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ON THE EXISTENCE OF TWO SOLUTIONS FOR A GENERAL CLASS OF JUMPING PROBLEMS

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Abstract. Via nonsmooth critical point theory we prove the existence of at least two solutions in $W_0^{1,p}(\Omega)$ for a jumping problem involving the Euler equation of multiple integrals of calculus of variations under natural growth conditions. Some new difficulties arise in comparison with the study of the semilinear and also the quasilinear case.

1. Introduction and main result

Let us consider the semilinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) = g(x,u) + \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 3$, $a_{ij} \in L^\infty(\Omega)$, $\omega \in H^{-1}(\Omega)$ and $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies

$$\lim_{s\to-\infty} \frac{g(x,s)}{s} = \alpha, \quad \lim_{s\to\infty} \frac{g(x,s)}{s} = \beta \quad \text{for some } \alpha, \beta \in \mathbb{R}.$$

Let (μ_h) be the sequence of eigenvalues, repeated according to multiplicity, of the linear operator $\{u \mapsto -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u)\}$ with homogeneous Dirichlet

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boundary conditions. Since 1972, starting from the celebrated paper of Ambrosetti and Prodi [1], the number of solutions of this jumping problem has been widely investigated, depending on the position of α and β with respect to the eigenvalues μ_h (see e.g. [19], [20], [22] and references therein).

On the other hand, since 1994, several efforts have been devoted to study the existence of weak solutions of the quasilinear problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u = g(x,u) + \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

by techniques of nonsmooth critical point theory (see [6], [9] and the subsequent papers [5], [10]; see also [2], [3] for a different approach).

In particular, a jumping problem for the previous equation has been successfully investigated in [7], [8]. More recently, existence results for the Euler equations of multiple integrals of calculus of variations

$$\begin{cases} -\text{div } (\nabla_\xi L(x,u,\nabla u)) + D_s L(x,u,\nabla u) = g(x,u) + \omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

have also been obtained in [21], [23] via techniques developed in [9], under suitable assumptions on L, $D_s L$ and $\nabla_\xi L$. In this paper we want to show that the results of [7] extend to the more general elliptic problem (1.1). It has to be noted that, in order to achieve this, some nontrivial new arguments have to be involved, in particular when dealing with the Palais–Smale condition and also, surprisingly, with the min–max estimates. We will tackle the problem from a variational point of view, that is looking for critical points of continuous functionals $f\colon W_0^{1,p}(\Omega) \to \mathbb{R}$ of type

$$f(u) = \int_\Omega L(x,u,\nabla u) - \int_\Omega G(x,u) - \langle \omega, u\rangle. \tag{1.2}$$

We point out that, in general, these functionals are not even locally Lipschitzian, so that classical critical point theory fails. Then we will employ the abstract framework of nonsmooth analysis developed in [9], [11], [13], [15], [16].

In our main result (Theorem 1.1), for a particular choice of ω, we will prove the existence of at least two solutions in $W_0^{1,p}(\Omega)$ of (1.1) by means of a classical min–max theorem in its nonsmooth version (Theorem 2.8).

More precisely, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 3$, $1 < p < n$, $\omega \in W^{-1,p'}(\Omega)$ and $L\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$ and of class C^1 in (s,ξ) a.e. in Ω. Moreover, the function

$$\{\xi \mapsto L(x,s,\xi)\}$$

is strictly convex and p-homogeneous. Furthermore, we assume the following conditions.

(A_1) there exist $\nu > 0$ and $b_1 > 0$ such that

$$\nu|\xi|^p \leq L(x,s,\xi) \leq b_1|\xi|^p \tag{1.3}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$,

(A_2) there exist $b_2, b_3 > 0$ such that

$$|D_s L(x,s,\xi)| \leq b_2|\xi|^p, \tag{1.4}$$

$$|\nabla_\xi L(x,s,\xi)| \leq b_3|\xi|^{p-1}, \tag{1.5}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$,

(A_3) there exist $R > 0$ and a bounded Lipschitzian map $\vartheta\colon \mathbb{R} \to [0,\infty[$ with

$$|s| \geq R \Rightarrow sD_s L(x,s,\xi) \geq 0, \tag{1.6}$$

$$sD_s L(x,s,\xi) \leq s\vartheta'(s)\nabla_\xi L(x,s,\xi) \cdot \xi \tag{1.7}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Without loss of generality, we may assume that $\vartheta(s) \to \overline{\vartheta} \in \mathbb{R}$ as $s \to \pm\infty$,

(A_4) $g(x,s)$ is a Carathéodory function and $G(x,s) = \int_0^s g(x,\tau)\,d\tau$. Moreover, there exist $\alpha, \beta \in \mathbb{R}$, $a \in L^{np/(n(p-1)+p)}(\Omega)$ and $b \in L^{n/p}(\Omega)$ such that

$$|g(x,s)| \leq a(x) + b(x)|s|^{p-1} \tag{1.8}$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and

$$\lim_{s\to-\infty} \frac{g(x,s)}{|s|^{p-2}s} = \alpha, \quad \lim_{s\to\infty} \frac{g(x,s)}{s^{p-1}} = \beta, \tag{1.9}$$

for a.e. $x \in \Omega$.

Let us now suppose that there exists $\ell \in L^\infty(\Omega)$ such that for a.e. $x \in \Omega$

$$\lim_{s\to\infty} L(x,s,\xi) = \lim_{s\to-\infty} L(x,s,\xi) = \ell(x)|\xi|^p, \tag{1.10}$$

$$s_h \to \infty,\ \xi_h \to \xi \Rightarrow \text{ the sequence } \nabla_\xi L(x,s_h,\xi_h) \text{ converges.} \tag{1.11}$$

Notice that both limits in (1.10) exist by virtue of (1.6). Moreover, in view of (1.3) we have $\operatorname{essinf}_{x\in\Omega}\ell(x) \geq \nu > 0$. From now on we will set $L_\infty(x,\xi) := \ell(x)|\xi|^p$ (observe that the limit in (1.11) necessarily has to be $\nabla_\xi L_\infty(x,\xi)$).

It is easily seen that, for instance, the Lagrangian $L(x,s,\xi) = (1+\arctan s^2) \cdot|\xi|^p/p$ satisfies all the previous assumptions. Let us now set

$$\lambda_1 := \min\left\{p\int_\Omega L_\infty(x,\nabla u) : u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^p = 1\right\},$$

be the first eigenvalue of

$$\{u \mapsto -\mathrm{div}\,(\nabla_\xi L_\infty(x, \nabla u))\}$$

with Dirichlet boundary data.

Observe that by [2, Lemma 1.4] the first eigenfunction ϕ_1 belongs to $L^\infty(\Omega)$ and by [24, Theorem 1.1] is strictly positive. We consider problem (1.1) with

$$\omega = t\phi_1^{p-1} + \omega_0, \quad \text{where } \omega_0 \in W^{-1,p'}(\Omega) \text{ and } t \in \mathbb{R}.$$

Under the previous assumptions, the following is the main result.

Theorem 1.1. *Assume that $\beta < \lambda_1 < \alpha$. Then there exist $\overline{t}, \underline{t} \in \mathbb{R}$ such that the problem*

$$\begin{cases} -\mathrm{div}\,(\nabla_\xi L(x, u, \nabla u)) + D_s L(x, u, \nabla u) & \\ \qquad\qquad = g(x, u) + t\phi_1^{p-1} + \omega_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

admits at least two solutions in $W_0^{1,p}(\Omega)$ for $t > \overline{t}$ and no solution for $t < \underline{t}$.

This result extends the main achievement of [7] dealing with the case $p = 2$ and

$$L(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^n a_{ij}(x, s)\xi_i\xi_j - G(x, s)$$

where the coefficients $a_{ij}(x, s)\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable in x, of class C^1 in s with $a_{ij}, D_s a_{ij} \in L^\infty(\Omega \times \mathbb{R})$ and satisfy

$$\sum_{i,j=1}^n a_{ij}(x, s)\xi_i\xi_j \geq \nu|\xi|^2, \qquad \sum_{i,j=1}^n sD_s a_{ij}(x, s)\xi_i\xi_j \geq 0,$$
$$\sum_{i,j=1}^n sD_s a_{ij}(x, s)\xi_i\xi_j \leq s\vartheta'(s) \sum_{i,j=1}^n a_{ij}(x, s)\xi_i\xi_j$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$, where $\vartheta\colon \mathbb{R} \to [0, \infty[$ is a bounded Lipschitzian map.

In this particular case, existence of at least three solutions has been recently proved in [8] assuming $\beta < \mu_1$ and $\alpha > \mu_2$, where μ_1 and μ_2 are the first and second eigenvalue of the operator

$$\Big\{u \mapsto -\sum_{i,j=1}^n D_j(A_{ij} D_i u)\Big\}, \quad A_{ij}(x) := \lim_{s \to \pm\infty} a_{ij}(x, s).$$

On the other hand, in our general setting, it is not clear how to define higher eigenvalues $\lambda_2, \lambda_3, \ldots$ with suitable properties. It must be noted that in [4] a possible characterization of the second eigenvalue is given for the p-Laplacian operator.

The plan of the paper is as follows: in Section 2 we recall some notions of nonsmooth critical point theory and a suitable Mountain Pass Theorem (Theorem 2.8); in Section 3 we state the variational formulation of the problem and prove that a suitable compactness condition is satisfied by the functional related to our problem; in Section 4 we show that also the required geometrical properties are satisfied; in Section 5 we end up the proof of the main result (Theorem 1.1).

2. Recalls of nonsmooth critical point theory

In this section we quote from [9], [11] some tools of nonsmooth critical point theory which we use in the paper.

Let us first recall the definition of weak slope for a continuous function.

Definition 2.1. Let X be a complete metric space, $F: X \to \mathbb{R}$ be a continuous function and $u \in X$. We denote by $|dF|(u)$ the supremum of the real numbers $\sigma \geq 0$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H}: B(u,\delta) \times [0,\delta] \to X,$$

such that, for every v in $B(u,\delta)$, and for every t in $[0,\delta]$ it results

$$d(\mathcal{H}(v,t),v) \leq t, \quad F(\mathcal{H}(v,t)) \leq F(v) - \sigma t.$$

The extended real number $|dF|(u)$ is called the *weak slope of F at u*.

The previous notion allows us to give the following definitions.

Definition 2.2. We say that $u \in X$ is a *critical point of F* if $|dF|(u) = 0$. We say that $c \in \mathbb{R}$ is a *critical value of F* if there exists a critical point $u \in X$ of F with $F(u) = c$.

Definition 2.3. Let $c \in \mathbb{R}$. We say that F satisfies the Palais–Smale condition at level c ($(\mathrm{PS})_c$ in short), if every sequence (u_h) in X such that $|dF|(u_h) \to 0$ and $F(u_h) \to c$ admits a subsequence converging in X.

Let us now turn to the concrete setting. Let $f: W_0^{1,p}(\Omega) \to \mathbb{R}$ be the functional defined in (1.2), which is continuous in view of (1.3). Notice that conditions (1.4) and (1.5) imply that for every $u \in W_0^{1,p}(\Omega)$ we have

$$\nabla_\xi L(x,u,\nabla u) \in L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^n), \quad D_s L(x,u,\nabla u) \in L^1_{\mathrm{loc}}(\Omega).$$

Therefore for each $u \in W_0^{1,p}(\Omega)$ we have

$$-\mathrm{div}\,(\nabla_\xi L(x,u,\nabla u)) + D_s L(x,u,\nabla u) \in \mathcal{D}'(\Omega).$$

Definition 2.4. We say that u is a *weak solution to* (1.1) if $u \in W_0^{1,p}(\Omega)$ and
$$-\mathrm{div}\,(\nabla_\xi L(x,u,\nabla u)) + \nabla_s L(x,u,\nabla u) = g(x,u) + \omega$$
in $\mathcal{D}'(\Omega)$.

Let us introduce the following variant of the $(\mathrm{PS})_c$ condition.

Definition 2.5. Let $c \in \mathbb{R}$. A sequence $(u_h) \subset W_0^{1,p}(\Omega)$ is said to be a *concrete Palais–Smale sequence at level* c ($(\mathrm{CPS})_c$-sequence, in short) for f, if $f(u_h) \to c$,
$$-\mathrm{div}\,(\nabla_\xi L(x,u_h,\nabla u_h)) + D_s L(x,u_h,\nabla u_h) \in W^{-1,p'}(\Omega),$$
eventually as $h \to \infty$ and
$$-\mathrm{div}\,(\nabla_\xi L(x,u_h,\nabla u_h)) + D_s L(x,u_h,\nabla u_h) - g(x,u_h) - \omega \to 0,$$
strongly in $W^{-1,p'}(\Omega)$.

We say that f *satisfies the concrete Palais–Smale condition at level* c ($(\mathrm{CPS})_c$ in short), if every $(\mathrm{CPS})_c$-sequence for f admits a strongly convergent subsequence.

Proposition 2.6. *For every* $u \in W_0^{1,p}(\Omega)$ *such that* $|df|(u) < \infty$ *we have*
$$\| - \mathrm{div}\,(\nabla_\xi L(x,u,\nabla u)) + D_s L(x,u,\nabla u) - g(x,u) - \omega\|_{-1,p'} \le |df|(u).$$

Proof. See [9, Theorem 2.1.3]. $\square$

The previous result implies the following remark.

Remark 2.7. The following facts hold:

(a) each critical point u of f is a weak solution to (1.1),
(b) if $c \in \mathbb{R}$ and f satisfies $(\mathrm{CPS})_c$ then f satisfies $(\mathrm{PS})_c$.

The next is the main tool in proving the existence of two solutions.

Theorem 2.8. *Let* $u_0, v_0, v_1 \in W_0^{1,p}(\Omega)$ *and* $r > 0$ *be such that*
$$\|v_0 - u_0\|_{1,p} < r, \quad \|v_1 - u_0\|_{1,p} > r, \quad \inf f(\overline{B_r(u_0)}) > -\infty$$
and
$$\inf\{f(u) : u \in W_0^{1,p}(\Omega),\ \|u - u_0\|_{1,p} = r\} > \max\{f(v_0), f(v_1)\}.$$
Let
$$\Gamma = \{\gamma\colon [0,1] \to W_0^{1,p}(\Omega) : \ \gamma \textit{ is continuous}, \gamma(0) = v_0 \textit{ and } \gamma(1) = v_1\},$$
and assume that $\Gamma \ne \emptyset$ *and that* f *satisfies the Palais–Smale condition at the two levels*
$$c_1 = \inf_{\overline{B_r(u_0)}} f, \qquad c_2 = \inf_{\gamma\in\Gamma} \max_{[0,1]} (f \circ \gamma).$$

Then it results $-\infty < c_1 < c_2 < \infty$ and there exist two solutions $u_1, u_2 \in W_0^{1,p}(\Omega)$ of (1.1) with $f(u_1) = c_1$ and $f(u_2) = c_2$.

Proof. See [13, Theorem 3.12]. □

3. Variational formulation and Palais–Smale condition

Let us now consider

$$g_0(x,s) := g(x,s) - \beta|s|^{p-2}s^+ + \alpha|s|^{p-2}s^-, \quad G_0(x,s) := \int_0^s g_0(x,\tau)\,d\tau.$$

Of course, g_0 is a Carathéodory function satisfying for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$

$$\lim_{|s|\to\infty} \frac{g_0(x,s)}{|s|^{p-2}s} = 0, \quad |g_0(x,s)| \le a(x) + \widetilde{b}(x)|s|^{p-1},$$

with $\widetilde{b} \in L^{n/p}(\Omega)$. Since we are interested in solutions $u \in W_0^{1,p}(\Omega)$ of the equation

$$-\mathrm{div}\,(\nabla_\xi L(x,u,\nabla u)) + D_s L(x,u,\nabla u) = g(x,u) + t\phi_1^{p-1} + \omega_0,$$

let us define the associated functional $f_t : W_0^{1,p}(\Omega) \to \mathbb{R}$, by setting

$$\begin{aligned} f_t(u) := {} & \int_\Omega L(x,u,\nabla u) - \frac{\beta}{p}\int_\Omega (u^+)^p - \frac{\alpha}{p}\int_\Omega (u^-)^p \\ & - \int_\Omega G_0(x,u) - |t|^{p-2}t\int_\Omega \phi_1^{p-1}u - \langle \omega_0, u\rangle. \end{aligned} \tag{3.1}$$

In order to prove our main result, the idea is to apply Theorem 2.8 to the functional f_t defined above. To this aim, we will prove in the following that f_t satisfies the concrete Palais–Smale condition (see Theorem 3.4) as well as the Mountain–Pass geometric assumptions (see Propositions 4.5 and 4.6).

Let now M be the positive constant such that

$$|D_s L(x,s,\xi)| \le M \nabla_\xi L(x,s,\xi)\cdot\xi \tag{3.2}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ (such a constant exists by (1.3) and (1.4)).

In the following result we prove one of the main tools of the paper.

Lemma 3.1. *Let $(u_h) \subset W_0^{1,p}(\Omega)$ and $(\varrho_h) \subset]0,\infty[$ with $\varrho_h \to \infty$ such that*

$$v_h = \frac{u_h}{\varrho_h} \rightharpoonup v \quad \text{in } W_0^{1,p}(\Omega).$$

Let $\gamma_h \rightharpoonup \gamma$ in $L^{n/p}(\Omega)$ with $|\gamma_h| \le c$ for some $c \in L^{n/p}(\Omega)$. Moreover, let

$$\mu_h \to \mu \quad \text{in } L^{np'/(n+p')}(\Omega), \qquad \delta_h \to \delta \quad \text{in } W^{-1,p'}(\Omega)$$

be such that

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla\varphi + \int_\Omega D_s L(x,u_h,\nabla u_h)\varphi = \int_\Omega \gamma_h |u_h|^{p-2}u_h\varphi + \varrho_h^{p-1}\int_\Omega \mu_h\varphi + \langle\delta_h,\varphi\rangle \tag{3.3}$$

for every $\varphi\in C_c^\infty(\Omega)$. Then, the following facts hold

(a) (v_h) *is strongly convergent to v in $W_0^{1,p}(\Omega)$,*
(b) $(\gamma_h|v_h|^{p-2}v_h)$ *is strongly convergent to $\gamma|v|^{p-2}v$ in $W^{-1,p'}(\Omega)$,*
(c) *there exist $\eta^+,\eta^-\in L^\infty(\Omega)$ such that*

$$\eta^+(x)=\begin{cases}\exp\{-\overline{\vartheta}\} & \text{if } v(x)>0,\\ \exp\{MR\} & \text{if } v(x)<0,\end{cases}$$
$$\exp\{-\overline{\vartheta}\}\le\eta^+(x)\le\exp\{MR\}\quad \text{if } v(x)=0,$$

and

$$\eta^-(x)=\begin{cases}\exp\{-\overline{\vartheta}\} & \text{if } v(x)<0,\\ \exp\{MR\} & \text{if } v(x)>0,\end{cases}$$
$$\exp\{-\overline{\vartheta}\}\le\eta^-(x)\le\exp\{MR\}\quad \text{if } v(x)=0.$$

Moreover,

$$\int_\Omega \eta^+\nabla_\xi L_\infty(x,\nabla v)\cdot\nabla\varphi \ge \int_\Omega \gamma\eta^+|v|^{p-2}v\varphi + \int_\Omega \mu\eta^+\varphi, \tag{3.4}$$

$$\int_\Omega \eta^-\nabla_\xi L_\infty(x,\nabla v)\cdot\nabla\varphi \le \int_\Omega \gamma\eta^-|v|^{p-2}v\varphi + \int_\Omega \mu\eta^-\varphi \tag{3.5}$$

for every $\varphi\in W_0^{1,p}(\Omega)$ with $\varphi\ge 0$.

Proof. Arguing as in [7, Lemma 3.1] assertion (b) immediately follows. Let us now prove assertion (a). Up to a subsequence, $v_h(x)\to v(x)$ for a.e. $x\in\Omega$. Consider now the map $\zeta\colon\mathbb{R}\to\mathbb{R}$ defined as

$$\zeta(s)=\begin{cases} Ms & \text{if } 0<s<R,\\ MR & \text{if } s\ge R,\\ -Ms & \text{if } -R<s<0,\\ MR & \text{if } s\le -R.\end{cases}$$

By [23, Proposition 3.1] we may choose $\varphi=v_h\exp\{\zeta(u_h)\}$ in (3.3), yielding

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla v_h\exp\{\zeta(u_h)\} + \int_\Omega [D_s L(x,u_h,\nabla u_h)+\zeta'(u_h)\nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla u_h]v_h\exp\{\zeta(u_h)\}$$

$$= \int_\Omega \gamma_h |u_h|^{p-2} u_h v_h \exp\{\zeta(u_h)\}$$

$$+ \varrho_h^{p-1} \int_\Omega \mu_h v_h \exp\{\zeta(u_h)\} + \langle \delta_h, v_h \exp\{\zeta(u_h)\}\rangle.$$

Therefore, taking into account conditions (1.6) and (3.2), we have

$$\varrho_h^{p-1} \int_\Omega \nabla_\xi L(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \leq \varrho_h^{p-1} \int_\Omega \gamma_h |v_h|^p \exp\{\zeta(u_h)\}$$
$$+ \varrho_h^{p-1} \int_\Omega \mu_h v_h \exp\{\zeta(u_h)\} + \langle \delta_h, v_h \exp\{\zeta(u_h)\}\rangle.$$

After division by ϱ_h^{p-1}, using the hypotheses on γ_h, μ_h and δ_h, we obtain

$$\limsup_h \int_\Omega \nabla_\xi L(x, u_h, \nabla v_h) \cdot \nabla v_h \exp\{\zeta(u_h)\} \leq \exp\{MR\}\bigg(\int_\Omega \gamma |v|^p + \int_\Omega \mu v\bigg). \tag{3.6}$$

Now, let us consider the function $\vartheta_1 : \mathbb{R} \to \mathbb{R}$ given by

$$\vartheta_1(s) = \begin{cases} \vartheta(s) & \text{if } s \geq 0, \\ Ms & \text{if } -R \leq s \leq 0, \\ -MR & \text{if } s \leq -R, \end{cases}$$

where the function ϑ satisfies condition (1.7).

Putting in (3.3) the test functions $(v^+ \wedge k) \exp\{-\vartheta_1(u_h)\}$ with $k \in \mathbb{N}$, we obtain

$$\begin{aligned} &\int_\Omega \nabla_\xi L(x, u_h, \nabla v_h) \cdot \nabla (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \\ &\quad + \varrho_h^{1-p} \int_\Omega [D_s L(x, u_h, \nabla u_h) - \vartheta_1'(u_h) \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h] \\ &\quad \cdot (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \\ &= \int_\Omega \gamma_h |v_h|^{p-2} v_h (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \\ &\quad + \int_\Omega \mu_h (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} + \varrho_h^{1-p} \langle \delta_h, (v^+ \wedge k) \exp\{-\vartheta_1(u_h)\}\rangle. \end{aligned} \tag{3.7}$$

By (1.6), (1.7) and (3.2) it results for every $h \in \mathbb{N}$

$$[D_s L(x, u_h, \nabla u_h) - \vartheta_1'(u_h) \nabla_\xi L(x, u_h, \nabla u_h) \cdot \nabla u_h](v^+ \wedge k) \exp\{-\vartheta_1(u_h)\} \leq 0.$$

Taking into account (1.5) and (1.11), one may apply [12, Theorem 5] and deduce that

$$\nabla v_h(x) \to \nabla v(x) \quad \text{for a.e. } x \in \Omega \setminus \{v = 0\}.$$

Being $u_h(x) \to \infty$ a.e. in $\Omega \setminus \{v = 0\}$, again recalling (1.11), we have

$$\nabla_\xi L(x, u_h(x), \nabla v_h(x)) \to \nabla_\xi L_\infty(x, \nabla v(x)) \quad \text{for a.e. } x \in \Omega \setminus \{v = 0\}.$$

Combining this pointwise convergence with (1.5), we obtain

$$\nabla_\xi L(x,u_h,\nabla v_h) \rightharpoonup \nabla_\xi L_\infty(x,\nabla v) \quad \text{in } L^{p'}(\Omega).$$

Therefore, for every $k \in \mathbb{N}$ we have

$$\lim_h \int_\Omega \nabla_\xi L(x,u_h,\nabla v_h)\cdot\nabla(v^+\wedge k)\exp\{-\vartheta_1(u_h)\} = \int_\Omega \nabla_\xi L_\infty(x,\nabla v)\cdot\nabla(v^+\wedge k)\exp\{-\overline{\vartheta}\},$$

$$\lim_h (v^+\wedge k)\exp\{-\vartheta_1(u_h)\} = (v^+\wedge k)\exp\{-\overline{\vartheta}\}$$

weakly in $W^{1,p}_0(\Omega)$,

$$\lim_h \int_\Omega \gamma_h|v_h|^{p-2}v_h(v^+\wedge k)\exp\{-\vartheta_1(u_h)\} = \int_\Omega \gamma|v|^{p-2}v(v^+\wedge k)\exp\{-\overline{\vartheta}\}$$

(by virtue of (b)) and

$$\lim_h \frac{1}{\varrho_h^{p-1}}(v^+\wedge k)\exp\{-\vartheta_1(u_h)\} = 0$$

weakly in $W^{1,p}_0(\Omega)$. Therefore, letting $h\to\infty$ in (3.7), for every $k\in\mathbb{N}$ we get

$$\int_\Omega \nabla_\xi L_\infty(x,\nabla v)\cdot\nabla(v^+\wedge k)\exp\{-\overline{\vartheta}\} \geq \int_\Omega \gamma|v|^{p-2}v(v^+\wedge k)\exp\{-\overline{\vartheta}\} + \int_\Omega \mu(v^+\wedge k)\exp\{-\overline{\vartheta}\}.$$

Finally, if we let $k\to\infty$, after division by $\exp\{-\overline{\vartheta}\}$, we have

$$\int_\Omega \nabla_\xi L_\infty(x,\nabla v^+)\cdot\nabla v^+ \geq \int_\Omega \gamma|v|^{p-2}(v^+)^2 + \int_\Omega \mu v^+. \tag{3.8}$$

Analogously, if we define a function $\vartheta_2\colon\mathbb{R}\to\mathbb{R}$ by

$$\vartheta_2(s) = \begin{cases} \vartheta(s) & \text{if } s\leq 0,\\ -Ms & \text{if } 0\leq s\leq R,\\ -MR & \text{if } s\geq R,\end{cases}$$

and consider in (3.3) the test functions $(v^-\wedge k)\exp\{-\vartheta_2(u_h)\}$ with $k\in\mathbb{N}$, we obtain

$$\int_\Omega \nabla_\xi L_\infty(x,\nabla v)\cdot\nabla v^- \leq -\int_\Omega \gamma|v|^{p-2}(v^-)^2 + \int_\Omega \mu v^-. \tag{3.9}$$

Thus, combining the inequalities (3.8) and (3.9), we get

$$\int_\Omega \nabla_\xi L_\infty(x,\nabla v)\cdot\nabla v \geq \int_\Omega \gamma|v|^p + \int_\Omega \mu v. \tag{3.10}$$

Putting together (3.6) and (3.10), we conclude that

$$\limsup_h \int_\Omega \nabla_\xi L(x,u_h,\nabla v_h)\cdot\nabla v_h \exp\{\zeta(u_h)\} \le \exp\{MR\}\int_\Omega \nabla_\xi L_\infty(x,\nabla v)\cdot\nabla v.$$

In particular, by Fatou's Lemma, it results

$$\begin{aligned}\exp\{MR\}\int_\Omega \nabla_\xi L_\infty(x,\nabla v)\cdot\nabla v &\le \liminf_h \int_\Omega \nabla_\xi L(x,u_h,\nabla v_h)\cdot\nabla v_h\exp\{\zeta(u_h)\}\\ &\le \exp\{MR\}\int_\Omega \nabla_\xi L_\infty(x,\nabla v)\cdot\nabla v,\end{aligned}$$

namely, as $h\to\infty$, we get

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla v_h)\cdot\nabla v_h\exp\{\zeta(u_h)\}\to\exp\{MR\}\int_\Omega \nabla_\xi L_\infty(x,\nabla v)\cdot\nabla v.$$

Therefore, since $\nu|\nabla v_h|^p\le\nabla_\xi L(x,u_h,\nabla v_h)\cdot\nabla v_h\exp\{\zeta(u_h)\}$, thanks to Lebesgue's Theorem, we obtain that

$$\lim_h\int_\Omega|\nabla v_h|^p=\int_\Omega|\nabla v|^p,$$

which concludes the proof of (a).

Let us now prove assertion (c). Up to a subsequence, $\exp\{-\vartheta_1(u_h)\}$ weakly* converges in $L^\infty(\Omega)$ to some η^+. Of course, we have

$$\eta^+(x)=\begin{cases}\exp\{-\overline{\vartheta}\} & \text{if } v(x)>0,\\ \exp\{MR\} & \text{if } v(x)<0,\end{cases}$$
$$\exp\{-\overline{\vartheta}\}\le\eta^+(x)\le\exp\{MR\}\quad\text{if } v(x)=0.$$

Then, let us consider in (3.3) as test functions:

$$\varphi\exp\{-\vartheta_1(u_h)\},\quad\varphi\in C_c^\infty(\Omega),\quad\varphi\ge0.$$

Whence, like in the previous arguments, we obtain

$$\int_\Omega\eta^+\nabla_\xi L_\infty(x,\nabla v)\cdot\nabla\varphi\ge\int_\Omega\gamma\eta^+|v|^{p-2}v\varphi+\int_\Omega\mu\eta^+\varphi,$$

for any positive $\varphi\in W_0^{1,p}(\Omega)$. Similarly, by means of the test functions

$$\varphi\exp\{-\vartheta_2(u_h)\},\quad\varphi\in C_c^\infty(\Omega),\quad\varphi\ge0,$$

we get for any positive $\varphi\in W_0^{1,p}(\Omega)$

$$\int_\Omega\eta^-\nabla_\xi L_\infty(x,\nabla v)\cdot\nabla\varphi\le\int_\Omega\gamma\eta^-|v|^{p-2}v\varphi+\int_\Omega\mu\eta^-\varphi,$$

where η^- is the weak* limit of some subsequence of $\exp\{-\vartheta_2(u_h)\}$. □

Arguing as in [7, Lemma 3.3], one obtains the following result.

Lemma 3.2. *Let (u_h) a sequence in $W_0^{1,p}(\Omega)$ and $\varrho_h \subset \,]0,\infty[$ with $\varrho_h \to \infty$. Assume that the sequence (u_h/ϱ_h) is bounded in $W_0^{1,p}(\Omega)$. Then*

$$\frac{g_0(x,u_h)}{\varrho_h^{p-1}} \to 0 \quad \text{in } L^{np'/(n+p')}(\Omega), \qquad \frac{G_0(x,u_h)}{\varrho_h^{p}} \to 0 \quad \text{in } L^1(\Omega)$$

as $h \to \infty$.

Lemma 3.3. *Let f_t be the functional defined in* (3.1). *Then for every $c,t \in \mathbb{R}$ the following facts are equivalent:*

(a) *f_t satisfies the* $(\mathrm{CPS})_c$ *condition,*
(b) *every* $(\mathrm{CPS})_c$*-sequence for f_t is bounded in $W_0^{1,p}(\Omega)$.*

Proof. The proof that (a)⇒(b) is trivial. Let us prove (b)⇒(a). Let (u_h) be a $(\mathrm{CPS})_c$-sequence for f_t. Since (u_h) is bounded in $W_0^{1,p}(\Omega)$, and the map

$$\{u \mapsto g(x,u) + t\phi_1^{p-1} + \omega_0\},$$

is completely continuous by (1.8), up to a subsequence $(g(x,u_h)+t\phi_1^{p-1}+\omega_0)$ is strongly convergent in $L^{np'/(n+p')}(\Omega)$, hence in $W^{-1,p'}(\Omega)$. By [23, Theorem 3.2] it follows that (u_h) is strongly convergent in $W_0^{1,p}(\Omega)$. □

We now come to one of the main tool of this paper.

Theorem 3.4. *Let f_t be the functional defined in* (3.1). *Then for every $c,t \in \mathbb{R}$ f_t satisfies the* $(\mathrm{CPS})_c$ *condition.*

Proof. If (u_h) is a $(\mathrm{CPS})_c$-sequence for f_t, we have $f_t(u_h) \to c$ and, for all $v \in C_c^\infty(\Omega)$ we have

$$\int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla v + \int_\Omega D_s L(x,u_h,\nabla u_h)v - \beta\int_\Omega (u_h^+)^{p-1}v \\ + \alpha\int_\Omega (u_h^-)^{p-1}v - \int_\Omega g_0(x,u_h)v - |t|^{p-2}t\int_\Omega \phi_1 v = \langle \omega_0+\sigma_h, v\rangle,$$

where $\sigma_h \to 0$ in $W^{-1,p'}(\Omega)$ as $h \to \infty$. Taking into account Lemma 3.3 it suffices to show that (u_h) is bounded in $W_0^{1,p}(\Omega)$. Assume by contradiction that, up to a subsequence, $\|u_h\|_{1,p} \to \infty$ as $h \to \infty$ and set

$$v_h = \frac{u_h}{\varrho_h}, \quad \varrho_h = \|u_h\|_{1,p}.$$

By Lemma 3.2, we can apply Lemma 3.1 choosing

$$\gamma_h(x) = \begin{cases} \beta & \text{if } u_h(x) \ge 0, \\ \alpha & \text{if } u_h(x) < 0, \end{cases}$$

$$\mu_h = \frac{g_0(x,u_h)}{\|u_h\|_{1,p}^{p-1}}, \quad \delta_h = |t|^{p-2}t\phi_1 + \omega_0 + \sigma_h.$$

Then, up to a subsequence, (v_h) strongly converges to some v in $W_0^{1,p}(\Omega)$. Moreover, putting $\varphi = v^+$ in (3.5) of Lemma 3.1, we get

$$\int_\Omega \eta^- \nabla_\xi L_\infty(x, \nabla v^+) \cdot \nabla v^+ \leq \int_\Omega \beta \eta^- (v^+)^p,$$

hence, taking into account (1.12), we have

$$\lambda_1 \int_\Omega (v^+)^p \leq \int_\Omega \nabla_\xi L_\infty(x, \nabla v^+) \cdot \nabla v^+ \leq \beta \int_\Omega (v^+)^p.$$

Since $\beta < \lambda_1$, then $v^+ = 0$. Using again (3.4) of Lemma 3.1, for every $\varphi \geq 0$ we get

$$\int_\Omega \eta^+ \nabla_\xi L_\infty(x, \nabla v) \cdot \nabla\varphi \geq \alpha \int_\Omega \eta^+ |v|^{p-2} v\varphi.$$

namely, since $v \leq 0$, we have

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla v) \cdot \nabla\varphi \geq \alpha \int_\Omega |v|^{p-2} v\varphi.$$

In a similar way, by (3.5) of Lemma 3.1 we get

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla v) \cdot \nabla\varphi \leq \alpha \int_\Omega |v|^{p-2} v\varphi.$$

Therefore we get

$$\int_\Omega \nabla_\xi L_\infty(x, \nabla v) \cdot \nabla\varphi = \alpha \int_\Omega |v|^{p-2} v\varphi,$$

which, in view of [18, Remark 1, p. 161] is not possible if α differs from λ_1. $\square$

4. Min–max estimates

In this section we will prove that our functional satisfies the geometrical assumptions required by the abstract multiplicity result (Theorem 2.8). Let us first introduce the "asymptotic functional" $f_\infty \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ by setting

$$f_\infty(u) := \int_\Omega L_\infty(x, \nabla u) - \frac{\beta}{p} \int_\Omega (u^+)^p - \frac{\alpha}{p} \int_\Omega (u^-)^p - \int_\Omega \phi_1^{p-1} u.$$

Then consider the functional $\widetilde{f}_t \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ given by $\widetilde{f}_t(u) = f_t(tu)/t^p$, namely

$$\begin{aligned}\widetilde{f}_t(u) := \int_\Omega L(x, tu, \nabla u) - \frac{\beta}{p} \int_\Omega (u^+)^p - \frac{\alpha}{p} \int_\Omega (u^-)^p \\ - \int_\Omega \frac{G_0(x, tu)}{t^p} - \int_\Omega \phi_1^{p-1} u - \frac{\langle \omega_0, u\rangle}{t^{p-1}}.\end{aligned}$$

Theorem 4.1. *The following facts hold.*

(a) *Assume that* $(t_h) \subset]0,\infty[$ *with* $t_h \to \infty$ *and* $u_h \to u$ *in* $W_0^{1,p}(\Omega)$. *Then*

$$\lim_h \widetilde{f}_{t_h}(u_h) = f_\infty(u).$$

(b) *Assume that* $(t_h) \subset]0,\infty[$ *with* $t_h \to \infty$ *and* $u_h \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$. *Then*

$$f_\infty(u) \le \liminf_h \widetilde{f}_{t_h}(u_h).$$

(c) *Assume that* $(t_h) \subset]0,\infty[$ *with* $t_h \to \infty$, $u_h \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *and*

$$\limsup_h \widetilde{f}_{t_h}(u_h) \le f_\infty(u).$$

Then (u_h) *strongly converges to* u *in* $W_0^{1,p}(\Omega)$.

Proof. (a) is easy to prove.

(b) Since $u_h \to u$ in $L^q(\Omega)$ for every $q < 2n/(n-2)$, it is sufficient to prove that

$$\int_\Omega L_\infty(x,\nabla u) \le \liminf_h \int_\Omega L(x,t_h u_h,\nabla u_h).$$

Let us define the Carathéodory function $\widetilde{L}\colon \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ by setting

$$\widetilde{L}(x,s,\xi) = \begin{cases} L(x,\tan(s),\xi) & \text{if } |s| < \pi/2, \\ L_\infty(x,\xi) & \text{if } |s| \ge \pi/2. \end{cases}$$

Note that $\widetilde{L} \ge 0$ and $\widetilde{L}(x,s,\,\cdot\,)$ is convex. Up to a subsequence we have

$$t_h u_h \to z \quad \text{for a.e. } x \in \Omega\setminus\{u=0\}, \quad \nabla u_h \rightharpoonup \nabla u \quad \text{in } L^p(\Omega\setminus\{u=0\}),$$

and

$$\arctan(t_h u_h) \to \arctan(z) \quad \text{in } L^p(\Omega\setminus\{u=0\}).$$

Therefore, by [14, Theorem 1] we deduce that

$$\int_{\Omega\setminus\{u=0\}} \widetilde{L}(x,\arctan(z),\nabla u) \le \liminf_h \int_{\Omega\setminus\{u=0\}} \widetilde{L}(x,\arctan(t_h u_h),\nabla u_h),$$

that implies

$$\begin{aligned}\int_\Omega L_\infty(x,\nabla u) &= \int_{\Omega\setminus\{u=0\}} L_\infty(x,\nabla u) \\ &\le \liminf_h \int_{\Omega\setminus\{u=0\}} L(x,t_h u_h,\nabla u_h) = \liminf_h \int_\Omega L(x,t_h u_h,\nabla u_h).\end{aligned}$$

Let us now prove (c). As above, we obtain

$$\liminf_h \int_\Omega L\bigg(x,t_h u_h,\frac{1}{2}\nabla u_h + \frac{1}{2}\nabla u\bigg) \ge \int_\Omega L_\infty(x,\nabla u).$$

Since we have

$$\lim_h \int_\Omega L(x,t_h u_h,\nabla u) = \int_\Omega L_\infty(x,\nabla u)$$

and

$$\limsup_h \int_\Omega L(x, t_h u_h, \nabla u_h) \le \int_\Omega L_\infty(x, \nabla u), \tag{4.1}$$

we get

$$\limsup_h \int_\Omega (L(x, t_h u_h, \nabla u_h) - L(x, t_h u_h, \nabla u)) \le 0.$$

On the other hand, the strict convexity implies that for every $h \in \mathbb{N}$

$$\frac{1}{2}L(x, t_h u_h, \nabla u_h) + \frac{1}{2}L(x, t_h u_h, \nabla u) - L\biggl(x, t_h u_h, \frac{1}{2}\nabla u_h + \frac{1}{2}\nabla u\biggr) > 0.$$

Therefore, the previous limits yield

$$\int_\Omega \biggl\{\frac{1}{2}L(x, t_h u_h, \nabla u_h) + \frac{1}{2}L(x, t_h u_h, \nabla u) - L\biggl(x, t_h u_h, \frac{1}{2}\nabla u_h + \frac{1}{2}\nabla u\biggr)\biggr\} \to 0.$$

In particular, up to a subsequence, we have

$$\frac{1}{2}L(x, t_h u_h, \nabla u_h) + \frac{1}{2}L(x, t_h u_h, \nabla u) - L\biggl(x, t_h u_h, \frac{1}{2}\nabla u_h + \frac{1}{2}\nabla u\biggr) \to 0,$$

a.e. in Ω. It easily verified that this can be true only if

$$\nabla u_h(x) \to \nabla u(x) \quad \text{for a.e. } x \in \Omega.$$

Then we have

$$L(x, t_h u_h(x), \nabla u_h(x)) \to L_\infty(x, \nabla u(x)) \quad \text{for a.e. } x \in \Omega.$$

Taking into account (4.1), we deduce

$$\int_\Omega L(x, t_h u_h, \nabla u_h) \to \int_\Omega L_\infty(x, \nabla u),$$

that by $\nu|\nabla u_h|^p \le L(x, t_h u_h, \nabla u_h)$ yields

$$\lim_h \int_\Omega |\nabla u_h|^p = \int_\Omega |\nabla u|^p,$$

namely the convergence of u_h to u in $W_0^{1,p}(\Omega)$. □

Remark 4.2. Assume that $\beta < \lambda_1 < \alpha$. Then the following facts hold:

(a) $f'_\infty(\overline{\phi_1})(\phi_1) = 0$,
(b) $\lim_{s\to-\infty} f_\infty(s\phi_1) = -\infty$, where we have set $\overline{\phi_1} = \phi_1/(\lambda_1 - \beta)^{1/(p-1)}$.

Proof. (a) is easy to prove.

(b) A direct computation yields that for $s < 0$

$$f_\infty(s\phi_1) = \frac{\lambda_1 - \alpha}{p}|s|^p - s.$$

Since $\alpha > \lambda_1$, assertion (b) follows. □

Lemma 4.3. *For every $M > 0$ there exists $\varrho > 0$ such that for every $w \in W_0^{1,p}(\Omega)$ with $\|w - \phi_1\|_{1,p} \le \varrho$ we have*

$$\int_\Omega L_\infty(x, -\nabla w^-) \ge M \int_\Omega (w^-)^p.$$

Proof. Argue as in [7, Lemma 4.1]. $\square$

Lemma 4.4. *There exists $r > 0$ such that*

(a) *if $\|w - \overline{\phi_1}\|_{1,p} \le r$ then $f_\infty(w) \ge f_\infty(\overline{\phi_1})$ for all $w \in W_0^{1,p}(\Omega)$,*
(b) *if $\|w - \overline{\phi_1}\|_{1,p} = r$ then $f_\infty(w) > f_\infty(\overline{\phi_1})$ for all $w \in W_0^{1,p}(\Omega)$.*

Proof. Let us fix a $u \in W_0^{1,p}(\Omega)$ and define $\eta_u :]0,\infty[\to \mathbb{R}$ by setting $\eta_u(t) = f_\infty(tu)$. It is easy to verify that η_u assumes the minimum value

$$\mathcal{M}(u) = -\frac{\left(1 - \dfrac{1}{p}\right)\left(\dfrac{1}{p}\right)^{1/(p-1)} \left[\displaystyle\int_\Omega \phi_1^{p-1} u\right]^{p/(p-1)}}{\left[\displaystyle\int_\Omega L_\infty(x, \nabla u) - \frac{\beta}{p}\int_\Omega (u^+)^p - \frac{\alpha}{p}\int_\Omega (u^-)^p\right]^{1/(p-1)}}.$$

Moreover, a direct computation yields for every $u \ne \overline{\phi_1}$

$$f_\infty(\overline{\phi_1}) < \mathcal{M}(u) \tag{4.2}$$

if and only if

$$p\int_\Omega L_\infty(x, \nabla u) > \beta\int_\Omega (u^+)^p + \alpha\int_\Omega (u^-)^p + (\lambda_1 - \beta)\left[\int_\Omega \phi_1^{p-1} u\right]^p. \tag{4.3}$$

If we now set $W = \{u \in W_0^{1,p}(\Omega) : \int_\Omega \phi_1^{p-1} u = 0\}$, we obtain

$$W_0^{1,p}(\Omega) = \mathrm{span}(\phi_1) \oplus W. \tag{4.4}$$

Let us now prove that (4.3) is really fulfilled in a neighbourhood of $\overline{\phi_1}$. Since (4.3) is homogeneous of degree p, we may substitute $\overline{\phi_1}$ with ϕ_1. Let us first consider the case $p \ge 2$ and $\beta > 0$. In view of (4.4), by strict convexity, there exists $\varepsilon_p > 0$ such that for any $w \in W$

$$\begin{aligned} &\beta\int_\Omega ((\phi_1 + w)^+)^p + (\lambda_1 - \beta)\int_\Omega \phi_1^p \\ &\quad \le \beta\int_\Omega ((\phi_1 + w)^+)^p + (\lambda_1 - \beta)\int_\Omega |\phi_1 + w|^p - (\lambda_1 - \beta)\varepsilon_p \int_\Omega |w|^p \\ &\quad \le \frac{\beta}{\lambda_1} p \int_\Omega L_\infty(x, \nabla(\phi_1 + w)^+) \\ &\qquad + \frac{\lambda_1 - \beta}{\lambda_1} p \int_\Omega L_\infty(x, \nabla(\phi_1 + w)) - (\lambda_1 - \beta)\varepsilon_p \int_\Omega |w|^p. \end{aligned} \tag{4.5}$$

On the other hand, by Lemma 4.3, for a sufficiently large M we get

$$\alpha\int_\Omega((\phi_1+w)^-)^p \le \frac{1}{M}\int_\Omega L_\infty(x,-\nabla(\phi_1+w)^-) \le \frac{\beta}{\lambda_1}p\int_\Omega L_\infty(x,-\nabla(\phi_1+w)^-), \tag{4.6}$$

for $\|w\|_{1,p}$ small enough. Combining (4.5) and (4.6) we obtain

$$\beta\int_\Omega((\phi_1+w)^+)^p+\alpha\int_\Omega((\phi_1+w)^-)^p+(\lambda_1-\beta)\int_\Omega\phi_1^p \le p\int_\Omega L_\infty(x,\nabla(\phi_1+w))-(\lambda_1-\beta)\varepsilon_p\int_\Omega|w|^p. \tag{4.7}$$

Therefore (4.3) holds in a neighbourhood of $\overline{\phi_1}$. In view of Lemma 4.4 of [18, Lemma 4.2], the case $1<p<2$ may be treated in a similar fashion. Let us now note that

$$\int_\Omega|\phi_1+w|^p\ge\int_\Omega\phi_1^p\quad\text{for all } w\in W.$$

In the case $\beta\le 0$ we have

$$\begin{aligned}\beta\int_\Omega&((\phi_1+w)^+)^p+\alpha\int_\Omega((\phi_1+w)^-)^p+(\lambda_1-\beta)\int_\Omega\phi_1^p\\ &\le\frac{\lambda_1}{2}\int_\Omega|\phi_1+w|^p+(\alpha-\beta)\int_\Omega\big((\phi_1+w)^-\big)^p+\left(\lambda_1-\frac{\lambda_1}{2}\right)\int_\Omega\phi_1^p\end{aligned}$$

so that we reduce to (4.7). □

Proposition 4.5. *Let $r>0$ be as in Lemma 4.4. Then there exist $\overline{t}\in\mathbb{R}^+$ and $\sigma>0$ such that for every $t\ge\overline{t}$ and $w\in W_0^{1,p}(\Omega)$*

$$\|w-\overline{\phi_1}\|_{1,p}=r\Rightarrow\widetilde{f}_t(w)\ge f_\infty(\overline{\phi_1})+\sigma.$$

Proof. By contradiction, let $(t_h)\subset\mathbb{R}$ and $(w_h)\subset W_0^{1,p}(\Omega)$ such that $t_h\ge h$ and

$$\|w_h-\overline{\phi_1}\|_{1,p}=r,\quad \widetilde{f}_{t_h}(w_h)<f_\infty(\overline{\phi_1})+\frac{1}{h}. \tag{4.8}$$

Up to a subsequence we have $w_h\rightharpoonup w$ with $\|w-\overline{\phi_1}\|_{1,p}\le r$. Then, by (4.8) and (a) of the previous lemma we get

$$\limsup_h\widetilde{f}_{t_h}(w_h)\le f_\infty(\overline{\phi_1})\le f_\infty(w). \tag{4.9}$$

In view of Theorem 4.1(c), w_h strongly converges to w and then $\|w-\overline{\phi_1}\|_{1,p}=r$. Combining (4.9) with (b) of Lemma 4.4, we get a contradiction. □

Proposition 4.6. *Let σ and $\bar t$ be as in the previous proposition. Then there exists $\widetilde t \ge \bar t$ such that for every $t \ge \widetilde t$ there exist $v_t, w_t \in W_0^{1,p}(\Omega)$ with*

$$\begin{aligned} \|v_t-\overline{\phi_1}\|_{1,p} &< r, \quad \widetilde f_t(v_t) \le \frac{\sigma}{2}+f_\infty(\overline{\phi_1}),\\ \|w_t-\overline{\phi_1}\|_{1,p} &> r, \quad \widetilde f_t(w_t) \le \frac{\sigma}{2}+f_\infty(\overline{\phi_1}). \end{aligned} \tag{4.10}$$

Moreover, we have $\sup_{s\in[0,1]} f_t(sv_t+(1-s)w_t) < \infty$.

Proof. We argue by contradiction. Set $\widetilde t = \bar t + h$ and suppose that there exists $(t_h)\subset\mathbb{R}$ with $t_h \ge \widetilde t$ such that for every v_{t_h} and w_{t_h} in $W_0^{1,p}(\Omega)$

$$\begin{aligned} \|v_{t_h}-\overline{\phi_1}\|_{1,p} &< r, \quad \widetilde f_{t_h}(v_{t_h}) > \frac{\sigma}{2}+f_\infty(\overline{\phi_1}),\\ \|w_{t_h}-\overline{\phi_1}\|_{1,p} &> r, \quad \widetilde f_{t_h}(w_{t_h}) > \frac{\sigma}{2}+f_\infty(\overline{\phi_1}). \end{aligned}$$

Take now (z_h) going strongly to $\overline{\phi_1}$ in $W_0^{1,p}(\Omega)$. By (a) of Theorem 4.1 we have $\widetilde f_{t_h}(z_h) \to f_\infty(\overline{\phi_1})$. On the other hand eventually $\|z_h-\overline{\phi_1}\|_{1,p} < r$ and $\widetilde f_{t_h}(z_h) \le \sigma/2 + f_\infty(\overline{\phi_1})$, that contradicts our assumptions. Recalling (b) of Remark 4.2, by arguing as in the previous step, it is easy to prove (4.10). The last statement is straightforward. □

5. Proof of the main result

We now come to the proof of the main result of the paper.

Proof of Theorem 1.1. From Theorem 3.4 we know that f_t satisfies the $(\mathrm{CPS})_c$ condition for any $c,t\in\mathbb{R}$. By Propositions 4.5 and 4.6 we may apply Theorem 2.8 with $u_0=\overline{\phi_1}$ and obtain existence of at least two solutions $u\in W_0^{1,p}(\Omega)$ of problem (1.13) for $t>\bar t$ for a suitable $\bar t$.

Let us now prove that there exists $\underline{t}$ such that (1.13) has no solutions for $t<\underline{t}$. If the assertion was false, then we could find a sequence $(t_h)\subset\mathbb{R}$ with $t_h\to-\infty$ and a sequence (u_h) in $W_0^{1,p}(\Omega)$ such that for every $v\in C_c^\infty(\Omega)$

$$\begin{aligned} &\int_\Omega \nabla_\xi L(x,u_h,\nabla u_h)\cdot\nabla v + \int_\Omega D_sL(x,u_h,\nabla u_h)v\\ &= \beta\int_\Omega (u_h^+)^{p-1}v - \alpha\int_\Omega (u_h^-)^{p-1}v + \int_\Omega g_0(x,u_h)v + |t_h|^{p-2}t_h\int_\Omega \phi_1^{p-1}v + \langle\omega_0,v\rangle\,. \end{aligned}$$

Let us first consider the case when, up to a subsequence, $t_h/\|u_h\|_{1,p}\to 0$ and set $v_h=u_h/\|u_h\|_{1,p}$. Applying Lemma 3.1 with $\varrho_h=\|u_h\|_{1,p}$, $\delta_h=\omega_0$ and

$$\gamma_h(x)=\begin{cases}\beta & \text{if } u_h(x)\ge 0,\\ \alpha & \text{if } u_h(x)<0,\end{cases}\qquad \mu_h=\frac{g_0(x,u_h)}{\|u_h\|_{1,p}^{p-1}}+\frac{|t_h|^{p-2}t_h}{\|u_h\|_{1,p}^{p-1}}\phi_1^{p-1},$$

up to a subsequence, (v_h) converges strongly to some v in $W_0^{1,p}(\Omega)$. Then using the same argument as in the proof of Theorem 3.4 we get a contradiction.

Assume now that there exists $M > 0$ such that $\|u_h\|_{1,p} \le -Mt_h$. Then setting $w_h = -u_h t_h^{-1}$, w_h weakly converges to some $w \in W_0^{1,p}(\Omega)$. By applying Lemma 3.1 with $\varrho_h = -t_h$, $\delta_h = \omega_0$ and

$$\gamma_h(x) = \begin{cases} \beta & \text{if } u_h(x) \ge 0, \\ \alpha & \text{if } u_h(x) < 0, \end{cases} \qquad \mu_h = -\frac{g_0(x, u_h)}{|t_h|^{p-2} t_h} - \phi_1^{p-1},$$

w_h strongly converges to w in $W_0^{1,p}(\Omega)$. The choice of the test function $\varphi = w^+$ gives, as in the first case, $w^+ = 0$. Arguing again as in the end of the proof of Theorem 3.4 we obtain a contradiction. □

Remark 5.1. Under suitable assumptions on g and ω_0, by [2, Lemma 1.4] the solutions $u \in W_0^{1,p}(\Omega)$ of (1.13) belong to $L^\infty(\Omega)$. Then, further regularity results can be found in [17].

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