Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 21, 2003, 369–374

ON SETS OF CONSTANT DISTANCE FROM A PLANAR SET

Piotr Pikuta

Abstract. In this paper we prove that d-boundaries

$$D_d = \{x : \operatorname{dist}(x, Z) = d\}$$

of a compact $Z \subset \mathbb{R}^2$ are closed absolutely continuous curves for d greater than some constant depending on Z. It is also shown that D_d is a trajectory of solution to the Cauchy Problem of a differential equation with a discontinuous right-hand side.

1. Introduction

Let $Z \subset \mathbb{R}^2$ be a compact set. For each $d > 0$ define the d-boundary D_d of Z, $D_d = \{x : \operatorname{dist}(x, Z) = d\}$. In [1], the first paper concerning with d-boundaries, M. Brown showed that for all but countable number of d, every component of D_d is a point, or a simple arc, or a simple closed curve. It was also proved that D_d is a 1-manifold for almost all d (see [4] and [6]). If Z is convex, it follows from [3, Theorem 4.7 (8), p. 434] and [3, Corollary 4.9, p. 440] (see also the statement before Definition 4.3 of [3]) that D_d is a boundary of a compact convex set, thus it is a closed curve of class $C^{1,1}$.

In this paper we prove that d-boundaries of a compact Z are closed absolutely continuous curves for d greater than some constant depending on Z. The result can be considered complementary to the Federer's concept of sets with positive

2000 *Mathematics Subject Classification.* 57N05, 34A36.

Key words and phrases. d-boundary, absolutely continuous curve, differential equation with discontinuous right-hand side.

reach (see [3, p. 432]). It is also shown that D_d is a trajectory of solution to the Cauchy Problem of a differential equation with a discontinuous right-hand side.

In Section 2 we familiarize the reader with the method used and prove lemmas which we need further to establish our result in Section 3. Some additional remarks are given in Section 4.

2. Definitions and auxiliary lemmas

Let D_d be a d-boundary of a compact set $Z \subset \mathbb{R}^2$ and $D[a,b] = \bigcup_{d\in[a,b]} D_d$ for each a, b such that $0 < a < b$. Let $Z_{x,d} = \{z \in Z : \operatorname{dist}(z,x) = d\}$, $x \in D_d$, $d > 0$.

Define a multivalued function $S\colon D[a,b] \to 2^{\mathbb{R}^2}$,

$$S(x) = \{y \in \mathbb{R}^2 : \text{ there exists } t \geq 0 \text{ such that } x + ty \in \operatorname{co} Z_{x,\operatorname{dist}(x,Z)}\},$$

where $\operatorname{co} X$ stands for the convex hull of X.

Lemma 2.1. *Multivalued function S defined above is upper semi-continuous in $U = \{u \in \mathbb{R}^2 : \operatorname{dist}(u,Z) \neq 0\}$.*

Sketch of proof. First prove that the multivalued function

$$T(x) = Z_{x,\operatorname{dist}(x,Z)}$$

is u.s.c. in U. Then prove that $\operatorname{co} T(x)$ and, finally, $S(x)$ are u.s.c. in U. $\square$

With every point $x \in D[a,b]$ we will associate a certain vector $f(x) \in S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ so that we can consider and solve the Cauchy problem with the right-hand side equal to $f(x)$. In general, the vector field f can be discontinuous, so we will go into differential inclusions to solve the problem.

Lemma 2.2. *If $(\operatorname{co} Z) \cap D[a,b] = \emptyset$, then there exists a unique single-valued function*

$$f\colon D[a,b] \to S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$$

such that

(a) *$\langle e, f(x)\rangle \leq 0$ for all $x \in D[a,b]$ and all $e \in S(x)$, and*
(b) *for fixed plane orientation $\overline{O}$ and $x \in D[a,b]$, there is a unique $e_x \in S(x)$, $\|e_x\| = 1$, for which the following conditions are fulfilled*
  (b1) *the pair of vectors $\{f(x), e_x\}$ has the orientation $\overline{O}$, and*
  (b2) *the equality $\langle e_x, f(x)\rangle = 0$ holds.*

Proof. Since Z is closed and $(\operatorname{co} Z) \cap D[a,b]$ is empty, it follows that the cone $S(x)$ and the dual cone $T(x) = \{f \in \mathbb{R}^2 : \langle e, f\rangle \leq 0, \text{ for each } e \in S(x)\}$ are closed and convex, and $S(x)$ do not contain any (straight) line. Consequently, in $(S(x) \cap S^1) \times (T(x) \cap S^1)$ one can find exactly two pairs $(e_x, f(x))$ for which (b2) is satisfied. Uniqueness of e_x and $f(x)$ follows from (b1). $\square$

For each $x \in D[a,b]$ let us define $G(x)$ consisting of all limit values of $f(x_n)$, when $x_n \to x$. Let $H(x)$, $x \in D[a,b]$, denote the set of all elements $k \in S^1$ such that there exists $e \in S(x)$ for which $\langle e, k\rangle = 0$ and the pair of vectors $\{e, k\}$ has the orientation $\overline{O}$. Moreover, let $F(x) = \operatorname{co} G(x)$ and $K(x) = \operatorname{co} H(x)$. Notice that Lemma 2.1 gives $G(x) \subset H(x)$ and, consequently, $F(x) \subset K(x)$.

The next two lemmas deal with local solutions to the problem

$$x'(t) \in K(x(t)), \quad \text{a.e. in } t \in \mathbb{R}. \tag{1}$$

We say that $x(t)$ is a solution to (1) in A if $x(t)$ is absolutely continuous (i.e. x has a finite derivative a.e. in A) and the inclusion is satisfied a.e. in $t \in A$.

LEMMA 2.3. *If* $(\operatorname{co} Z) \cap D[a,b] = \emptyset$ *and* $x_0 \in D_d$, $d \in (a,b)$, *then there exists a local solution to the problem*

$$\begin{gathered} x'(t) \in K(x(t)), \quad \text{a.e. in } t \in \mathbb{R}, \\ x(t_0) = x_0 \in D[a,b], \end{gathered} \tag{2}$$

and the trajectory of this (absolutely continuous) solution lies in $D[a,b]$.

PROOF. Since $F(x) \subset K(x)$, $x \in D[a,b]$, we can restrict our attention to the inclusion $x'(t) \in F(x(t))$. Obviously, $f(x) \in F(x)$, $F(x)$ is closed (because $G(x)$ is closed) and bounded for each $x \in D[a,b]$. Moreover, $D[a,b]$ is closed, hence, by [5, p. 53, Lemma 16], F is u.s.c. in $D[a,b]$. Applying [5, p. 60, Theorem 1] we obtain that the problem (2) has a local solution which trajectory lies in $D[a,b]$. $\square$

REMARK 2.4. Let $\overline{V}$ be a closed neighbourhood of t_0. Obviously, $\overline{V} \times D[a,b]$ is compact, thus by theorem [5, p. 61, Theorem 2] the solution to (2) can be extended to the boundary of $\overline{V} \times D[a,b]$. Since $0 \notin K(x)$ for each $x \in D[a,b]$, the trajectory of the solution do not have stationary points and there exists a neighbourhood $W \subset D[a,b]$ of x_0 such that the trajectory passing through x_0 reaches the boundary of W.

LEMMA 2.5. *If* $x_0 \in D_d$ *and* $(\operatorname{co} Z) \cap D_d = \emptyset$, *then there exists a local solution to the problem*

$$\begin{gathered} x'(t) \in K(x(t)), \quad \text{a.e. in } t \in \mathbb{R}, \\ x(t_0) = x_0, \end{gathered} \tag{3}$$

and the trajectory of this solution lies in D_d.

PROOF. Let us take a, $b \in \mathbb{R}$ such that $0 < a < b$, $d \in (a,b)$ and $(\operatorname{co} Z) \cap D[a,b] = \emptyset$. By Lemma 2.3, there exists a local solution $y(t)$ to (2). We claim that the trajectory of $y(t)$ lies in D_d.

Let $w(t) = \operatorname{dist}(y(t), Z)$. Since $w(t_0) > 0$, we have $w(t) > 0$ in a neighbourhood V of t_0 and $y'(t)$ exists almost everywhere in V. From [2, Theorem 2.2.1] it follows that

$$w'(t) = \min\left\{\frac{\langle y(t) - z, y'(t)\rangle}{w(t)} : z \in Z_{y(t),w(t)}\right\}$$

a.e. in V, where $Z_{y(t),w(t)} = \{z \in Z : \|y(t) - z\| = w(t)\}$. For each $f \in K(y(t))$ there exists $z \in \operatorname{co} Z_{y(t),w(t)}$ such that $\langle z - y(t), f\rangle = 0$, so $w'(t) \leq 0$ a.e. in V.

Now, let us change the plane orientation $\overline{O}$ to the opposite one $-\overline{O}$, and consider the problem

$$x'(t) \in K_1(x), \quad x(t_0) = x_0 \in D_d, \tag{4}$$

with $K_1(x)$ being constructed analogously to $K(x)$. Notice that $K_1(x) = -K(x)$, $x \in D[a, b]$. Thus the solution $y(t)$ to (3) corresponds to a solution (let us call it $y_1(t)$) to the problem (4) in such a way that both y and y_1 have the same trajectory in a neighbourhood of x_0, i.e. $y_1(t_0 + t) = y(t_0 - t)$. For $w_1(t) = \operatorname{dist}(y_1(t), Z)$ we have

$$w_1'(t) = \min\left\{\frac{\langle y_1(t) - z, y_1'(t)\rangle}{w_1(t)} : z \in Z_{y_1(t),w_1(t)}\right\} \leq 0$$

a.e. in a neighbourhood V_1 of t_0. Since

$$w'(t_0 + t) = -w_1'(t_0 - t)$$

we conclude that $w'(t) = 0$ for almost all $t \in V \cap V_1$. The distance function is absolutely continuous, so $w(t) = \operatorname{dist}(y(t_0), Z) = d$ in $V \cap V_1$. $\square$

3. Main result

THEOREM 3.1. *There exists $a_0 > 0$ such that d-boundaries D_d of $Z \subset \mathbb{R}^2$ are closed absolutely continuous curves for all $d > a_0$.*

PROOF. Since Z is compact there exists a closed ball $\overline{K}_{s,r}$ of radius r centered at $s \in \operatorname{co} Z$ such that $Z \subset \overline{K}_{s,r}$. Let

$$a_0 = \sup_{k \in \partial\overline{K}_{s,r}} \inf_{z \in Z} \|k - z\|,$$

where $\partial\overline{K}_{s,r}$ is the boundary of $\overline{K}_{s,r}$. Suppose $d > a_0$ and notice that $(\operatorname{co} Z) \cap D_d = \emptyset$.

The distance function is continuous, so the intersection of D_d and every half-line starting at s is not empty. Using elementary geometry it is easy to prove that every such intersection has no more than one point.

For each $p \in D_d$ we consider the problem $x'(t) \in K(x(t))$ with $x(t_0) = p$ and obtain the solution which trajectory is an absolutely continuous curve lying in a neighbourhood W_p of p. By Remark 2.4 the trajectory passing through p reaches the boundary of W_p. Let C_p be the smallest cone with vertex s containing two endpoints of the trajectory passing through p in W_p. The intersection of W_p, C_p and the trajectory of the solution is a simple arc. Since D_d is compact only a finite number of $(W_p \cap C_p)$'s are needed to cover D_d. Thus D_d is a collection of a finite number of simple arcs which form a simple closed curve. $\square$

THEOREM 3.2. *If $x_0 \in D_d$ and $(\operatorname{co} Z) \cap D_d = \emptyset$, then*

(a) *the solution $y(t)$ to the problem* (3) *solves*

$$x'(t) = f(x(t)), \quad \text{a.e. in } t \in \mathbb{R}, \qquad x(t_0) = x_0. \tag{5}$$

(b) *D_d is the trajectory of the solution to the problem* (5).

PROOF. We will use the notations introduced in the proof of Lemma 2.5.

(a) Since $w'(t) = 0$ we have

$$\min\left\{\left\langle \frac{y(t)-z}{w(t)}, y'(t)\right\rangle : z \in Z_{y(t),w(t)}\right\} = 0$$

where

$$\frac{y(t)-z}{\|y(t)-z\|} = \frac{y(t)-z}{w(t)} \in S(x) \cap S^1.$$

We only need to apply Lemma 2.2 to prove $y'(t) = f(y)$. Actually, condition (a) of Lemma 2.2 is fulfilled with $e = (z - y(t))/w(t)$, $z \in \operatorname{co} Z_{y(t),w(t)}$. Because $Z_{y(t),w(t)}$ is closed there exists $z_y \in Z_{y(t),w(t)}$ satisfying $\langle (y(t)-z_y)/w(t), y'(t)\rangle = 0$ – condition (b2). Finally, the orientation condition (b1) follows from $y'(t) \in K(x)$.

(b) follows from (a) and the proof of Theorem 3.1. $\square$

4. Final remarks

Following [4, Theorem 6.1 and its Corollary] we prove the following theorem.

THEOREM 4.1. *For fixed $d > a_0$ the set $\{x \in D_d : Z_{x,d}$ has at least 2 points$\}$ is countable.*

PROOF. For each $x \in D_d$ denote by z_{x_1} and z_{x_2} two distinct points of $Z_{x,d}$ such that $\{y \in \mathbb{R}^2 :$ there exists $t \geq 0$ such that $x + ty \in \operatorname{co}\{z_{x_1}, z_{x_2}\}\} = S(x)$. Let T_x be the triangle with vertices x, z_{x_1} and z_{x_2}. Notice that $\operatorname{Int} T_x \neq \emptyset$. It is easy to see that for each $x, y \in D_d$, $x \neq y$, T_y cannot contain any vertex of T_x. Thus $\operatorname{Int} T_x \cap \operatorname{Int} T_y = \emptyset$. A collection of disjoint open subsets of $\mathbb{R}^2$ is countable. $\square$

COROLLARY 4.2. *The set of points of discontinuity of f along the trajectory of solution to* (5) *is countable.*

PROOF. Proof follows from Theorem 4.1 and Lemma 2.1. $\square$

References

[1] M. Brown, *Sets of constant distance from a planar set*, Michigan Math. J. **19** (1972), 321–323.

[2] V. F. Demyanov and V. N. Maloziemov, *On the theory of nonlinear min-max problems*, Uspekhi Mat. Nauk **XXVI** (1971), 53–104. (Russian)

[3] H. Federer, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[4] S. Ferry, *When ε-boundaries are manifolds*, Fund. Math. **90** (1976), 199–210.

[5] A. F. Filippov, *Differential Equations with Discontinuous Right-Hand Side*, Nauka, Moscow, 1985. (Russian)

[6] E. Gariepy and W. D. Pepe, *On the level sets of a distance functions in a Minkowski space*, Proc. Amer. Math. Soc. **31** (1972), 255–259.

Manuscript received February 12, 2001

Piotr Pikuta
Institute of Mathematics
Maria Curie-Skłodowska University
pl. Marii Curie-Skłodowskiej 1
20-031 Lublin, POLAND

E-mail address: ppikuta@golem.umcs.lublin.pl