

ON A RADIAL POSITIVE SOLUTION TO A NONLOCAL ELLIPTIC EQUATION

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ABSTRACT. The existence of a solution to the Dirichlet boundary value problem for nonlinear Poisson equations with the nonlocal nonlinear term

$$-\Delta u = f\left(u, \int (g \circ u)\right), \quad u|_{\partial U} = 0,$$

is proved by means of fixed point theorems for increasing compact operators.

1. Introduction

We study the following boundary value problem:

$$(1.1) \quad -\Delta u = f\left(u, \int_U g \circ u\right),$$

$$(1.2) \quad u|_{\partial U} = 0,$$

and look for its positive solutions. The domain $U \subset \mathbb{R}^n$ is assumed to be an annulus $U(R, \rho) = B(0, R) \setminus \bar{B}(0, \rho)$, for $0 < \rho < R$ or a ball $U(R, 0) = B(0, R)$ for $\rho = 0$, what enables us to seek a radial solution

$$u(x) = v(|x|)$$

where $v: [\rho, R] \rightarrow \mathbb{R}$.

2000 *Mathematics Subject Classification.* 35J65.

Key words and phrases. Nonlocal equation, radial solution, fixed point.

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The study of such problems is motivated by a lot of physical applications starting from the well-known Poisson–Boltzmann equation (see [1], [8], [13]), however our assumptions are not satisfied for many physically important examples.

The method we use is typical for local boundary value problems. We shall formulate an equivalent fixed point problem and look for its solution in the cone of nonnegative function in an appropriate Banach space. Here, we work in the space of continuous functions and obtain at least one solution. If one works in the space of integrable functions, then the existence of multiple solutions can be obtained under relatively strong assumptions on the function g (see [3], [4]). We will use the following theorems:

THEOREM 1.1 ([6, p. 41]). *Let P be a cone in a Banach space X , i.e. P is closed convex set such that:*

- (a) $\lambda P \subset P$ for $\lambda \geq 0$,
- (b) $P \cap (-P) = \{0\}$,

and this cone is normal, i.e. there exists a positive constant C such that $\|v\| \leq C\|w\|$, for $v, w \in P$, $v \leq w$. Let, for $v, w \in X$, the relation $v \leq w$ denotes that $w - v \in P$. Suppose that a mapping $T: P \rightarrow X$ is completely continuous and nondecreasing, i.e.

$$T(v) \leq T(w), \quad \text{for } v \leq w.$$

If there exist points $v_1, v_2 \in P$, $v_1 \leq v_2$, for which $v_1 \leq T(v_1)$ and $T(v_2) \leq v_2$, then the mapping T has a fixed point $tv_0 \in P$, for which $v_1 \leq v_0 \leq v_2$.

THEOREM 1.2 ([6, p. 94]). *Let P be a cone in a Banach space, Ω_1 and Ω_2 two bounded open neighbourhoods of zero such that $\overline{\Omega_1} \subset \Omega_2$. Let $T: P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P$ be a completely continuous mapping. If (the case of expansion)*

$$\|T(v)\| \leq \|v\| \quad \text{for } v \in \partial\Omega_1 \quad \text{and} \quad \|T(v)\| \geq \|v\| \quad \text{for } v \in \partial\Omega_2,$$

or reversely (the case of compression)

$$\|T(v)\| \geq \|v\| \quad \text{for } v \in \partial\Omega_1 \quad \text{and} \quad \|T(v)\| \leq \|v\| \quad \text{for } v \in \partial\Omega_2,$$

then operator T has a fixed point.

2. Main results

Suppose that functions $f: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$, $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ ($\mathbb{R}_+ = [0, \infty)$) are continuous. If we look for the radial solutions of BVP (1.1)–(1.2), then function v should satisfy the following BVP for ordinary differential equation:

$$(2.1) \quad -v''(r) - \frac{n-1}{r}v'(r) = f\left(v(r), \omega_n \int_{\rho}^R s^{n-1}g(v(s)) ds\right),$$

with the boundary conditions:

$$(2.2) \quad v(R) = 0 = v(\rho) \quad \text{for } \rho > 0,$$

or

$$(2.3) \quad v(R) = 0 = \lim_{r \rightarrow 0^+} v'(r) \quad \text{for } \rho = 0,$$

where ω_n stands for the measure of the unit sphere in \mathbb{R}^n . The linear homogeneous equation $-v'' - (n-1)v'/r = 0$ with both boundary conditions has only the trivial solution, thus there exists the Green function G . By standard calculations one can verify that

$$G(r, t) = \frac{t(R^{n-2} - \max(r, t)^{n-2})(\min(r, t)^{n-2} - \rho^{n-2})}{(n-2)(R^{n-2} - \rho^{n-2})r^{n-2}}$$

for $\rho > 0$ and $n > 2$,

$$G(r, t) = \frac{t(\ln R - \ln(\max(r, t)))(\ln(\min(r, t)) - \ln \rho)}{\ln R - \ln \rho}$$

for $\rho > 0$ and $n = 2$,

$$G(r, t) = \frac{t^{n-1}}{n-2} \left(\frac{1}{\max(r, t)^{n-2}} - \frac{1}{R^{n-2}} \right)$$

for $\rho = 0$ and $n > 2$,

$$G(r, t) = t(\ln R - \ln(\max(r, t)))$$

for $\rho = 0$ and $n = 2$.

The BVP is then equivalent to the integral equation

$$(2.4) \quad v(r) = \int_{\rho}^R G(r, t) f \left(v(t), \omega_n \int_{\rho}^R s^{n-1} g(v(s)) ds \right) dt.$$

Let $X = C([\rho, R])$ denote the space of real continuous functions on $[\rho, R]$ with the sup-norm, P denote the cone of nonnegative functions in X and let $T: P \rightarrow P$ be defined by the formula

$$(2.5) \quad T(v)(r) = \int_{\rho}^R G(r, t) f \left(v(t), \omega_n \int_{\rho}^R s^{n-1} g(v(s)) ds \right) dt.$$

We look for fixed points of T , since they are positive radial solutions of BVP (1.1)–(1.2).

Notice that, from the Arzela–Ascoli Theorem, our operator T is completely continuous. It is clear that the cone P is normal. Making certain supplementary assumptions and using Theorem 1.1 we obtain a fixed point of T :

THEOREM 2.1. Let $f: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ and $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be continuous and satisfy the following conditions:

$$\begin{aligned} f(v_1, y_1) &\leq f(v_2, y_2) && \text{if } v_1 \leq v_2 \text{ and } y_1 \leq y_2, \\ g(v_1) &\leq g(v_2) && \text{if } v_1 \leq v_2. \end{aligned}$$

If there exists a positive number c_1 such that

$$f\left(c_1, \omega_n g(c_1) \frac{R^n - \rho^n}{n}\right) \leq \frac{c_1}{\gamma},$$

where

$$\gamma = \sup_{r \in [\rho, R]} \int_{\rho}^R G(r, t) dt,$$

then BVP (1.1)–(1.2) has a positive radial solution u_0 with the norm $\|u_0\| = \sup_{x \in U} |u_0(x)| \leq c_1$.

PROOF. Let $v_1(t) = 0$ and $v_2(t) = c_1$, $t \in [\rho, R]$. We have $v_1, v_2 \in P$ and $v_1 \leq v_2$. It is clear that $v_1 \leq T(v_1)$. The assumptions concerning monotonicity of f and g imply that T is nondecreasing. We shall estimate values of $T(v_2)$

$$\begin{aligned} T(v_2)(r) &= \int_{\rho}^R G(r, t) f\left(v_2(t), \omega_n \int_{\rho}^R s^{n-1} g(v_2(s)) ds\right) dt \\ &\leq \int_{\rho}^R G(r, t) f\left(c_1, \omega_n g(c_1) \frac{R^n - \rho^n}{n}\right) dt \leq \frac{c_1}{\gamma} \gamma = c_1 = v_2(r). \end{aligned}$$

Thus $T(v_2) \leq v_2$.

Hence Theorem 1.1 implies the existence of a fixed point v_0 of T . We can define $u_0(x) := v_0(|x|)$. □

After simple though long calculations, one can compute

$$\begin{aligned} \gamma &= \frac{1}{2n} \left(\frac{R^n - \rho^n}{R(n-2) - \rho^{n-2}} - r_0^2 - \frac{R^{n-2} \rho^{n-2} (R^2 - \rho^2)}{r_0^{n-2} (R^{n-2} - \rho^{n-2})} \right) \\ &\leq \frac{1}{2n} \frac{R^n + \rho^n - \rho 2R^{n-2} - \rho^{n-2} R^2}{R^{n-2} - \rho^{n-2}} \end{aligned}$$

for $\rho > 0$ and $n > 2$, where

$$r_0 = \left(\frac{n-2}{2} \right)^{1/n} \frac{R^{(n-2)/n} \rho^{(n-2)/n} (R^2 - \rho^2)^{1/n}}{(R^{n-2} - \rho^{n-2})^{1/n}},$$

and

$$\gamma = \frac{\rho^2(\ln R - \ln r_0) + R^2(\ln r_0 - \ln \rho) - r_0^2(\ln R - \ln \rho)}{8(\ln R - \ln \rho)} \leq \frac{R^2 - \rho^2}{8}$$

for $\rho > 0$ and $n = 2$, where for $\rho = 0$

$$r_0 = \left(\frac{R^2 - \rho^2}{2(\ln R - \ln \rho)} \right)^{1/2}, \quad \gamma = \frac{R^2}{2n}.$$

We shall demonstrate an application of the cone-compression version of Theorem 1.2 to obtain a positive radial solution of BVP (1.1)–(1.2) under appropriate assumptions on the function f :

THEOREM 2.2. *Let $f: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ and $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ be continuous. Suppose there exist constants $0 < A < \gamma^{-1}$, where γ is described above, and B such that:*

$$(2.6) \quad f(v, y) \leq Av + B$$

for $v \geq 0$ and $y \in \mathbb{R}$. Then BVP (1.1)–(1.2) has a positive radial solution.

PROOF. We shall demonstrate that operator T defined by (2.5) has a fixed point in P . If $f(0, \omega_n(R^n - \rho^n)g(0)/n) = 0$, then the function $v_0(t) = 0, t \in [\rho, R]$ is a fixed point of T . Hence we shall consider the case of

$$f(0, \omega_n(R^n - \rho^n)g(0)/n) > 0.$$

From the continuity of f and g there exist positive constants K and δ , for which

$$f\left(v(t), \omega_n \int_{\rho}^R s^{n-1} g(v(s)) ds\right) \geq K$$

for any $v \in P, \|v\| \leq \delta$. Let $v \in P, \|v\| \leq \delta$. We have

$$\begin{aligned} \|Tv\| &= \sup_{r \in [\rho, R]} |T(v)(r)| \\ &= \sup_{r \in [\rho, R]} \int_{\rho}^R G(r, t) f\left(v(t), \omega_n \int_{\rho}^R s^{n-1} g(v(t)) ds\right) dt \geq \gamma K. \end{aligned}$$

Setting $\Omega_1 = \{v \in X : \|v\| < \min(\gamma K, \delta)\}$, we have $\|T(v)\| \geq \|v\|$ for any $v \in P \cap \partial\Omega_1$.

Define $\Omega_2 := \{v \in X : \|v\| \leq B\gamma/(1 - A\gamma)\}$. Taking $v \in P \cap \partial\Omega_2$, we obtain:

$$\begin{aligned} \|T(v)\| &= \sup_{r \in [\rho, R]} |T(v)(r)| \\ &= \sup_{r \in [\rho, R]} \int_{\rho}^R G(r, t) f\left(v(t), \omega_n \int_{\rho}^R s^{n-1} g(v(s)) ds\right) dt \\ &\leq \sup_{r \in [\rho, R]} \int_{\rho}^R G(r, t) (Av(t) + B) dt \\ &\leq \gamma A \|v\| + \gamma B = \frac{AB\gamma^2}{1 - A\gamma} + B\gamma = \frac{B\gamma}{1 - A\gamma} = \|v\|. \end{aligned}$$

Thus the cone-compression version of Theorem 1.2 implies the assertion. □

The growth condition (2.6) in the above theorem does not depend on the second variable of f . This fact implies that none growth conditions for the

function g are required. Moreover, the proof of Theorem 2.2 can be repeated without changes in the case of an equation

$$-v''(r) - \frac{n-1}{r}v'(r) = f(v(r), \lambda(v))$$

instead (2.1) with any continuous (nonlinear) functional λ on the space $C([\rho, R])$.

We shall demonstrate an example of existence theorem with weakened growth condition for f but with one on g .

THEOREM 2.3. *Let $f: \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ and $g: \mathbb{R}_+ \rightarrow \mathbb{R}$ be continuous. Suppose that there exist positive constants $A < \gamma^{-1}$, $B, C, D, p, q, pq \leq 1$ such that*

$$(2.7) \quad f(v, y) \leq Av + B + C|y|^p$$

for $v \geq 0, y \in \mathbb{R}$ and

$$(2.8) \quad |g(t)| \leq D|t|^q$$

for $t \in \mathbb{R}$. Then BVP (1.1)–(1.2) has a positive radial solution.

PROOF. The theorem can be proved as Theorem 2.2 with certain changes in the definition of the set Ω_2 . Before describing Ω_2 , we estimate using (2.7) and (2.8):

$$\begin{aligned} \|Tv\| &= \sup_{r \in [\rho, R]} \int_{\rho}^R G(r, t) f\left(v(t), \omega_n \int_{\rho}^R s^{n-1} g(v(s)) ds\right) dt \\ &\leq \gamma \left(A\|v\| + B + C \left(\omega_n \int_{\rho}^R s^{n-1} |g(v(s))| ds \right)^p \right) \\ &\leq \gamma \left(A\|v\| + B + CB^p \omega_n^p \left(\frac{R^n - \rho^n}{n} \right)^p D^p \|v\|^{pq} \right) \\ &= (\gamma A + \gamma B \|v\|^{-1} + \gamma E \|v\|^{pq-1}) \|v\| \end{aligned}$$

with $E = CB^p \omega_n^p ((R^n - \rho^n)/n)^p D^p$.

Thus there exists a positive constant M , for which

$$\|T(v)\| \leq \|v\|$$

whenever $\|v\| \geq M$ and we can define $\Omega_2 := \{v \in X : \|v\| < M\}$. The required inequality for $v \in \partial\Omega_1$ is obtained as in the proof of Theorem 2.2.

3. Solutions that change sign

Let BVP (1.1)–(1.2) be studied without the restriction of the positivity of solutions. Then, we obtain the existence of a solution under the classical Bernstein type assumptions on f (see [2] or [5]):

THEOREM 3.1. *Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Suppose there exists a positive constant M such that*

$$(3.1) \quad vf(v, y) \leq 0 \quad \text{for } |v| \geq M, y \in \mathbb{R}.$$

Then, there exists a radial solution u_0 of BVP (1.1)–(1.2) with the norm $\|u_0\| = \sup_{x \in U} |u_0(x)| \leq M$.

PROOF. Assume that inequality in (3.1) is sharp, i.e.

$$vf(v, y) < 0 \quad \text{for } |v| \geq M, y \in \mathbb{R}.$$

We will use the Leray–Schauder degree theory (see for instance [9]). In the first step, we obtain *a priori* bound for solutions of ODE:

$$(3.2) \quad -v'' - \frac{n-1}{r}v' = \mu f\left(v, \omega_n \int_{\rho}^R t^{n-1}g(v(t)) dt\right),$$

with boundary conditions (2.2) or (2.3) for $\mu \in [0, 1]$. Let $v = v(r)$ be a solution of (3.2) and suppose that $|v(r)|$ attains its maximum at $r = r_0$. If $v(r_0) > M$ then $v'(r_0) = 0$ and $v''(r_0) \leq 0$, if $v(r_0) < -M$, then $v'(r_0) = 0$ and $v''(r_0) \geq 0$; both cases contradict (3.2). Thus $\|v\| \leq M$.

The BVP (3.2), (2.2) or (2.3) is equivalent to the equation

$$(3.3) \quad (I - \mu T)v = 0,$$

where I stands for the identity mapping. We treat $I - \mu T$ as a mapping from the ball $B(0, M + \varepsilon) \subset X$ into X , where $X = C([\rho, R])$ with sup-norm. We know, that the above equation has no solution in $X \setminus B(0, M + \varepsilon)$, so the Leray–Schauder degree $\deg(I - T, B(0, M + \varepsilon)) = \deg(I, B(0, M + \varepsilon)) = 1$. Therefore, the BVP (2.1)–(2.2) or (2.3) has a solution, which proves the assertion.

The general case (3.1) is proved by applying the perturbation of f by v/n . New right hand sides satisfy sharp inequality and we get a sequence v_n of solutions to the perturbed equations. One can take a convergent subsequence from this sequence and its limit is a solution to the problem (compare [11]). \square

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Manuscript received October 10, 2003

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