 and Y_2 and F a multifunction. We say $F \in \mathrm{PK}(Z, W)$ if W is convex, and there exists a map $S: Z \to W$ with

$$Z = \bigcup \{\mathrm{int}\, S^{-1}(w) : w \in W\}, \quad \mathrm{co}(S(x)) \subseteq F(x) \quad \text{for } x \in Z,$$

and $S(x) \neq \emptyset$ for each $x \in Z$; here $S^{-1}(w) = \{z : w \in S(z)\}$. Finally we recall the following selection theorem (see [10]).

Theorem 1.6. *If Z is paracompact, W is convex, and $F \in \mathrm{PK}(Z, W)$. Then there exists a continuous* (*single valued*) *function $f: Z \to W$ with $f(x) \in F(x)$ for each $x \in Z$. Moreover, if Z is compact, then $f \subseteq \mathrm{co}(A)$ for some finite subset A of W.*

2. Essential maps

In this section we present a homotopy type result for essential $\mathcal{U}_c^\kappa$ maps. Here E is a Hausdorff locally convex topological vector space, C is a closed convex subset of E, $U \subseteq C$ is convex, U is an open subset of E, and $0 \in U$. Notice $\mathrm{int}_C U = U$ since U is open in C. We will consider maps $F: \overline{U} \to K(C)$ (here $K(C)$ denotes the family of nonempty compact subsets of C and $\overline{U}$ denotes the closure of U in C). Throughout our map $F: \overline{U} \to K(C)$ will satisfy *one* of the following conditions:

(H1) F is compact,

(H2) if $D \subseteq \overline{U}$ and $D \subseteq \overline{\mathrm{co}}(\{0\} \cup F(D))$ then $\overline{D}$ is compact, (H3) F is countably P-concentrative and E is Fréchet (here P is a defining system of seminorms),

(H4) F is lower semicontinuous and if $D \subseteq \overline{U}$ and $D \subseteq \mathrm{co}(\{0\} \cup F(D))$ then $\overline{D}$ is compact, or

(H5) F is lower semicontinuous, F maps compact sets into relatively compact sets, and if $D \subseteq \overline{U}$, $D \subseteq \mathrm{co}(\{0\} \cup F(D))$ with $K \subseteq D$ countable and $\overline{K} = \overline{D}$ then $\overline{D}$ is compact and in this case we also assume
 (a) for any relatively compact convex set A of E there exists a countable set $B \subseteq A$ with $\overline{B} = \overline{A}$, and
 (b) if Q is a compact subset of E then $\overline{\mathrm{co}}(Q)$ is compact.

Fix $i \in \{1, 2, 3, 4, 5\}$.

Definition 2.1. We say $F \in \mathrm{LS}^i(\overline{U}, C)$ if $F \in \mathcal{U}_c^\kappa(\overline{U}, C)$ satisfies (Hi).

Definition 2.2. We say $F \in \mathrm{LS}^i_{\partial U}(\overline{U}, C)$ if $F \in \mathrm{LS}^i(\overline{U}, C)$ with $x \notin Fx$ for $x \in \partial U$; here ∂U denotes the boundary of U in C.

Definition 2.3. A map $F \in \mathrm{LS}^i_{\partial U}(\overline{U}, C)$ is essential in $\mathrm{LS}^i_{\partial U}(\overline{U}, C)$ if for every $G \in \mathrm{LS}^i_{\partial U}(\overline{U}, C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

Theorem 2.1. *Fix $i \in \{1, \ldots, 5\}$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, $F \in \mathrm{LS}^i(\overline{U}, C)$ and assume the following condition holds:*

$$x \notin \lambda F x \quad \text{for } x \in \partial U \text{ and } \lambda \in (0, 1]. \tag{2.1}$$

Then F is essential in $\mathrm{LS}^i_{\partial U}(\overline{U}, C)$ (in particular F has a fixed point in U).

Proof. Let $H \in \mathrm{LS}^i_{\partial U}(\overline{U}, C)$ with $H|_{\partial U} = F|_{\partial U}$. We must show H has a fixed point in U. Let μ be the Minkowski functional on $\overline{U}$ and let $r\colon E \to \overline{U}$ be given by

$$r(x) = \frac{x}{\max\{1, \mu(x)\}} \quad \text{for } x \in E.$$

Let $G = rH$. Now $G \in \mathcal{U}_c^\kappa(\overline{U}, \overline{U})$ since $\mathcal{U}_c^\kappa$ is closed under compositions. Next we show G satisfies the compactness criteria in Theorem's 1.1–1.5 (i.e. G satisfies the conditions in Theorem 1.i). We will just consider the case $i = 5$ since the cases $i = 2, 4$ are similar and the cases $i = 1, 3$ are immediate. Let $i = 5$. Now $H \in \mathrm{LS}^5(\overline{U}, C)$ implies H is lower semicontinuous so G is lower semicontinuous (since r is continuous), so (1.2) holds with F replaced by G. Now let $D \subseteq \overline{U}$, $D = \mathrm{co}(\{0\} \cup G(D))$, $K \subseteq D$ countable and $\overline{K} = \overline{D}$. Now since $r(A) \subseteq \mathrm{co}(\{0\} \cup A)$ for any subset A of E we have

$$D \subseteq \mathrm{co}(\{0\} \cup \mathrm{co}(\{0\} \cup H(D))) = \mathrm{co}(\{0\} \cup H(D)).$$

This implies $\overline{D}$ is compact since $H \in \mathrm{LS}^5(\overline{U}, C)$. As a result G satisfies (1.4). Now Theorem 1.5 guarantees that there exists $x \in \overline{U}$ with $x \in G(x) = rH(x)$ (the same result holds if $i \in \{1, \ldots, 4\}$). Thus $x = r(y)$ for some $y \in H(x)$ with $x \in \overline{U} = U \cup \partial U$ (note $\mathrm{int}_C U = U$ since U is open in E). Now either $y \in \overline{U}$ or $y \notin \overline{U}$. If $y \in \overline{U}$ then $r(y) = y$, so $x = y \in H(x)$, and we are finished since $x = y \in U$ since (2.1) with $H|_{\partial U} = F|_{\partial U}$ implies

$$x \notin \lambda Hx \quad \text{for } x \in \partial U \text{ and } \lambda \in (0,1]. \tag{2.2}$$

If $y \notin \overline{U}$ then $r(y) = y/\mu(y)$ with $\mu(y) > 1$. Thus $x = \lambda y$ (i.e. $x \in \lambda H(x)$) with $0 < \lambda = 1/\mu(y) < 1$. Note $x \in \partial U$ since $\mu(x) = \mu(\lambda y) = 1$ (note $\partial U = \partial_E U$ since $\mathrm{int}_C U = U$). As a result $x \in \lambda H(x)$ with $\lambda = 1/\mu(y) \in (0,1)$ and $x \in \partial U$, and this contradicts (2.2). □

Remark 2.1. In (H4) and (H5), F lower semicontinuous can be replaced by

$$\begin{cases} \text{for any relatively compact, convex subset } A \text{ of } \overline{U} \\ \text{with } \mathrm{co}(F(A)) \subseteq A \text{ we have } F(\overline{A}) \subseteq \overline{F(A)}. \end{cases} \tag{2.3}$$

To see this suppose G is as in Theorem 2.4 and suppose A is a relatively compact, convex subset of $\overline{U}$ with $\mathrm{co}(G(A)) \subseteq A$. Then since H satisfies (2.3) (with F replaced by H) and since r is continuous we have

$$G(\overline{A}) = rH(\overline{A}) \subseteq r(\overline{H(A)}) \subseteq \overline{rH(A)} = \overline{G(A)}.$$

Thus G satisfies (2.3) (with F replaced by G).

Remark 2.2. In fact Theorem 2.1 is a homotopy result since we will now show that the zero map is essential in $\mathrm{LS}^i_{\partial U}(\overline{U}, C)$. Then the zero map essential in $\mathrm{LS}^i_{\partial U}(\overline{U}, C)$ with $F \cong 0$ and (2.1) guarantees (Theorem 2.1) that F is essential in $\mathrm{LS}^i_{\partial U}(\overline{U}, C)$.

To show the zero map is essential in $\mathrm{LS}^i_{\partial U}(\overline{U}, C)$ let $\theta \in \mathrm{LS}^i_{\partial U}(\overline{U}, C)$ with $\theta|_{\partial U} = \{0\}$. Let μ and r be as in Theorem 2.1 and let $J = r\theta$. As in Theorem 2.1, Theorem's 1.1–1.5 guarantee that there exists $x \in \overline{U}$ with $x \in J(x) = r\theta(x)$. Thus $x = r(y)$ for some $y \in \theta(x)$ and essentially the same argument as in Theorem 2.1 yields $x \in U$ with $x \in \theta(x)$.

In fact we can generalize the theory presented above as follows. We assume $F: \overline{U} \to K(C)$ satisfies either (H1), (H2), (H3) or one of the following conditions:

- (H6) if $D \subseteq \overline{U}$ and $D \subseteq \mathrm{co}(\{0\} \cup F(D))$ then $\overline{D}$ is compact, or
- (H7) if $D \subseteq \overline{U}$, $D \subseteq \mathrm{co}(\{0\} \cup F(D))$ with $K \subseteq D$ countable and $\overline{K} = \overline{D}$ then $\overline{D}$ is compact and in this case we also assume
  - (a) for any relatively compact convex set A of E there exists a countable set $B \subseteq A$ with $\overline{B} = \overline{A}$, and
  - (b) if Q is a compact subset of E then $\overline{\mathrm{co}}(Q)$ is compact.

Fix $i \in \{1, 2, 3, 6, 7\}$.

Definition 2.4. We say $F \in \mathrm{GLS}^i(\overline{U}, C)$ if $F \in \mathcal{U}_c^\kappa(\overline{U}, C)$ satisfies condition (A).

We assume condition (A) is such that

$$(2.4)\qquad \begin{cases} \text{for any map } F \in \mathrm{GLS}(\overline{U}, C) \text{ and any continuous single valued} \\ \text{map } r\colon E \to \overline{U} \text{ we have that } rF \text{ satisfies condition (A).} \end{cases}$$

Example 2.1. If condition (A) means either the map is

(a) approximable, or
(b) admissible in the sense of Gorniewicz, or
(c) in $\mathcal{U}_c^\kappa$,

then clearly (2.4) holds.

Definition 2.5. We say $F \in \mathrm{GLS}^i(\overline{U}, C)$ if $F \in \mathrm{GLS}(\overline{U}, C)$ satisfies (Hi).

Definition 2.6. We say $F \in \mathrm{GLS}^i_{\partial U}(\overline{U}, C)$ if $F \in \mathrm{GLS}^i(\overline{U}, C)$ with $x \notin Fx$ for $x \in \partial U$.

Definition 2.7. A map $F \in \mathrm{GLS}^i_{\partial U}(\overline{U}, C)$ is essential in $\mathrm{GLS}^i_{\partial U}(\overline{U}, C)$ if for every $G \in \mathrm{GLS}^i_{\partial U}(\overline{U}, C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

Essentially the same reasoning as in Theorem 2.1 establishes the following result.

Theorem 2.2. *Fix $i \in \{1, 2, 3, 6, 7\}$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, $F \in \mathrm{GLS}^i(\overline{U}, C)$ and assume* (2.1) *and* (2.4) *hold. Also suppose the following condition holds:*

$$(2.5)\qquad \textit{any map } \Phi \in \mathrm{GLS}^i(\overline{U}, \overline{U}) \textit{ has a fixed point.}$$

Then F is essential in $\mathrm{GLS}^i_{\partial U}(\overline{U}, C)$.

Next we present a more natural "homotopy" type property for PK maps. Again E is a Hausdorff locally convex topological vector space, C is a closed convex subset of E, $U \subseteq C$ is convex, U is an open subset of E, $0 \in U$, and $\overline{U}$ is paracompact.

Remark 2.3. We refer the reader to [2] where a different essential map approach is presented for certain subclasses of $\mathcal{U}_c^\kappa$ maps in Hausdorff topological vector spaces (see also Theorem 2.4 in this paper).

Throughout our map $F \in \mathrm{PK}(\overline{U}, C)$ will satisfy either (H1), (H2), (H3), (H6) or (H7).

Remark 2.4. If $F \in \mathrm{PK}(\overline{U}, C)$ satisfies (H3) then the assumption that $\overline{U}$ is paracompact is redundant since E is a Fréchet space in this case.

Fix $i \in \{1, 2, 3, 6, 7\}$.

Definition 2.8. We say $F \in \mathrm{PK}^i(\overline{U}, C)$ if $F \in PK(\overline{U}, C)$ satisfies (Hi).

Definition 2.9. We say $F \in \mathrm{PK}^i_{\partial U}(\overline{U}, C)$ if $F \in PK^i(\overline{U}, C)$ with $x \notin Fx$ for $x \in \partial U$.

Definition 2.10. We say $f \in \mathrm{CPK}^i(\overline{U}, C)$ if $f: \overline{U} \to C$ is a single valued continuous map with f satisfying (Hi) (with F replaced by f).

Definition 2.11. A map $F \in \mathrm{PK}^i_{\partial U}(\overline{U}, C)$ is essential in $\mathrm{PK}^i_{\partial U}(\overline{U}, C)$ if for any continuous selection $f: \overline{U} \to C$ of F there exists $x_g \in U$ with $x_g = g(x_g)$ for every $g \in \mathrm{CPK}^i(\overline{U}, C)$ with $x \neq g(x)$ for $x \in \partial U$ and $g|_{\partial U} = f|_{\partial U}$.

Theorem 2.3. *Fix $i \in \{1, 2, 3, 6, 7\}$ and let E be a Hausdorff locally convex topological vector space, C a closed convex subset of E, $U \subseteq C$ convex, U an open subset of E, $0 \in U$, $\overline{U}$ paracompact, $F \in \mathrm{PK}^i(\overline{U}, C)$ and assume* (2.1) *holds. Then F is essential in $\mathrm{PK}^i_{\partial U}(\overline{U}, C)$ (in particular F has a fixed point in U).*

Proof. Let $f: \overline{U} \to C$ be any continuous selection of F (guaranteed of course from Theorem 1.6). Let $g \in \mathrm{CPK}^i(\overline{U}, C)$ with $x \neq g(x)$ for $x \in \partial U$ and $g|_{\partial U} = f|_{\partial U}$. Now (2.1) implies $x \neq \lambda f(x)$ for $x \in \partial U$ and $\lambda \in (0, 1]$ and this together with $g|_{\partial U} = f|_{\partial U}$ implies

$$x \neq \lambda g(x) \quad \text{for } x \in \partial U \text{ and } \lambda \in (0, 1]. \tag{2.6}$$

Let μ and r be as in Theorem 2.1 and let $h = rg$. Notice $h: \overline{U} \to \overline{U}$ is continuous and it is easy to check (as in Theorem 2.1) that h satisfies the compactness criteria in Theorem 1.i since $g \in \mathrm{CPK}^i(\overline{U}, C)$. Now Theorem's 1.1–1.5 (see also Remark 2.1) guarantees that there exists $x \in \overline{U}$ with $x = rg(x)$. Thus $x = r(y)$ where $y = g(x)$ and $x \in \overline{U}$. Now either $y \in \overline{U}$ or $y \notin \overline{U}$. If $y \in \overline{U}$ then $r(y) = y$, so $x = y = g(x)$, and we are finished since $x = y \in U$ (see (2.6)). If $y \notin \overline{U}$ then $r(y) = y/\mu(y)$ with $\mu(y) > 1$. Thus $x = \lambda y = \lambda g(x)$ with $0 < \lambda = 1/\mu(y) < 1$. Note $x \in \partial U$ since $\mu(x) = \mu(\lambda y) = 1$, and so (2.6) is contradicted. □

Remark 2.5. As in Remark 2.2 we will show Theorem 2.3 is a homotopy result. To see this we need only to show that the zero map is essential in $\mathrm{PK}^i_{\partial U}(\overline{U}, C)$. To see this let $\theta \in \mathrm{CPK}^i(\overline{U}, C)$ with $x \neq \theta(x)$ for $x \in \partial U$ and $\theta|_{\partial U} = 0$ (here we are considering the zero map so $f(x) = 0$ for $x \in \overline{U}$). Let μ and r be as in Theorem 2.1 and let $j = r\theta$. Now Theorems 1.1–1.5 guarantee that there exists $x \in \overline{U}$ with $x = j(x) = r\theta(x)$. Essentially the same argument as in Theorem 2.3 yields $x \in U$ with $x = \theta(x)$.

Next we present another approach motivated from the ideas in [2]. Here E will be a Hausdorff topological vector space (we do not assume locally convex), C is a closed convex subset of E, U is an open subset of C, $0 \in U$ and $\overline{U}$ is paracompact.

Fix $i \in \{1,2,3,6,8\}$. Here $F \in \mathrm{PK}(\overline{U}, C)$ satisfies (H8) if the following holds:

(H8) if $D \subseteq \overline{U}$, $D \subseteq \mathrm{co}(\{0\} \cup F(D))$ with $K \subseteq D$ countable and $\overline{K} = \overline{D}$ then $\overline{D}$ is compact and in this case we also assume
- (a) either E is a normal space or E is such that any closed subset is compact if and only if it is sequentially compact,
- (b) for any relatively compact convex set A of E there exists a countable set $B \subseteq A$ with $\overline{B} = \overline{A}$, and
- (c) if Q is a compact subset of E then $\overline{\mathrm{co}}(Q)$ is compact.

Theorem 2.4. *Fix $i \in \{1,2,3,6,8\}$ and let E be a Hausdorff topological vector space, C a closed convex subset of E, U an open subset of C, $0 \in U$, $\overline{U}$ paracompact, $F \in \mathrm{PK}^i(\overline{U}, C)$ and assume* (2.1) *holds. In addition suppose*

$$\text{(2.7)} \qquad \text{the zero map is essential in } PK^i_{\partial U}(\overline{U}, C).$$

Then F is essential in $\mathrm{PK}^i_{\partial U}(\overline{U}, C)$ (in particular F has a fixed point in U).

Proof. Let $f: \overline{U} \to C$ be any continuous selection of F (guaranteed of course from Theorem 1.6) and let $g \in CPK^i(\overline{U}, C)$ with $x \neq g(x)$ for $x \in \partial U$ and $g|_{\partial U} = f|_{\partial U}$. Let

$$Q = \{x \in \overline{U} : x = \lambda g(x) \text{ for some } \lambda \in [0,1]\}.$$

Now $Q \neq \emptyset$ is closed. In fact it is easy to check (see the argument in [2]) that Q is compact. Also $Q \cap \partial U = \emptyset$ since (2.1) implies $x \neq \lambda g(x)$ for $x \in \partial U$ and $\lambda \in (0,1]$. Now since Hausdorff topological vector spaces are completely regular there exists a continuous map $\mu: \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(Q) = 1$. Define a map R_μ by $R_\mu(x) = \mu(x) g(x)$. It is easy to see (see the argument in [2]) that $R_\mu \in \mathrm{CPK}^i(\overline{U}, C)$ with $R_\mu|_{\partial U} = 0$. Now (2.7) guarantees that there exists $x \in U$ with $x = R_\mu(x)$. Thus $x \in Q$ so $\mu(x) = 1$, i.e. $x = g(x)$. □

Notice Remark 2.5 gives an example of when (2.7) is satisfied. Here are other examples.

Example 2.2. Here E is a Hausdorff topological vector space, C a closed convex subset of E, U an open subset of C, $0 \in U$ and $\overline{U}$ paracompact.

(a) If $i = 1$ and C is Schauder admissible then (2.7) holds (with $i = 1$).

To see this let $\theta \in \mathrm{CPK}^i(\overline{U}, C)$ with $x \neq \theta(x)$ for $x \in \partial U$ and $\theta|_{\partial U} = 0$. Let

$$J(x) = \begin{cases} \theta(x) & \text{for } x \in \overline{U}, \\ 0 & \text{for } x \in C \backslash \overline{U}. \end{cases}$$

Clearly $J: C \to C$ is a continuous compact map. Theorem 1.1 (applied to J) guarantees that there exists $x \in C$ with $x = J(x)$. If $x \notin U$ we have $x = J(x) = 0$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x = J(x) = \theta(x)$.

(b) If $i = 2$, C is q-Schauder admissible and assume the following condition is satisfied:

$$\overline{\text{co}}(A) \text{ is compact for any compact subset } A \text{ of } E. \tag{2.8}$$

Then (2.7) holds (with $i = 2$).

To see this let θ and J be as in (a). Let $D \subseteq C$ with $D = \overline{\text{co}}(\{0\} \cup J(D))$. Then

$$D \subseteq \overline{\text{co}}(\{0\} \cup \theta(D \cap U)) \tag{2.9}$$

and so $D \cap U \subseteq \overline{\text{co}}(\{0\} \cup \theta(D \cap U))$.

Now since $\theta \in \text{CPK}^2(\overline{U}, C)$ (i.e. θ satisfies (H2)) we have that $\overline{D \cap U}$ is compact, and so θ continuous guarantees that $\theta(\overline{D \cap U})$ is compact. This together with (2.8) implies $\overline{\text{co}}(\{0\} \cup \theta(\overline{D \cap U}))$ is compact. Now (2.9) implies $\overline{D}$ is compact. Theorem 1.2 (applied to J) guarantees that there exists $x \in C$ with $x = J(x)$, and as in (a) we have $x \in U$ so $x = \theta(x)$.

(c) If $i = 3$ (so automatically E is a Fréchet space) then (2.7) holds (with $i = 3$).

To see this let θ and J be as in (a). It is clear that J is a countably P-concentrative map. Now apply Theorem 1.3.

(d) If $i = 6$, C is q-Schauder admissible and assume (2.8) holds. Then (2.7) holds (with $i = 6$).

To see this let θ and J be as in (a) and essentially the same reasoning as in (b) guarantees that J satisfies (1.3). Now apply Theorem 1.4.

(e) If $i = 8$, C is q-Schauder admissible and assume (2.8) holds. Then (2.7) holds (with $i = 8$).

To see this let θ and J be as in (a). Now let $D \subseteq C$, $D = \overline{\text{co}}(\{0\} \cup J(D))$, $\overline{A} = \overline{D}$ and $A \subseteq D$ countable. Then

$$D \cap U \subseteq \overline{\text{co}}(\{0\} \cup \theta(D \cap U).$$

Also $A \cap U$ is countable, $A \cap U \subseteq D \cap U$ and $\overline{A \cap U} = \overline{D \cap U}$ since

$$A \cap U \subseteq D \cap U \subseteq \overline{D} \cap U = \overline{A} \cap U \subseteq \overline{A \cap U}$$

since U is open. Now since $\theta \in CPK^8(\overline{U}, C)$ we have that $\overline{D \cap U}$ is compact, and as in (b) we have that $\overline{D}$ is compact. Now apply Theorem 1.5.

Remark 2.6. The PK maps in Theorems 2.3 and 2.4 could be replaced by other classes of maps in the literature which have a continuous selection theorem similar to Theorem 1.6.

References

[1] R.P. Agarwal and D. O'Regan, *An essential map approach for multivalued A-proper, weakly inward,* DKT*and weakly closed maps*, Comput. Math. Appl. **38** (1999), 131–139.

[2] ———, *Homotopy and Leray–Schauder principles for multimaps*, Nonlinear Anal. Forum **7** (2002), 103–111.

[3] ———, *Fixed point theory of Mönch type for general classes of admissible maps*, Nonlinear Funct. Anal. Appl. (to appear).

[4] ———, *Collectively fixed point theorems*, Nonlinear Anal. Forum (to appear).

[5] R. P. Agarwal, D. O'Regan and S. Park, *Fixed point theory for multimaps in extension type spaces*, J. Korean Math. Soc. **39** (2002), 579–591.

[6] H. Ben–El–Mechaiekh and P. Deguire, *Approachability and fixed points for nonconvex set valued maps*, J. Math. Anal. Appl. **170** (1992), 477–500.

[7] J. Daneš, *Generalized contractive mappings and their fixed points*, Comment. Math. Univ. Carolin. **11** (1970), 115–136.

[8] L. Górniewicz and M. losarski, *Topological essentiality and differential inclusions*, Bull. Australian Math. Soc. **45** (1992), 177–193.

[9] D. O'Regan, *A unified fixed point theory for countably P-concentrative multimaps*, Appl. Anal. **81** (2002), 565–574.

[10] S. Park, *Fixed points, intersection theorems, variational inequalities and equilibrium theorems*, Internat. J. Math. Math. Sci. **24** (2000), 79–93.

[11] ———, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), 803–829.

[12] M. Vath, *Fixed point theorems and fixed point index for countably condensing maps*, Topol. Methods Nonlinear Anal. **13** (1999), 341–363.

Manuscript received October 29, 2002

Ravi P. Agarwal
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, Florida 32901–6975, USA

E-mail address: agarwal@fit.edu

Donal O'Regan
Department of Mathematics
National University of Ireland
Galway, IRELAND

E-mail address: donal.oreagan@nuigalway.ie