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PARABOLIC EQUATIONS WITH CRITICAL NONLINEARITIES

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Abstract. As well known the problem of global continuation of solutions to semilinear parabolic equations is completely solved when the nonlinear term is subordinated to an α-power of the main linear operator with $\alpha \in [0,1)$. In this paper we study three examples of *critical problems* in which the mentioned subordination takes place with $\alpha = 1$, i.e. the nonlinearity has the same *order of magnitude* as the linear main part. We use specific techniques of proving global solvability that fit well the considered examples for which general abstract methods fail.

1. Introduction

Studying abstract semilinear evolutionary problems of the form:

$$u_t + Au = F(u), \ u(0) = u_0, \tag{1.1}$$

in a Banach space X, where A is a sectorial operator in X and F stands for the nonlinear term we face twice the necessity of limiting the growth of the nonlinearity with respect to u. The first time is when we build local in time solutions. In particular, using the semigroup technique we need to show Lipschitz continuity on bounded sets of F acting between some fractional power space X^α, $\alpha \in (0,1)$ and the base space X (see [10]). The second time is when we want to assure global in time solvability of our problem. The growth limitations appearing in both of the mentioned questions have been frequently studied and

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through the last 15 years some progress has been achieved. We shall recall here the results of H. Amann ([1]), W. von Wahl ([14]) and the notion of an ε-*regular solution* that allow us to weaken restrictions for local solvability of parabolic equations (see [2]).

In this paper we study equations with nonlinear terms F having critical growth in the sense of [14] or [5]. In Section 2 we establish a suitable subordination condition allowing the continuation of solutions within the technique of [10]. In Section 3 using the concept of *monotone operators* (see [4]) we study in a "large" space of initial data global solvability of a problem involving perturbations of subdifferentials. Finally in Section 4, based on the approach of [9], we discuss the Navier–Stokes system and obtain a global description of solutions in case of arbitrary space dimension n and small external force.

2. The case of subordinated nonlinearity

As a simple example shows:

$$u_t = \Delta u - 2\Delta u,$$

considered with Dirichlet boundary condition, when the nonlinear term has "the same order of magnitude" as Au, the solution (even local in time) need not to exist. There is known, however, a number of particular (sufficient) conditions on the nonlinearity F or the set of initial data u_0 guaranteeing even global in time solvability of the problem (1.1).

The sufficient condition we discuss in this section is rather connected with smallness of initial data (see [13]), than with the growth of the nonlinearity F (in that case fast growth of F is even desirable). It will have the form:

$$\|F(w)\|_{X^0} \leq \varepsilon \|w\|_{X^1} + C(\varepsilon)\|w\|_{X^\alpha}, \tag{2.1}$$

valid for a sufficiently small positive constant ε with a certain $\alpha \in (0,1)$. If (2.1) holds for all $w \in X^1$, we are allowed to take fairly general initial data. It may happen however that the structure condition (2.1) holds only for certain solutions and in fact it has a more general form involving appropriate "introductory" estimate. This is just the case studied below in Assumption A.

2.1. An abstract global existence result. We will consider the abstract equation (1.1) with a *positive definite selfadjoint* operator A in a Hilbert space H:

$$u_t = -Au + F(u), \quad u(0) = u_0.$$

We set $X^0 = H$, $X^1 = D(A)$ and assume that $F: X^{1/2} \to X$ is Lipschitz continuous on bounded sets (for the definition of X^α see [10]). Thus, for any $u_0 \in X^{1/2}$, problem (1.1) possesses a unique local solution defined on a maximal interval of existence $[0, \tau_{u_0})$ (see e.g. [5]) .

We will assume that the following condition is satisfied.

ASSUMPTION A. There exists an auxiliary Banach space $Y, D(A) \subset Y$, such that a global in time estimate of all possible $X^{1/2}$ solutions of (1.1) in Y is known. Also:

(2.2) if Y is not embedded in $X^{1/2}$ we consider these $X^{1/2}$ solutions $u(t,u_0)$ for which:

$$\|F(u(t,u_0))\|^2_{X^0} \le \beta(\|u(t,u_0)\|_Y)\|u(t,u_0)\|^2_{X^1} + \gamma(\|u(t,u_0)\|_Y),$$

for $t \in (0,\tau_{u_0})$, with $\beta(s) \le b < 1$ for $s \ge 0$, and $\gamma\colon \mathbb{R}^+ \to \mathbb{R}^+$ being a continuous function.

REMARK 2.1. Note, that the second condition above is very similar to the notion of a "relatively bounded perturbation" as introduced in [11], however in the case of a *linear* perturbation F.

We have the following observation:

LEMMA 2.2. *If $u = u(t,u_0)$ is an $X^{1/2}$ solution to* (1.1) *for which Assumption* A *holds, then it exists globally in time.*

PROOF. Indeed, if we assume local $X^{1/2}$ solvability of (1.1), the only property we need to check is a global in time (that means; uniform in any bounded time interval $[0,T]$) a priori estimate of the solution in $X^{1/2}$. But such an estimate is easy to obtain in the presence of Assumption A. Multiplying equation (1.1) by Au, we find that

$$\langle u_t, Au\rangle_H + \langle Au, Au\rangle_H = \langle F(u), Au\rangle_H \tag{2.3}$$

or, after using Assumption A and the Cauchy inequality, that

$$\frac{d}{dt}\|u\|^2_{X^{1/2}} + \|u\|^2_{X^1} \le \beta(\|u\|_Y)\|u\|^2_{X^1} + \gamma(\|u\|_Y). \tag{2.4}$$

Thanks to the estimate $\|u\|^2_{X^{1/2}} \le c^{-1}\|u\|^2_{X^1}$, denoting

$$y(t) = \|u(t,u_0)\|^2_{X^{1/2}}, \qquad g(t) = \gamma(\|u(t,u_0)\|_Y),$$

we arrive at a linear differential inequality:

$$y'(t) \le -(1-b)cy(t) + g(t), \quad t > 0. \tag{2.5}$$

Solving (2.5), we obtain the bound:

$$\begin{aligned} \|u(t,u_0)\|^2_{X^{1/2}} &\le \|u_0\|^2_{X^{1/2}} \exp(-(1-b)ct) \\ &\quad + \int_0^t \gamma(\|u(s,u_0)\|_Y)\exp((1-b)cs)\,ds \exp(-(1-b)ct), \end{aligned} \tag{2.6}$$

which is the required $X^{1/2}$ estimate of the solution. □

Remark 2.3. In particular, if we have the bound $\gamma(\|u(t)\|_Y) \le m, t \ge 0$, the last estimate extends to:

$$\|u(t,u_0)\|^2_{X^{\frac{1}{2}}} \le \|u_0\|^2_{X^{\frac{1}{2}}} \exp\left(-(1-b)ct\right) + \frac{m}{(1-b)c}. \tag{2.7}$$

It is also clear from (2.5) that if we weaken the condition on β in (2.2) to $\beta(s) \le b = 1$ for $s \ge 0$, we will still have an estimate

$$\frac{d}{dt}\|u(t,u_0)\|^2_{X^{1/2}} \le \gamma(\|u(t,u_0)\|_Y)$$

sufficient for the global in time $X^{1/2}$ solvability of (1.1).

2.2. An application. The Cahn–Hilliard equation with an auxiliary $Y = H^1(\Omega)$ estimate and critical growth of the nonlinearity respectively to such estimate will serve as an example. In a bounded $C^{4+\varepsilon}$-smooth ($\varepsilon > 0$) domain in R^3 consider the problem:

$$\begin{cases} u_t = -\Delta^2 u + \mu\Delta(f(u)), \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \\ \dfrac{\partial u}{\partial N} = \dfrac{\partial(\Delta u)}{\partial N} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $f \in C^3(R)$ is such that

$$|f''(s)| \le c(|s|^3 + 1), \quad s \in R, \tag{2.9}$$

and the constant $\mu > 0$ will be later chosen sufficiently small. Note that the approach of [12] (see also [5]) does not work here since we do not assume any kind of monotonicity of f.

Besides the smallness of μ and (2.9) we need only to assume that there exists $M > 0$ such that

$$F(v) := \int_0^v f(s)\,ds \ge -M, \tag{2.10}$$

for all $v \in \mathbb{R}$, the last condition being sufficient to obtain a uniform estimate of solutions in $H^1(\Omega)$:

$$\|u(t)\|^2_{H^1(\Omega)} = \|\nabla u(t)\|^2_{L^2(\Omega)} + |\overline{u}(t)|^2 \le 2(\mathcal{L}(u_0) + \mu M|\Omega|) + |\overline{u}_0|^2. \tag{2.11}$$

Here $\mathcal{L}$ is a Lyapunov function:

$$\mathcal{L}(v) = \frac{1}{2}\|\nabla v\|^2_{L^2(\Omega)} + \mu\int_\Omega F(v)\,dx. \tag{2.12}$$

It is also well known that the problem (2.8) preserves in time the spatial average of solutions:

$$\overline{u}(t) = |\Omega|^{-1}\int_\Omega u(t,x)\,dx = \overline{u}_0, \tag{2.13}$$

as long as the solutions exist. Local $X^{1/2}$ solvability of (2.8) is a simple consequence of the embeddings:

$$H^2(\Omega) \subset L^\infty(\Omega), \quad H^2(\Omega) \subset W^{1,4}(\Omega), \quad n \leq 3, \tag{2.14}$$

since $X^{1/2} \subset H^2$ and the nonlinear term will be Lipschitz continuous from bounded subsets of $X^{1/2}$ (bounded as well in $L^\infty(\Omega)$) into $L^2(\Omega)$. Also the operator $A = \Delta^2$ defined on

$$D(A) = \left\{ \phi \in H^4(\Omega) : \frac{\partial \phi}{\partial N} = \frac{\partial(\Delta\phi)}{\partial N} = 0 \text{ on } \partial\Omega \right\} \tag{2.15}$$

is selfadjoint on $L^2(\Omega)$ and the quantity

$$(\|\Delta^2\phi\|^2_{L^2(\Omega)} + |\overline{\phi}|^2)^{1/2} \tag{2.16}$$

defines an equivalent norm on $D(A)$ (see [5] for details).

The first condition of the Assumption A is fulfilled with $Y = H^1(\Omega)$ thanks to (2.11). We need to check validity of the second condition. Integrating and using the Young inequality we find:

$$\begin{aligned} \mu\|\Delta(f(u))\|^2_{L^2(\Omega)} &= \mu \int_\Omega (f'(u)\Delta u + f''(u)|\nabla u|^2)^2\, dx \\ &\leq \mu \operatorname{const} \int_\Omega (|\Delta u|^{9/5} + |\nabla u|^3 + |u|^9 + 1)^2\, dx \\ &\leq 4\mu \operatorname{const} \left(\int_\Omega (|\Delta u|^{18/5} + |\nabla u|^6 + |u|^{18})\, dx + |\Omega| \right). \end{aligned} \tag{2.17}$$

The subsequent components will be next estimated using versions of the Nirenberg–Gagliardo inequality ([5, p. 26]):

$$\left(\int_\Omega |\Delta u|^{18/5}\, dx \right)^{5/18} \leq c\|u\|^{5/9}_{H^4(\Omega)} \|u\|^{4/9}_{H^1(\Omega)}, \tag{2.18}$$

$$\left(\int_\Omega |\nabla u|^6\, dx \right)^{1/6} \leq c\|u\|^{1/3}_{H^4(\Omega)} \|u\|^{2/3}_{H^1(\Omega)}, \tag{2.19}$$

$$\left(\int_\Omega |u|^{18}\, dx \right)^{1/18} \leq c\|u\|^{1/9}_{H^4(\Omega)} \|u\|^{8/9}_{H^1(\Omega)}. \tag{2.20}$$

We will thus extend (2.17), using (2.16), to an estimate:

$$\begin{aligned} &\mu\|\Delta(f(u))\|^2_{L^2(\Omega)} \\ &\quad\leq \mu \operatorname{const}' \|u\|^2_{H^4(\Omega)} (\|u\|^{8/5}_{H^1(\Omega)} + \|u\|^4_{H^1(\Omega)} + \|u\|^{16}_{H^1(\Omega)}) \\ &\qquad + \mu \operatorname{const}'' (|\Omega| + |\overline{u}|^2 (\|u\|^{8/5}_{H^1(\Omega)} + \|u\|^4_{H^1(\Omega)} + \|u\|^{16}_{H^1(\Omega)})), \end{aligned} \tag{2.21}$$

corresponding to (2.2).

We need yet to fulfill the condition in Assumption A:

$$\begin{aligned}(2.22)\quad &\beta(\|u(t,u_0)\|_{H^1(\Omega)})\\ &= \mu C'(\|u(t,u_0)\|^{8/5}_{H^1(\Omega)} + \|u(t,u_0)\|^4_{H^1(\Omega)} + \|u(t,u_0)\|^{16}_{H^1(\Omega)}) \le b < 1.\end{aligned}$$

The function $\beta(s) = \mu C'(s^{8/5} + s^4 + s^{16})$ defined above fulfills $\beta(0) = 0$ and is increasing for $s \ge 0$. Fixing $b \in (0,1)$, close to 1, we set $s_0 = \beta^{-1}(b)$. Choose $\mu > 0$ in equation (2.8) such that $2\mu M|\Omega| < s_0$. Such μ are admissible in our theory.

Observe next that, due to the embedding $H^1(\Omega) \subset L^6(\Omega)$ and the growth limitation (2.9), the term $\int_\Omega F(v(x))\,dx$ is well defined for $v \in H^1(\Omega)$, ($|F(v)|$ grows like const $(|v| + |v|^6)$). Also the map $\mathcal{F}$ defined for $v \in H^1(\Omega)$;

$$(2.23)\qquad H^1(\Omega) \ni v \xrightarrow{\mathcal{F}} \int_\Omega F(v(x))\,dx$$

is continuous at 0.

If we limit our considerations to initial data u_0 belonging to a sufficiently small ball $B_{X^{1/2}}(0,\alpha) \subset X^{1/2}$, such that

$$(2.24)\qquad 2\mathcal{L}(u_0) + |\overline{u}_0|^2 \le s_0 - 2\mu M|\Omega| \quad \text{for } u_0 \in B_{X^{1/2}}(0,\alpha),$$

then, thanks to (2.11), the solution $u(t,u_0)$ will be bounded uniformly in $Y = H^1(\Omega)$ as long as it exists by the constant s_0 fixed above.

The $X^{1/2}$ solution corresponding to such data will be global in time as a consequence of the *continuation property* (e.g. [5, p. 55]) and the estimate (2.6) guaranteeing boundedness of the $X^{1/2}$ norm of such solutions.

3. Global solutions of the $2m$-th order semilinear parabolic equation involving critical exponent

Our further concern is the $2m$-th order equation

$$(3.1)\qquad u_t + Au + f(u) = 0, \quad t > 0,\ x \in \Omega,$$

where

$$(3.2)\qquad A = \sum_{|\xi|,|\zeta| \le m} (-1)^{|\zeta|} D^\zeta (a_{\xi,\zeta}(x) D^\xi)$$

is a uniformly strongly elliptic operator in a bounded domain $\Omega \subset \mathbb{R}^n$ with $\partial\Omega \in C^{2m}$.

Equation (3.1) will be studied together with the homogeneous boundary conditions

$$(3.3)\qquad B_0 u = \ldots = B_{m-1} u = 0, \quad t \ge 0,\ x \in \partial\Omega,$$

such that the triple $(A, \{B_j\}, \Omega)$ forms a *regular elliptic boundary value problem.* We suppose additionally that

(3.4) an unbounded operator A in $L^2(\Omega)$ with the domain $D(A) = W^{2m,2}_{\{B_j\}}(\Omega)$ is symmetric and bounded below.

Recall that, for A, $\{B_j\}$, Ω as above and for a certain $\lambda \geq 0$, $A_\lambda := A+\lambda I$ is a selfadjoint positive definite operator in the Hilbert space $L^2(\Omega)$. Furthermore, A_λ considered in any space $L^p(\Omega)$, $p \in [2,\infty)$, defines a sectorial positive operator whose fractional powers will be denoted further by X^α_p, $\alpha \geq 0$, $p \in [2,\infty)$. Also, the resolvent of A_λ is compact as a result of the Calderon–Zygmund type estimate.

3.1. Critically growing nonlinearities. In [14] the $2m$-th order Dirichlet boundary value problem was studied, which corresponds to boundary conditions $B_j = \partial^j/\partial\nu^j$ for $j = 0,\dots,m-1$. Concerning nonlinearity, the assumptions of [14] read:

$$f\colon R \to R, \quad f \in C^1(R), \tag{3.5}$$

$$F(r) := \int_0^r f(s)\,ds \geq -cr^2, \quad r \in R, \tag{3.6}$$

and

$$\begin{aligned} sf(s) &\leq c(|s|^{(n+2m)/(n-2m)+1} + 1), & |s| \geq 1,\\ sf(s) &\geq -c(|s|^{(n+2m)/(n-2m)+1-\varepsilon} + 1), & |s| \geq 1, \end{aligned} \tag{3.7}$$

for certain $n > 2m$, $c > 0$, $\varepsilon > 0$.

As a result of (3.5), f may be viewed as a Lipschitz continuous on bounded sets map acting between X^α_p and X^0_p, provided that $\alpha > 1/2m$ and $p > n$. Evidently, in the latter case there exists a unique local X^α_p solution $u = u(\,\cdot\,,u_0)$ through $u_0 \in X^\alpha_p$, defined on a maximal interval of existence $[0,\tau_{u_0})$ (see [10]).

If condition (3.6) is fulfilled with c sufficiently small, we get the bound

$$\|u(t,u_0)\|^2_{X_2^{1/2}} \leq \operatorname{const}\mathcal{L}(u_0), \quad t \in [0,\tau_{u_0}), \tag{3.8}$$

where $\mathcal{L}$ denotes a Lyapunov function for (3.1); i.e.

$$\mathcal{L}(v) = \frac{1}{2}\|v\|^2_{X_2^{1/2}} + \int_\Omega F(v)\,dx. \tag{3.9}$$

Restricting further the growth of f according to a condition ($n > 2m$)

$$|f(s)| \leq c(|s|^{(n+2m)/(n-2m)} + 1), \quad s \in \mathbb{R}, \tag{3.10}$$

(cf. (3.7)) we observe the following consequence of the Nirenberg–Gagliardo inequality:

$$\|f(u(t,u_0))\|_{L^p(\Omega)} \leq g(p, \|u(t,u_0)\|_{X_2^{1/2}})(1 + \|u(t,u_0)\|_{X^1_p}), \tag{3.11}$$

with $p \in [2,\infty)$, $t \in [0,\tau_{u_0})$, and a function $g(p,s)$ increasing in s.

Remark 3.1. It follows from the above considerations that with the growth rate (3.10) the problem (3.1)–(3.3) is *essentially critical*. Namely, the local existence theory of [10] cannot be directly applied for initial values from $X_2^{1/2}$ since f does not take $X_2^{1/2}$ into $L^2(\Omega)$ (for this, the highest exponent acceptable in an estimate like (3.10) is $n/(n-2m)$). As a consequence, $X_2^{1/2}$ estimate of the solutions to (3.1)–(3.3) is insufficient within the mentioned theory for justification of their global existence. In [14] this difficulty was overcome with the aid of the already mentioned structure assumptions imposed on f. More precisely, with $\{B_j\} = \{\partial^j/\partial\nu^j\}$ and A, Ω as above, and with assumptions (3.5)–(3.7), *uniform continuity* of a map $[0,\tau_{u_0}) \ni t \to u(t,u_0) \in X_2^{1/2}$ has been proved there, which property was in turn "translated" into global in time continuation of the solutions corresponding to $u_0 \in X_p^1$ with $p > n+1$ (see Theorem I.2 therein).

Below we restrict our attention to polynomial nonlinearities satisfying (3.5)–(3.7). Instead of the analytic semigroup theory we will use the concept of maximal monotone operators in Hilbert space ([4], [3]).

Theorem 3.2. *Suppose that the triple $(A,\{B_j\},\Omega)$ forms a regular elliptic boundary value problem and* (3.4) *holds. If a polynomial f satisfies* (3.5)–(3.7), *then the problem* (3.1)–(3.3) *generates a C^0 semigroup on $L^2(\Omega)$.*

Proof. Since f is a polynomial, condition (3.6) is equivalent to

$$f(0)=0 \quad \text{and} \quad f'(s) \geq -C \quad \text{for } s \in R.$$

Thus, let $V = X_2^{1/2}$, $f_0(s) = f(s) + Cs - f(0)$ for $s \in R$, and define a functional

$$V \ni v \to \mathcal{J}(v) = \frac{1}{2}\int_\Omega |A_\lambda^{1/2} v|^2\,dx + \int_\Omega \int_0^v f_0(s)\,ds\,dx \in \mathbb{R}. \tag{3.12}$$

Under our assumptions $\mathcal{J}$ is convex, Gateaux differentiable and $\nabla\mathcal{J}$ coincides with the nonlinear operator $\mathcal{M}\colon V \to V^*$;

$$\langle \mathcal{M}(v), w\rangle_{V^*,V} = \langle A_\lambda^{1/2} v, A_\lambda^{1/2} w\rangle_{L^2(\Omega)} + \langle f_0(v), w\rangle_{L^2(\Omega)}, \quad w \in V. \tag{3.13}$$

Furthermore

$$\langle \mathcal{M}(v), v\rangle_{V^*,V} \geq \|v\|^2_{X_2^{1/2}}, \tag{3.14}$$

which shows that $\mathcal{M}$ is *monotone, hemicontinuous and coercive.*

Consequently $\widetilde{\mathcal{M}}\colon L^2(\Omega) \to L^2(\Omega)$ defined by

$$\widetilde{\mathcal{M}}(u) = \mathcal{M}(u) \quad \text{for } u \in D(\widetilde{\mathcal{M}}) := \{v \in V : \mathcal{M}(v) \in L^2(\Omega)\}$$

is a *maximal monotone* operator in $L^2(\Omega)$ (see [3, Theorem 1.3] and thus (3.1)–(3.3) may be rewritten in an abstract form

$$\frac{du}{dt}(t) + \widetilde{\mathcal{M}}(u(t)) + \mathcal{N}(u(t)) = 0, \quad t > 0, \tag{3.15}$$

with globally Lipschitz term $\mathcal{N}(v) = -(\lambda + c)v + f(0)$. By [4, Theorem 3.17, Remark 3.14] we may conclude that there exists a global *weak solution* $u(\,\cdot\,, u_0) \in C([0,\infty); L^2(\Omega))$ to (3.15) through each $u_0 \in H$. If $u_0 \in D(\widetilde{\mathcal{M}})$, then $u = u(\,\cdot\,, u_0)$ is Lipschitz continuous on each $[0,\tau]$ and hence is a *strong solution* in the sense of [4] such that $u(0) = u_0$, $u(t, u_0)$ satisfies (3.15) for a.e. $t \in (0,\infty)$, and $u(t, u_0) \in D(\widetilde{\mathcal{M}})$ for a.e. $t \in (0,\infty)$. □

REMARK 3.3. Since (3.15) has a subgradient form, additional regularity of the solutions can be obtained following the results of [3, Chapter 4, Section 2]; in particular the estimate (3.8) holds. Also, for more specific f, the existence of a global attractor may be shown based on [5, Theorem 8.6.1].

3.2. Local well posedness of (3.1)–(3.3) in $X_2^{1/2}$. Since we noticed above that in the case of critical exponent f does not take $X_2^{1/2}$ into $L^2(\Omega)$, a result concerning ε-regular solutions to (3.1) should be mentioned. It indicates that (3.1)–(3.3) is locally well posed in $X_2^{1/2}$ under the assumptions like (3.5) and (3.10).

PROPOSITION 3.4. *If the triple $(A, \{B_j\}, \Omega)$ forms regular elliptic boundary value problem and* (3.4) *holds, then conditions* (3.5) *and*

$$|f'(s)| \le c(|s|^{(n+2m)/(n-2m)-1} + 1), \quad s \in \mathbb{R} \tag{3.16}$$

(*where $n > 2m$*) *both imply that to each $u_0 \in X_2^{1/2}$ corresponds a unique local ε-regular solution to* (3.1)–(3.3).

PROOF. The proof of the above proposition is a consequence of abstract results reported in [2]. It involves properties of a Hilbert scale generated by (X_2^0, A_λ) (see [1, Chapter V, Theorem 1.5.15]) and relies on the fact that f may be viewed as a Lipschitz map on bounded sets from $X_2^{1/2+\varepsilon}$ into $X_2^{-1/2+\gamma(\varepsilon)}$ (i.e. an *ε-regular map* relatively to the pair $(X_2^{1/2}, X_2^{-1/2})$; see [2, Definition 2]). □

If, using the smoothing properties of sectorial equation, one shows that ε-regular solutions from Proposition 3.4 enter for $t > 0$ the space $X_{n+1}^{1/2}$ then, with further assumptions (3.4)–(3.7), the result of [14] will ensure that these solutions may be extended to the whole $[0,\infty)$. Without the sophisticated procedure of [14], the question whether ε-regular solutions from Proposition 3.4 can be continued to the whole $[0,\infty)$ remains open, since it is generally unknown whether the $X_2^{1/2}$ estimate is sufficient for such a purpose (see [2, Proposition 1]).

4. Stability of equilibrium for n-D Navier–Stokes system

This section is devoted to the n-dimensional Navier–Stokes system describing incompressible viscous fluid flow. Using a concept of the Lyapunov function we will give a natural explanation of the stability of equilibria under small perturbation. The problem considered in this part is critical (see (4.4)) and existence of the global in time smooth solutions for arbitrary data u_0, ν, h is so far unknown.

Consider the problem

$$\begin{cases} u_t = \nu\Delta u - \nabla p - (u,\nabla)u + h, & t>0,\ x\in\Omega,\\ \operatorname{div} u = 0, & t>0,\ x\in\Omega,\\ u=0,\ t>0, & x\in\partial\Omega,\\ u(0,x)=u_0(x), & x\in\Omega, \end{cases} \tag{4.1}$$

where $n\geq 2$, $\nu>0$ is a viscosity constant and Ω is a bounded subdomain of $\mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^{2+\rho}$; $\rho\in(0,1)$ fixed from now on.

For any $h\in[L^r(\Omega)]^n$ the system (4.1) may be studied as an abstract Cauchy problem:

$$u_t + A_r u = F_r u + P_r h, \quad t>0,\ u_{|t=0} = u_0. \tag{4.2}$$

Operator $A_r = -\nu P_r\Delta$ with the domain $D(A_r) = X_r \cap \{\phi\in[W^{2,r}(\Omega)]^n : \phi_{|\partial\Omega}=0\}$ is sectorial in $X_r = \mathrm{cl}_{[L^r(\Omega)]^n}\{\phi\in[C_0^\infty(\Omega)]^n : \operatorname{div}\phi=0\}$, and $F_r u = -P_r(u,\nabla)u$. Moreover, for $\alpha\in[1/2,1)$ and $r>n$ the nonlinear term F_r, acting from X_r^α into X_r, is Lipschitz continuous on bounded sets. For the description of the projector P_r and characterization of the domains X_r^α of fractional powers of A_r we refer the reader to [7] and [8]. Here we claim that:

Proposition 4.1. *For any $\alpha\in[1/2,1)$ and $r>n\geq 2$ the problem* (4.2) *is locally well posed in X_r^α and*

$$u(\,\cdot\,,u_0)\in C([0,\tau_{u_0}),X_r^\alpha)\cap C^1((0,\tau_{u_0}),X_r^{1^-})\cap C((0,\tau_{u_0}),X_r^1), \tag{4.3}$$

where $[0,\tau_{u_0})$ denotes the maximal interval of existence of solution corresponding to initial data $u_0\in X_r^\alpha$.

It may be seen that

$$\|F_r w\|_{X_r} \leq c_r\|w\|^2_{X_r^{1/2}}, \quad w\in[W^{1,r}(\Omega)]^n,\ r>n\geq 2,$$

(see [8]) and consequently, thanks to the interpolation inequality,

$$\|F_r(u(t,u_0)) + h\|_{X_r} \leq C(\|u(t,u_0)\|_{X_r},\|h\|_{X_r})(1+\|u(t,u_0)\|_{X_r^1}), \tag{4.4}$$

$t\in[0,\tau_{u_0})$, which shows that nonlinearity in (4.2) has the same order of magnitude as A_r relatively to an $[L^r(\Omega)]^n$ estimate of solution with $r>n$. Such an estimate is however generally unknown. Below we will thus refer to the known

$[L^2(\Omega)]^n$ estimate, which will allow us to obtain conclusions concerning stability of stationary solutions.

Lemma 4.2. *Suppose that $r > n \geq 2$ and $u_S \in X^1_r$ is a stationary solution to* (4.2) *such that*

$$\|u_S\|_{[W^{1,\infty}(\Omega)]^n} \leq \frac{\nu}{C^2_\Omega}, \tag{4.5}$$

where C_Ω denotes a constant appearing in the Poincaré inequality. Then, (4.2) *admits a Lyapunov functional.*

Proof. Suppose that u is a solution to (4.2) resulting from Proposition 4.1, which moreover is defined for all $t > 0$. Note that the equation for $v = u - u_S$ reads

$$v_t = \nu A_r v - P_r(v, \nabla)v - P_r(v, \nabla)u_S - P_r(u_S, \nabla)v, \quad t > 0. \tag{4.6}$$

Following considerations of [7] observe next that

$$P_r v = P_2 v \quad \text{for } v \in [L^r(\Omega)]^n,\ r \geq 2,$$

P_2 being a selfadjoint bounded operator on $[L^2(\Omega)]^n$. Thus, for $\phi, \psi, \eta \in X^1_r$,

$$\langle P_r(\phi, \nabla)\psi, \eta\rangle_{[L^2(\Omega)]^n} = \langle P_2(\phi, \nabla)\psi, \eta\rangle_{[L^2(\Omega)]^n} = \langle(\phi, \nabla)\psi, \eta\rangle_{[L^2(\Omega)]^n}$$

and

$$\langle A_r\eta, \eta\rangle_{[L^2(\Omega)]^n} = \langle\Delta\eta, \eta\rangle_{[L^2(\Omega)]^n}.$$

Therefore, multiplying both sides of (4.6) in $[L^2(\Omega)]^n$ by v (belonging to the class described in (4.3)), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v\|^2_{[L^2(\Omega)]^n} &\leq -\nu\sum_{i=1}^n \|\nabla v_i\|^2_{L^2(\Omega)} + \|u_S\|_{[W^{1,\infty}(\Omega)]^n}\sum_{i=1}^n \|v_i\|^2_{L^2(\Omega)} \\ &\leq \bigg(-\frac{\nu}{C^2_\Omega} + \|u_S\|_{[W^{1,\infty}(\Omega)]^n}\bigg)\|v\|^2_{[L^2(\Omega)]^n} \leq 0. \end{aligned} \tag{4.7}$$

On the other hand, if we have

$$\mathcal{L}(u(t, u_0)) := \frac{1}{2}\|u(t, u_0) - u_S\|^2_{[L^2(\Omega)]^n} \equiv \text{const} \tag{4.8}$$

where $u = u(\,\cdot\,, u_0)$ is a solution to (4.2), then $v = u - u_S$ solves (4.6), which now leads to the relation:

$$0 = \frac{d}{dt}[\mathcal{L}(u(t, u_0))] = \langle v_t, v\rangle_{[L^2(\Omega)]^n} \leq \bigg(-\frac{\nu}{C^2_\Omega} + \|u_S\|_{[W^{1,\infty}(\Omega)]^n}\bigg)\|v\|^2_{[L^2(\Omega)]^n} \leq 0.$$

Evidently, the above inequality implies that $\|v\|_{[L^2(\Omega)]^n} \equiv 0$ and, consequently, $u \equiv u_S$. □

Remark 4.3. Condition (4.5) is trivially satisfied in the case when the external force is zero; i.e. $h \equiv 0$. However, in the latter case the Grashof number, which measures the dynamical complexity of the solutions, is equal to zero. Note also that (4.7) necessitates the exponential decay of any smooth global solution to (hypothetical) equilibrium u_S in $[L^2(\Omega)]^n$ norm.

For $h \in [L^r(\Omega)]^n$, $r > n \geq 2$ and $\alpha \in [1/2, 1)$ we introduce further the set

$$\mathcal{U}_{h,r,\alpha} := \{u_0 \in X_r^\alpha : \sup_{t\in[0,\tau_{u_0})} \|u(t,u_0)\|_{X_r^\alpha} < \infty\},$$

where τ_{u_0} is the right end of the maximal interval of existence of a local solution $u(\,\cdot\,,u_0)$ of (4.2) resulting from Proposition 4.1. Hence, due to the continuation property this set contains all globally bounded in time X_r^α solutions.

For $n = 2$, $\mathcal{U}_{h,r,\alpha}$ is the whole of X_r^α (see e.g. [5, Section 6.6]), whereas in higher space dimensions the answer how reach is $\mathcal{U}_{h,r,\alpha}$ strongly depends on the Grashof number. More precisely, if the quantity $\|h\|_{[L^r(\Omega)]^n}/\nu^2\lambda_1$ is sufficiently small it may be easily seen that $\mathcal{U}_{h,r,\alpha}$ contains a neighborhood of zero (see e.g. [6]). However, for arbitrarily large initial velocity and external force, the existence of global smooth solutions is known to be an open problem.

Below we will focus on the case when $\mathcal{U}_{h,r,\alpha}$ is not void. For $u_0 \in \mathcal{U}_{h,r,\alpha}$ we set $\gamma^+(u_0) = \{u(t,u_0),\ t \geq 0\}$ and $\omega(u_0) = \bigcap_{s\geq 0} \mathrm{cl}_{X_r^\alpha}\gamma^+(u(s,u_0))$.

Lemma 4.4. *Suppose that* (4.2) *possesses a stationary solution* u_S *for which* (4.5) *holds and let* $u_0 \in \mathcal{U}_{h,r,\alpha}$. *Then, the solution* $u(\,\cdot\,,u_0)$ *to* (4.2) *converges to* u_S *in* X_r^α.

Proof. For $u_0 \in \mathcal{U}_{h,r,\alpha}$ it is evident that $\tau_{u_0} = \infty$ and $\gamma^+(u_0)$ is bounded in X_r^α. Thus $\mathrm{cl}_{X_r^\alpha}\gamma^+(u_0)$ is also bounded and is contained in $\mathcal{U}_{h,r,\alpha}$. Since the resolvent operators $(\lambda I - A_r)^{-1}$ are compact for $\lambda \in \rho(A_r)$, this implies further compactness of $\mathrm{cl}_{X_r^\alpha}\gamma^+(u(1+t,u_0))$ in X_r^α ([9]). Lemma 4.2 implies now that $\omega(u_0) = \{u_S\}$, which completes the proof. □

Global behavior of orbits of the Navier–Stokes system may be now described as follows.

Corollary 4.5. *If* (4.2) *possesses a stationary solution* u_S *for which* (4.5) *holds, then* $\{u_S\} \subset \mathcal{U}_{h,\alpha,r}$ *is a maximal compact invariant subset of* X_r^α *and any solution* $u(\,\cdot\,,u_0)$ *of* (4.2) *either blows-up in* X_r^α *in finite or infinite time or converges to* $\{u_S\}$. *In particular, there exists* $\mu_0 > 0$ *such that* u_S *is asymptotically stable whenever* $\|h\|_{[L^r(\Omega)]^n} < \mu_0$. *If in addition* $n = 2$, *and* $h \equiv 0$, *then zero is a globally asymptotically stable equilibrium.*

Proof. The alternative is an immediate consequence of Lemma 4.4. Next, if $\|h\|_{[L^r(\Omega)]^n}$ is sufficiently small (see [6, (18) or (28)] for detail calculations of an appropriate upper bound), there exists a compact invariant set $\mathcal{A}$ attracting certain neighbourhood of zero. In the light of our previous considerations we conclude that $\mathcal{A}$ lies in $\mathcal{U}_{h,\alpha,r}$ and coincides with $\{u_S\}$. Indeed, if we take any complete invariant orbit contained in $\mathcal{A}$, then we observe that the Lyapunov functional defined by

$$\mathcal{L}(w) := \frac{1}{2}\|w - u_S\|^2_{[L^2(\Omega)]^n}, \quad w \in \mathcal{U}_{h,\alpha,r},$$

must be constant along this orbit. Otherwise among elements of $\mathcal{U}_{h,\alpha,r}$ there would be two different equilibria, which is excluded by Lemma 4.2.

In particular, if $h \equiv 0$, then $\mathcal{U}_{h,r,\alpha} = X^\alpha_r$ for $n = 2$, and the compact global attractor, which is known to exist for the Navier–Stokes system in this case, is a single point set $\{0\}$. □

REMARK 4.6. Note that if $r > n$ and the $[L^r(\Omega)]^n$ norm of h fulfills appropriate smallness restriction, then the Navier–Stokes system possesses a stationary solution $u_S \in D(A_r)$ such that $\|u_S\|_{[W^{2,r}(\Omega)]^n}$ tends to zero if $\|h\|_{[L^r(\Omega)]^n} \to 0$ (see [5, Theorem 8.3.1] for detail calculations). Thus, for small perturbations, condition (4.5) is naturally satisfied.

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