

## THE JUMPING NONLINEARITY PROBLEM REVISITED: AN ABSTRACT APPROACH

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ABSTRACT. We consider a class of nonlinear problems of the form

$$Lu + g(x, u) = f,$$

where  $L$  is an unbounded self-adjoint operator on a Hilbert space  $H$  of  $L^2(\Omega)$ -functions,  $\Omega \subset \mathbb{R}^N$  an arbitrary domain, and  $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a “jumping nonlinearity” in the sense that the limits

$$\lim_{s \rightarrow -\infty} \frac{g(x, s)}{s} = a \quad \text{and} \quad \lim_{s \rightarrow \infty} \frac{g(x, s)}{s} = b$$

exist and “jump” over an eigenvalue of the operator  $-L$ . Under rather general conditions on the operator  $L$  and for suitable  $a < b$ , we show that a solution to our problem exists for any  $f \in H$ . Applications are given to the beam equation, the wave equation, and elliptic equations in the whole space  $\mathbb{R}^N$ .

### 1. Introduction

The so-called *jumping nonlinearity* problem has a long and rich history starting with the pioneering paper by Ambrosetti–Prodi [2] in 1973. In the early years following the appearance of [2] a number of authors contributed to the study of such problems, notably Berger and Podolak ([7]), Kazdan and Warner ([14]),

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Amann and Hess ([1]), Dancer ([9]) and Fučík ([11]) who coined the term *jumping nonlinearity* in the case of ordinary differential equations. During the 1980's, we could cite the contributions of de Figueiredo ([10]), Gallouet and Kavian ([13]), Lazer and McKenna ([15]), Ruf ([19]), and Solimini ([22]), among others. The interested reader will find a more complete bibliography up to 1990 in [17].

In its simplest form, the *jumping nonlinearity* problem consists in studying the question of existence of solution for the Dirichlet problem

$$(1.1) \quad u'' + g(u) = f(x), \quad u(0) = u(\pi) = 0,$$

where  $f(x)$  is a given function in  $L^2(0, \pi)$  and the nonlinearity  $g(s)$  is a  $C^1$  function crossing some eigenvalue  $\lambda_k = k^2$  ( $k = 1, 2, \dots$ ) of the problem  $u'' + \lambda u = 0$ ,  $u(0) = u(\pi) = 0$ , in the sense that

$$(1.2) \quad 0 < a = \lim_{s \rightarrow -\infty} g'(s) < \lambda_k < \lim_{s \rightarrow \infty} g'(s) = b.$$

Of particular importance here is the *problem at infinity*,  $(F)_{a,b}$ , given by  $u'' + bu^+ - au^- = 0$ ,  $u(0) = u(\pi) = 0$ , and its so-called *Fučik spectrum*  $\Sigma = \{(a, b) \in \mathbb{R}^2 \mid (F)_{a,b} \text{ has a nonzero solution}\}$ . In fact, since the Fučík spectrum is completely known in the above ODE case, it can be shown by topological arguments ([12]) that, if  $(a, b) \notin \Sigma$  then, depending on the particular connected component of  $\mathbb{R}^2 \setminus \Sigma$  where the point  $(a, b)$  given in (1.2) is located, problem (1.2) (which could be considered a *non-resonant* problem)

- (I) *either* has at least one solution for any given  $f$ ,
- (II) *or else* may have a solution for certain  $f$ 's and no solution for others.

Our main goal in the present work is to obtain an existence result that holds for *any* given right hand side  $f$  and that can be applied to general classes of operators. In order to motivate our main theorem, we recall a few basic results in the case of the Laplace operator on a *bounded domain*. In fact, since the appearance of [2] many authors have considered existence results for the problem

$$(1.3) \quad \Delta u + g(u) = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where  $\Omega \subset \mathbb{R}^N$  is a bounded smooth domain and  $g$  satisfies (1.2), with  $0 < \lambda_1 < \dots < \lambda_k < \dots$  denoting the spectrum of the Laplace operator with Dirichlet boundary condition. It is worth noting that, in this case, the principal eigenvalue  $\lambda_1$  plays a special role. Namely, alternative (II) above occurs whenever  $k = 1$  in (1.2). Therefore, a general existence result for any right hand side can only be expected in the case  $a > \lambda_1$ . Although there are a number of existence and multiplicity theorems when  $a > \lambda_1$ , we would like to mention the following result of Lazer and McKenna (Theorems 2.4 and 2.5 in [15]):

THEOREM 1 ([15]). Assume (1.2) with  $b \in (\lambda_k, \lambda_{k+1})$ , for some  $k \geq 2$ , and that  $g'(s) \leq b_1 < \lambda_{k+1}$  for all  $s \in \mathbb{R}$  (where  $b_1 > b$ ). There exists  $\lambda_{k-1} < \alpha = \alpha(b) < \lambda_k$  such that, if  $a \in (\alpha, \lambda_k)$ , then problem

$$\Delta u + g(u) = t\phi_1 + h \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

has at least three solutions for  $t$  large negative and a unique solution for  $t$  large positive (here,  $\phi_1 > 0$  denotes a normalized  $\lambda_1$ -eigenfunction).

In this paper we will prove an existence result in the spirit of Theorem 1 under reasonably weak conditions so that it is applicable to a large class of problems on bounded and unbounded domains (clearly, the uniqueness assertion in Theorem 1 precludes multiplicity for an arbitrary right hand side  $f$ ).

More precisely, we consider a class of nonlinear problems of the form

$$(P) \quad Lu + g(x, u) = f,$$

where  $L$  is an unbounded self-adjoint operator on a Hilbert space  $H$  of  $L^2(\Omega)$ -functions,  $\Omega \subset \mathbb{R}^N$  an arbitrary domain, and  $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a ‘‘jumping nonlinearity’’ in the sense that the limits  $\lim_{s \rightarrow -\infty} g(x, s)/s = a$ ,  $\lim_{s \rightarrow \infty} g(x, s)/s = b$  exist and ‘‘cross’’ a (possibly multiple) eigenvalue of the operator  $-L$  according to the following assumptions (where  $\sigma(L)$  denotes the spectrum of  $L$ ,  $\sigma_e(L)$  the essential spectrum of  $L$  and  $s^+ = \max\{s, 0\}$ ,  $s^- = s^+ - s$ ):

- (L<sub>1</sub>)  $\widehat{\lambda} > 0$  is an isolated point of  $\sigma(-L)$ .
- (L<sub>2</sub>)  $1 \leq \dim \ker(L + \widehat{\lambda}) < \infty$  and every  $0 \neq u \in \ker(L + \widehat{\lambda})$  changes sign.
- (L<sub>3</sub>)  $Lu + bu^+ - \widehat{\lambda}u^- = 0$  has no nonzero solution, where  $\widehat{\lambda} < b$ .
- (L<sub>4</sub>)  $(-\infty, -b) \cap \sigma_e(L) = \emptyset$  (so  $L$  has a discrete point spectrum in  $(-\infty, -b)$ ).
- (G<sub>1</sub>)  $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a ‘‘Carathéodory function’’ such that

$$0 < \lim_{s \rightarrow -\infty} \frac{g(x, s)}{s} = a < b = \lim_{s \rightarrow \infty} \frac{g(x, s)}{s}.$$

- (G<sub>2</sub>) Set  $g_0(x, s) := g(x, s) - (bs^+ - as^-)$ . Then, for every  $\varepsilon > 0$  there exists  $0 \leq b_\varepsilon(x) \in L^2(\Omega)$  such that

$$|g_0(x, s)| \leq b_\varepsilon(x) + \varepsilon|s| \quad \text{a.e. } x \in \Omega, s \in \mathbb{R}.$$

- (G<sub>3</sub>)  $g_0(x, \cdot)$  is nondecreasing for a.a.  $x \in \Omega$ .

Furthermore, we write  $\widehat{\lambda}_-$  (resp.  $\widehat{\lambda}_+$ ) to denote the closest point in  $\sigma(-L)$  to the left (resp. right) of  $\widehat{\lambda}$ . Our main result in this paper is the following theorem:

THEOREM 2. Assume (L<sub>1</sub>)–(L<sub>4</sub>) for some  $\widehat{\lambda} < b < \widehat{\lambda}_+$ . There exists  $\widehat{\lambda}_- < \alpha = \alpha(b, L) < \widehat{\lambda}$  such that, if  $a \in (\alpha, \widehat{\lambda}]$  and (G<sub>1</sub>)–(G<sub>3</sub>) are satisfied, then problem (P) possesses a solution for any  $f \in H$ .

A few comments about the hypotheses are in order.

REMARK. (a) Hypotheses (L<sub>1</sub>) and (L<sub>2</sub>) say that the closed interval  $[a, b]$  contains a single eigenvalue of  $-L$  (of any finite multiplicity) which cannot be a principal eigenvalue (as every  $0 \neq u \in \ker(L + \widehat{\lambda})$  changes sign).

(b) Hypothesis (L<sub>3</sub>) says that the point  $(b, \widehat{\lambda})$  does not belong to the *Fučik Spectrum* of  $-L$  (see Remark 3.8).

(c) Hypothesis (L<sub>4</sub>) allows applications to problems with selfadjoint operators which may have continuous spectrum or eigenvalues of infinity multiplicity.

(d) Hypothesis (G<sub>3</sub>) is needed for the proof of our result in the present generality. Nevertheless, in some situations (G<sub>3</sub>) can be removed.

Theorem 2 will be applied to the question of finding time-periodic solutions of the *beam equation* and the *vibrating string equation*, *without any symmetry assumption*. We will also consider existence of  $H^2$ -solutions of the *Schrödinger equation* in the whole space  $\mathbb{R}^N$ . In particular, in this case (G<sub>3</sub>) is not needed and we can prove the following result:

THEOREM 3. *Assume that  $V: \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies the conditions:*

(V<sub>1</sub>)  $V \in C(\mathbb{R}^N)$  and  $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ .

(V<sub>2</sub>) *There exists  $\varphi \in C_0^\infty(\mathbb{R}^N)$  such that  $\int (|\nabla\varphi|^2 + V(x)\varphi^2) < 0$ .*

(V<sub>3</sub>)  $\widehat{\lambda} > 0$  *is an isolated point of  $\sigma(\Delta - V(x))$  with  $\sigma(\Delta - V(x)) \cap [a, b] = \{\widehat{\lambda}\}$ .*

Let  $\lambda_0 < \lambda_1 < \dots < 0$  denote the distinct eigenvalues of  $L = -\Delta + V(x)$  in  $H^2(\mathbb{R}^N)$ . Then, given  $b$  with  $\widehat{\lambda} = |\lambda_k| < b < |\lambda_{k-1}|$ , for some  $k \geq 1$ , there exists  $\alpha$  with  $|\lambda_{k+1}| < \alpha < |\lambda_k|$  (or  $0 < \alpha < |\lambda_k|$  if  $\lambda_k$  is the largest negative eigenvalue of  $L$ ) such that, for  $a \in (\alpha, |\lambda_k|)$ , and  $g: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  satisfying conditions (G<sub>1</sub>), (G<sub>2</sub>) above, equation

$$-\Delta u + V(x)u + g(x, u) = f(x), \quad u \in H^2(\mathbb{R}^N)$$

has a solution for any given  $f \in L^2(\mathbb{R}^N)$ .

The organization of this paper is as follows. In Section 2 we provide the abstract framework for problem (P) while Theorem 2 is proved in Section 3. Applications and proofs of the corresponding theorems are given in Section 4.

We thank the referee for informing us of the recent papers [5], [6] and asking for a comparison of the results. In [6] the authors give a description of the Fučík spectrum near  $(\widehat{\lambda}, \widehat{\lambda})$ , where  $\widehat{\lambda}$  is an isolated point of the spectrum  $\sigma(L)$  of a general selfadjoint operator  $L$  defined on a Hilbert space  $H$  of  $L^2(\Omega)$ , with  $\Omega$  a bounded subset of  $\mathbb{R}^N$  (their work complements or extends other results in this direction, see e.g. [3], [18], [20] and references therein). Such a description enables them to apply topological arguments (in particular, degree computations) and prove some existence results for solutions of (P). However, we remark that both their approach and existence results are of a different nature from ours.

**2. The abstract framework**

Let  $H$  be a Hilbert space of  $L^2(\Omega)$ -functions, with innerproduct and norm denoted by  $(\cdot, \cdot)_2$  and  $|\cdot|_2$ , respectively. Let  $L: D(L) \subset H \rightarrow H$  be an unbounded self-adjoint operator. In view of the spectral theorem, we recall that, for any continuous function  $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ , we can define the self-adjoint operator

$$\varphi(L) = \int_{-\infty}^{\infty} \varphi(\lambda) dE(\lambda) = \int_{\sigma(L)} \varphi(\lambda) dE(\lambda),$$

where  $\{E(\lambda) \mid \lambda \in \mathbb{R}\}$  is a spectral family for  $L$ ,  $\sigma(L)$  is the spectrum of  $L$  and

$$(2.1) \quad D(\varphi(L)) = \left\{ u \in H \mid \int_{-\infty}^{\infty} |\varphi(\lambda)|^2 d(E(\lambda)u, u)_2 < \infty \right\},$$

$$(2.2) \quad (\varphi(L)u, v)_2 = \int_{-\infty}^{\infty} \varphi(\lambda) d(E(\lambda)u, v)_2, \quad \text{for all } u \in D(\varphi(L)), v \in H.$$

In particular, we have that

$$L = \int_{-\infty}^{\infty} \lambda dE(\lambda),$$

where

$$(2.3) \quad D(L) = \left\{ u \in H \mid \int_{-\infty}^{\infty} \lambda^2 d(E(\lambda)u, u)_2 < \infty \right\},$$

$$(2.4) \quad (Lu, v)_2 = \int_{-\infty}^{\infty} \lambda d(E(\lambda)u, v)_2 \quad \text{for all } u \in D(L), v \in H.$$

We note that (2.4) implies

$$|Lu|_2^2 = \int_{-\infty}^{\infty} \lambda^2 d(E(\lambda)u, u)_2 \quad \text{for all } u \in D(L).$$

Next, we pick  $\varphi(\lambda) = |\lambda|^{1/2}$  and define the unbounded self-adjoint operator

$$A = |L|^{1/2} = \int_{-\infty}^{\infty} |\lambda|^{1/2} dE(\lambda),$$

that is,

$$(2.5) \quad D(A) = \left\{ u \in H \mid \int_{-\infty}^{\infty} |\lambda| d(E(\lambda)u, u)_2 < \infty \right\},$$

$$(2.6) \quad (Au, v)_2 = \int_{-\infty}^{\infty} |\lambda|^{1/2} d(E(\lambda)u, v)_2 \quad \text{for all } u \in D(A), v \in H.$$

As is well-known, the closedness of the operator  $A$  implies that  $E := D(A)$  is a Hilbert space with the graph-innerproduct

$$(u, v)_E := (u, v)_2 + (Au, Av)_2$$

and corresponding *graph-norm*

$$\|u\|_E = (|u|_2^2 + |Au|_2^2)^{1/2} = \left( \int_{-\infty}^{\infty} (1 + |\lambda|) d(E(\lambda)u, u)_2 \right)^{1/2}.$$

LEMMA 2.1 (Poincaré-type inequality). *Assume that  $0 \notin \sigma(L)$  and define*

$$\|u\| := \left( \int_{-\infty}^{\infty} |\lambda| d(E(\lambda)u, u)_2 \right)^{1/2} = |Au|_2.$$

*Then, there exists  $\delta_0 > 0$  such that*

$$\|u\|^2 \geq \delta_0 |u|_2^2 \quad \text{for all } u \in E.$$

*In particular, the norm  $\|\cdot\|$  is equivalent to the graph-norm  $\|\cdot\|_E$  on  $E$ .*

PROOF. It follows at once by letting  $\delta_0 = \text{dist}(0, \sigma(L))$  and noticing that

$$\int_{-\infty}^{\infty} |\lambda| d(E(\lambda)u, u)_2 = \int_{\sigma(L)} |\lambda| d(E(\lambda)u, u)_2 \geq \delta_0 |u|_2^2. \quad \square$$

From now on we assume that  $0 \notin \sigma(L)$ . It follows that  $D(L) \subset E$ . We shall consider the Hilbert space  $E$  equipped with the norm  $\|\cdot\|$  defined above, which comes from the innerproduct

$$(2.7) \quad \langle u, v \rangle = \int_{-\infty}^{\infty} |\lambda| d(E(\lambda)u, v)_2 = (Au, Av)_2.$$

Let us write  $\mathbb{R} = \mathbb{R}_+ \cup \mathbb{R}_- = \{\lambda \mid \lambda \geq 0\} \cup \{\lambda \mid \lambda < 0\}$  and denote

$$(2.8) \quad P_+ = E(\mathbb{R}_+) = \chi_{\mathbb{R}_+}(L), \quad P_- = E(\mathbb{R}_-) = \chi_{\mathbb{R}_-}(L),$$

where  $\chi_S$  denotes the characteristic function of a set  $S \subset \mathbb{R}$ . Then,  $P_+$  and  $P_-$  are orthogonal projections such that  $P_+ \oplus P_- = I$ . Correspondingly, we have the orthogonal decomposition

$$(2.9) \quad E = E_+ \oplus E_-,$$

where  $E_+ = P_+(E)$ ,  $E_- = P_-(E)$ .

Also note that the spectral theorem implies that  $A$  commutes with both  $P_+$  and  $P_-$ : in other words, if  $u \in D(A)$  then  $P_{\pm}u \in D(A)$  and  $P_{\pm}Au = AP_{\pm}u$ . Moreover, from (2.1), (2.2), (2.6) and (2.8), it follows that  $D(L) = D(A^2)$  and, for any  $u \in D(L)$ ,  $v \in E$ ,

$$(2.10) \quad ((P_+ - P_-)Au, Av)_2 = ((P_+ - P_-)A^2u, v)_2 = (Lu, v)_2.$$

Next, let  $\Omega \subset \mathbb{R}^N$  be an arbitrary domain and let  $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  be a *Carathéodory function* (i.e.  $g(\cdot, s)$  is measurable for all  $s \in \mathbb{R}$  and  $g(x, \cdot)$  is continuous for almost all  $x \in \Omega$ ) satisfying

$$(2.11) \quad |g(x, s)| \leq A(x) + B|s|, \quad \text{a.a. } x \in \Omega, \quad s \in \mathbb{R},$$

where  $0 \leq A(x) \in L^2(\Omega)$  and  $B \geq 0$ . From this, it follows that the *Nemytskiĭ operator*

$$u(x) \mapsto g(x, u(x))$$

is well-defined and continuous from  $L^2(\Omega)$  to  $L^2(\Omega)$  (see [23]). Given  $f \in H$  and an unbounded self-adjoint operator  $L: D(L) \subset H \rightarrow H$  (as in the beginning of this section), we consider the equation

$$(P) \quad Lu + g(x, u) = f, \quad u \in D(L).$$

A solution of (P) is a function  $u \in D(L)$  that satisfies the equation in  $H$ . We now verify that equation (P) has a natural variational structure in the sense that its solutions correspond exactly to the critical points of a related functional  $I: E \rightarrow \mathbb{R}$ . Indeed, consider the quadratic form  $Q: E \rightarrow \mathbb{R}$  defined by

$$(2.12) \quad Q(u) = \frac{1}{2}(\|P_+u\|^2 - \|P_-u\|^2), \quad u \in E.$$

(Note that (2.10) implies  $Q(u) = (1/2)(Lu, u)_2$  whenever  $u \in D(L) \subset E$ ). Next, let  $G(x, s) = \int_0^s g(x, t) dt$ , and consider the functional

$$(2.12) \quad I(u) = Q(u) + \int_{\Omega} G(x, u) dx - \int_{\Omega} fu dx.$$

Then, the following result holds true.

LEMMA 2.2 (Variational structure and regularity). *The functional  $I: E \rightarrow \mathbb{R}$  is well-defined and of class  $C^1$  on  $E$ . In addition, its critical points  $u \in E$  are precisely the solutions  $u \in D(L)$  of equation (P).*

PROOF. Under assumption (2.11) it follows that  $G(x, s)$  satisfies the estimate

$$|G(x, s)| \leq A(x)|s| + \frac{1}{2}B|s|^2,$$

for a.a.  $x \in \Omega$ , for all  $s \in \mathbb{R}$ , so that the Nemytskiĭ operator  $u(x) \mapsto G(x, u(x))$  is well-defined and continuous from  $L^2(\Omega)$  to  $L^1(\Omega)$ , and it is easy to see that the functional  $I: E \rightarrow \mathbb{R}$  is also well-defined. In fact, it is not hard to show in this case that  $I$  is of class  $C^1$  with

$$I'(u) \cdot h = \langle P_+u, h \rangle - \langle P_-u, h \rangle + (g(x, u), h)_2 - (f, h)_2 \quad \text{for all } u, h \in E$$

or

$$I'(u) \cdot h = ((P_+ - P_-)Au, Ah)_2 + (g(x, u) - f, h)_2 \quad \text{for all } u, h \in E.$$

Now, let  $\hat{u} \in E$  be a critical point of  $I$ , that is,  $I'(\hat{u}) \cdot h = 0$  for all  $h \in E$ . From the above expression we obtain

$$(Ah, (P_+ - P_-)A\hat{u})_2 = (h, f - g(x, \hat{u}))_2 \quad \text{for all } h \in E = D(A).$$

This shows that  $(P_+ - P_-)A\hat{u} \in D(A^*)$  and  $A^*(P_+ - P_-)A\hat{u} = f - g(x, \hat{u})$ . Using the fact that  $A^* = A$ , we conclude that  $(P_+ - P_-)\hat{u} \in D(A^2) = D(L)$ , that is,  $\hat{u} \in D(L)$ , and

$$(P_+ - P_-)A^2\hat{u} = f - g(x, \hat{u})$$

so that, in view of (2.10),

$$L\hat{u} = f - g(x, \hat{u}).$$

We have shown that a critical point  $\hat{u} \in E$  of the functional  $I$  belongs in fact to  $D(L)$  and is a solution of equation (P). On the other hand, it is straightforward to see that a solution  $\hat{u} \in D(L)$  of equation (P) is a critical point of  $I$ .  $\square$

### 3. Proof of main result

In this section we prove our main abstract result on existence of solution for

$$(P) \quad Lu + g(x, u) = f, \quad u \in D(L).$$

Keeping in mind the abstract framework developed in the previous section, we recall the following assumptions that we are making on the unbounded self-adjoint operator  $L: D(L) \subset H \rightarrow H$  and on the function  $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ :

- (L<sub>1</sub>)  $\hat{\lambda} > 0$  is an isolated point of  $\sigma(-L)$ ,
- (L<sub>2</sub>)  $1 \leq \dim \ker(L + \hat{\lambda}) < \infty$  and every  $0 \neq u \in \ker(L + \hat{\lambda})$  changes sign,
- (L<sub>3</sub>)  $Lu + bu^+ - \hat{\lambda}u^- = 0$  has no nonzero solution, where  $\hat{\lambda} < b$ ,
- (L<sub>4</sub>)  $(-\infty, -b) \cap \sigma_e(L) = \emptyset$  (so  $L$  has a discrete point spectrum in  $(-\infty, -b)$ ).
- (G<sub>1</sub>)  $g: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a “Carathéodory function” such that

$$0 < \lim_{s \rightarrow -\infty} \frac{g(x, s)}{s} = a < b = \lim_{s \rightarrow \infty} \frac{g(x, s)}{s},$$

- (G<sub>2</sub>) Set  $g_0(x, s) := g(x, s) - (bs^+ - as^-)$ . Then, for every  $\varepsilon > 0$  there exists  $0 \leq b_\varepsilon(x) \in L^2(\Omega)$  such that

$$|g_0(x, s)| \leq b_\varepsilon(x) + \varepsilon|s| \quad \text{a.e. } x \in \Omega, \quad s \in \mathbb{R},$$

- (G<sub>3</sub>)  $g_0(x, \cdot)$  is nondecreasing for a.a.  $x \in \Omega$ .

Furthermore, we write  $\hat{\lambda}_-$  (resp.  $\hat{\lambda}_+$ ) to denote the closest point in  $\sigma(-L)$  to the left (resp. right) of  $\hat{\lambda}$ .

**THEOREM 2.** *Assume (L<sub>1</sub>)–(L<sub>4</sub>) for some  $\hat{\lambda} < b < \hat{\lambda}_+$ . There exists  $\hat{\lambda}_- < \alpha = \alpha(b, L) < \hat{\lambda}$  such that, if  $a \in (\alpha, \hat{\lambda}]$  and (G<sub>1</sub>)–(G<sub>3</sub>) are satisfied, then problem (P) possesses a solution for any  $f \in H$ .*

The proof of Theorem 2 will follow from a saddle-point type theorem due to Silva [21] (cf. also Brezis and Nirenberg [8]). We start by recalling a Palais–Smale type condition.



Let a  $C^1$  functional  $I: E \rightarrow \mathbb{R}$  be given on the Hilbert space  $E$ . Let

$$E = V \oplus W$$

be an orthogonal decomposition of the space  $E$ ,  $P_V: E \rightarrow V$ ,  $P_W: E \rightarrow W$  the corresponding orthogonal projections, and assume that

$$W_1 \subset W_2 \subset \dots \subset W$$

is a sequence of finite-dimensional subspaces of  $W$  such that  $\bigcup_{n=1}^\infty W_n$  is dense in  $W$ . A sequence  $(u_n) \subset E$  is called a  $(PS)_c^*$  sequence w.r.t.  $\{W_n\}$  at the level  $c \in \mathbb{R}$  if and only if there exists  $k_n \rightarrow \infty$  such that

$$P_W(u_n) \in W_{k_n}, \quad I(u_n) \rightarrow c, \quad \text{and} \quad \|I'(u_n)|_{V \oplus W_{k_n}}\| \rightarrow 0$$

as  $n \rightarrow \infty$ . And we say that the functional  $I$  satisfies condition  $(PS)_c^*$  w.r.t.  $\{W_n\}$  if and only if any  $(PS)_c^*$  sequence w.r.t.  $\{W_n\}$  possesses a subsequence that converges to a critical point of  $I$ .

**THEOREM 3.1** ([21], [8]). *Let  $E = V \oplus W$  be an orthogonal decomposition of the Hilbert space  $E$ . Assume that  $I: E \rightarrow \mathbb{R}$  is a  $C^1$  functional such that*

- (i)  $\sup_{u \in W} I(u) := I^\infty < \infty$ ,
- (ii)  $\inf_{u \in V} I(u) := I_\infty > -\infty$ ,
- (iii)  $I$  satisfies condition  $(PS)_c^*$  w.r.t.  $\{W_n\}$  for all  $I_\infty \leq c \leq I^\infty$ .

*Then  $I$  has a critical point  $\hat{u} \in E$ .*

**REMARK 3.2.** We remark that, as stated, the above result is a slight variation of Theorem 7 in [8]. Moreover, similarly to [8], condition (ii) can be replaced by the weaker requirement that  $I$  is bounded below on finite-dimensional subspaces of  $V$ , as long as the Palais–Smale condition (iii) is strengthened.

We start by pointing out that, without loss of generality, we may assume that  $0 \notin \sigma(L)$  and  $\hat{\lambda}$  is the least positive eigenvalue of  $-L$ . Indeed, by  $(L_1)$ , we have that  $(-\hat{\lambda}, -\hat{\lambda} + \delta] \cap \sigma(L) = \emptyset$  for some  $\delta > 0$  and, by letting  $\tilde{L}u := Lu + (\hat{\lambda} - \delta)u$ ,  $\tilde{g}(x, s) := (b - \hat{\lambda} + \delta)s^+ - (a - \hat{\lambda} + \delta)s^- + g_0(x, s)$ , we obtain the equivalent equation

$$\tilde{L}u + \tilde{g}(x, u) = f,$$

where  $0 \notin \sigma(\tilde{L})$ ,  $\tilde{\lambda} := \delta$  is the least positive eigenvalue of  $-\tilde{L}$ , and assumptions corresponding to  $(G_1)$ – $(G_3)$ ,  $(L_1)$ – $(L_4)$  hold for  $\tilde{g}$  and  $\tilde{L}$ .

Therefore, we will prove all the results that follow from now on under the assumption that  $0 \notin \sigma(L)$  and  $\hat{\lambda}$  is the least positive eigenvalue of  $-L$ . Then, for  $\hat{\lambda} < b < \hat{\lambda}_+$  given,  $-\hat{\lambda}$  is the only point of  $\sigma(L)$  in the interval  $[-b, 0]$ . We also let

$$(3.1) \quad \begin{aligned} P_1 &= \chi_{(-\infty, -b)}(L), & P_2 &= \chi_{[-b, 0]}(L), \\ (3.2) \quad E_1 &= P_1(E), & E_2 &= P_2(E), \end{aligned}$$

so that  $E_- = P_-(E) = E_1 \oplus E_2$  and  $E_2 = \ker(L + \widehat{\lambda})$ . In what follows we will consider the orthogonal decomposition

$$E = V \oplus W,$$

where  $V := E_+ \oplus E_2$  and  $W := E_1$ .

Finally, we consider  $0 < a \leq \widehat{\lambda}$  and recall the definition of the functional  $I: E \rightarrow \mathbb{R}$  whose critical points are the solutions  $u \in D(L)$  of equation (P):

$$I(u) = \frac{1}{2} (\|P_+u\|^2 - \|P_-u\|^2) + \int G(x, u) - \int fu.$$

From now on, unless otherwise stated, all integrals are taken over  $\Omega$ .

In the next Lemmas 3.3–3.6 we verify the *geometric conditions* (i), (ii) of Theorem 3.1

LEMMA 3.3.  $\sup_{u \in W} I(u) := I^\infty < \infty$ .

PROOF. In view of  $(G_2)$  we have  $G(x, s) = (b/2)|s^+|^2 + (a/2)|s^-|^2 + G_0(x, s)$ , where

$$|G_0(x, s)| \leq b_\varepsilon(x)|s| + \frac{\varepsilon}{2}|s|^2.$$

Therefore, using the fact that  $0 < a < b$  and  $u \in W = E_1 \subset E_-$ , we obtain

$$\begin{aligned} (3.3) \quad I(u) &= -\frac{1}{2}\|u\|^2 + \frac{b}{2} \int |u^+|^2 + \frac{a}{2} \int |u^-|^2 + \int G_0(x, u) - \int fu \\ &\leq -\frac{1}{2}\|u\|^2 + \frac{b}{2} \int |u|^2 + \varepsilon \int |u|^2 + C_1(\varepsilon)\|f\|_2^2 + C_2(\varepsilon). \end{aligned}$$

On the other hand, applying Lemma 2.1 with  $L|_{E_1}$  and noticing that

$$\text{dist}(0, \sigma(L|_{E_1})) = b + \delta, \quad \text{for some } \delta > 0,$$

we have the Poincaré-type inequality

$$(3.4) \quad \|u\|^2 = \|P_1u\|^2 \geq (b + \delta)\|u\|_2^2.$$

Taking  $\varepsilon = \delta/4$  in (3.3) and using (3.4), it follows that

$$I(u) \leq -\frac{\delta}{4(b + \delta)}\|u\|^2 + C$$

for some  $0 < C < \infty$ . □

Next, we will estimate  $I(u)$  from below on the subspace  $V = E_+ \oplus E_2$  and show that  $\inf_{u \in V} I(u) := I_\infty > -\infty$ . We need the following lemmas:

LEMMA 3.4. *Let  $B > 0$ , assume condition  $(L_2)$ ,  $1 \leq \dim \ker(L + \widehat{\lambda}) < \infty$ , and that every nonzero element  $\psi \in \ker(L + \widehat{\lambda})$  changes sign. If  $(\psi_n) \subset \ker(L + \widehat{\lambda})$ ,  $(w_n) \subset E_+$  and  $(\delta_n) \subset \mathbb{R}_+$  are sequences such that*

- (i)  $\|\psi_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ ,
- (ii)  $\|w_n\| \leq 1$  for all  $n \geq 1$ ,
- (iii)  $0 \leq \delta_n \leq \delta_0 < \mu B / (\mu + 1)$ , where

$$\mu = \inf_{\psi \neq 0} \frac{|\psi^+|_2^2}{|\psi^-|_2^2} > 0, \quad \psi \in \ker(L + \widehat{\lambda}),$$

then

$$\lim_{n \rightarrow \infty} \left[ -\delta_n |\psi_n|_2^2 + B \int |(w_n + \psi_n)^+|^2 \right] = \infty.$$

PROOF. Let  $s_n = \|\psi_n\|$  and  $\varphi_n = \psi_n / \|\psi_n\|$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[ -\delta_n |\psi_n|_2^2 + B \int |(w_n + \psi_n)^+|^2 \right] \\ = \lim_{n \rightarrow \infty} s_n^2 \left[ -\delta_n |\varphi_n|_2^2 + B \int \left| \left( \frac{w_n}{s_n} + \varphi_n \right)^+ \right|^2 \right]. \end{aligned}$$

Since  $\dim \ker(L + \widehat{\lambda}) < \infty$ , we have (by passing to a subsequence, if necessary) that  $\varphi_n \rightarrow \varphi$  in  $E$  for some  $0 \neq \varphi \in \ker(L + \widehat{\lambda})$ : in particular  $\varphi_n \rightarrow \varphi$  in  $L^2$ . Also, we have from (ii) that  $w_n/s_n \rightarrow 0$  in  $E$  (hence in  $L^2$ ), and we may assume that  $\delta_n \rightarrow \widehat{\delta}$ , in view of (iii). Therefore,

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n^2 \left[ -\delta_n |\varphi_n|_2^2 + B \int \left| \left( \frac{w_n}{s_n} + \varphi_n \right)^+ \right|^2 \right] \\ = \lim_{n \rightarrow \infty} s_n^2 [-\widehat{\delta} |\varphi|_2^2 + B |\varphi^+|_2^2 + o(1)] \\ = \lim_{n \rightarrow \infty} s_n^2 [(B - \widehat{\delta}) |\varphi^+|_2^2 - \widehat{\delta} |\varphi^-|_2^2 + o(1)] \\ \geq \lim_{n \rightarrow \infty} s_n^2 [(B - \widehat{\delta})\mu - \widehat{\delta}] |\varphi^-|_2^2 + o(1) = \infty, \end{aligned}$$

since  $(B - \widehat{\delta})\mu - \widehat{\delta} = \mu B - (\mu + 1)\widehat{\delta} > 0$ , in view of (iii). □

LEMMA 3.5. *Consider the function  $f(\rho, \varphi, w) = 1/2 - \rho |\varphi|_2^2 + B \int |(w + \varphi)^+|^2$  defined for  $\rho \in \mathbb{R}_+$ ,  $\varphi \in \ker(L + \widehat{\lambda})$ ,  $w \in S_+ := \{u \in E_+ \mid \|u\| = 1\}$ . Then, there exists  $\rho_0 = \rho_0(B, L) > 0$  such that*

$$m_\rho := \inf_{\varphi, w} f(\rho, \varphi, w) > 0, \quad 0 < \rho \leq \rho_0.$$

PROOF. Indeed, suppose there exists  $\rho_n > 0$  such that  $\rho_n \rightarrow 0$  and

$$\inf_{\varphi, w} f(\rho_n, \varphi, w) \leq 0.$$

Then, there exist sequences  $\varphi_n \in \ker(L + \widehat{\lambda})$  and  $w_n$  such that  $\|w_n\| = 1$  and

$$f(\rho_n, \varphi_n, w_n) \leq \frac{1}{n},$$

that is,

$$\frac{1}{2} - \rho_n |\varphi_n|_2^2 + B \int |(w_n + \varphi_n)^+|^2 \leq \frac{1}{n}.$$

In view of Lemma 3.4, there must exist  $M > 0$  such that  $|\varphi_n|_2 \leq M$  for all  $n \geq 1$ . But then, passing to the limit in the above expression leads to a contradiction. Therefore, it follows that

$$\inf_{\varphi, w} f(\rho, \varphi, w) > 0, \quad 0 < \rho \leq \rho_0.$$

for some  $\rho_0 > 0$ . □

LEMMA 3.6. *There exists  $0 < \alpha_1 = \alpha_1(b, L) < \widehat{\lambda}$  such that if  $a \in (\alpha_1, \widehat{\lambda}]$  then*

$$\inf_{u \in V} I(u) := I_\infty > -\infty.$$

PROOF. Let  $u = w + \psi \in E_+ \oplus \ker(L + \widehat{\lambda}) = V$ . We have

$$\begin{aligned} I(u) &= \frac{1}{2} \|w\|^2 - \frac{1}{2} \|\psi\|^2 + \frac{b}{2} \int |u^+|^2 + \frac{a}{2} \int |u^-|^2 + \int G_0(x, u) - \int fu \\ &= \frac{1}{2} \|w\|^2 - \frac{1}{2} \|\psi\|^2 + \frac{a}{2} \int |u|^2 + \frac{b-a}{2} \int |u^+|^2 + \int G_0(x, u) - \int fu \\ &\geq \frac{1}{2} \|w\|^2 - \frac{\widehat{\lambda}}{2} |\psi|_2^2 + \frac{a}{2} |u|_2^2 + \frac{b-a}{2} \int |(w + \psi)^+|^2 - \varepsilon |u|_2^2 - C(\varepsilon), \end{aligned}$$

hence

$$I(u) \geq \frac{1}{2} \|w\|^2 + \frac{a-2\varepsilon}{2} |w|_2^2 + \frac{a-\widehat{\lambda}-2\varepsilon}{2} |\psi|_2^2 + \frac{b-a}{2} \int |(w + \psi)^+|^2 - C(\varepsilon),$$

which gives

$$(3.5) \quad I(u) \geq \frac{1}{2} \|w\|^2 - \frac{\widehat{\lambda}-a+2\varepsilon}{2} |\psi|_2^2 + \frac{b-\widehat{\lambda}}{2} \int |(w + \psi)^+|^2 - C(\varepsilon).$$

We note that the above estimate is of the form

$$I(u) \geq \frac{1}{2} \|w\|^2 - A |\psi|_2^2 + B \int |(w + \psi)^+|^2 - C(\varepsilon), \quad u = w + \psi \in V.$$

where  $A = (\widehat{\lambda} - a + 2\varepsilon)/2 > 0$ ,  $B = (b - \widehat{\lambda})/2 > 0$ ,  $w \in E_+$  and  $\psi \in \ker(L + \widehat{\lambda})$ .

We will consider two cases:

*Case 1.*  $\|w\| = 0$ . In this case we have

$$I(u) = I(\psi) \geq -A |\psi|_2^2 + B \int |\psi^+|^2 - C(\varepsilon),$$

and, if we take  $A < \mu B/(\mu + 1)$ , then Lemma 3.4 implies that  $I(\psi) \rightarrow \infty$  as  $\|\psi\| \rightarrow \infty$ . In particular,  $I(\psi)$  is bounded from below. In other words, if

$$(3.6) \quad \frac{\widehat{\lambda} - a + 2\varepsilon}{2} < \frac{\mu}{\mu + 1} \left( \frac{b - \widehat{\lambda}}{2} \right)$$

then,  $I(\psi)$  is bounded from below.

*Case 2.*  $\|w\| \neq 0$ . In this case we can write

$$I(u) \geq \|w\|^2 \left[ \frac{1}{2} - A \left| \frac{\psi}{\|w\|} \right|_2^2 + B \int \left| \left( \frac{w}{\|w\|} + \frac{\psi}{\|w\|} \right)^+ \right|^2 \right] - C(\varepsilon).$$

and, with  $\rho_0 > 0$  given in Lemma 3.5, if

$$(3.7) \quad \frac{\widehat{\lambda} - a + 2\varepsilon}{2} < \rho_0$$

then Lemma 3.6 implies that  $I(u) \geq m_\rho \|w\|^2 - C(\varepsilon) \geq -C(\varepsilon)$ , hence  $I(u)$  is again bounded from below.

Combining Cases 1 and 2 and in view of (3.6), (3.7), we define

$$(3.8) \quad \alpha_1 := \max \left\{ \frac{(2\mu + 1)\widehat{\lambda} - \mu b}{\mu + 1}, \widehat{\lambda} - 2\rho_0, 0 \right\}.$$

If  $a \in (\alpha_1, \widehat{\lambda}]$  then, by picking  $\varepsilon = \widehat{\varepsilon} > 0$  suitably small, it follows that

$$\inf_{u \in V} I(u) := I_\infty > -\infty. \quad \square$$

Next, we will consider the compactness condition  $(PS)_c^*$  of Theorem 3.1.

LEMMA 3.7. *There exists  $0 < \alpha_2 = \alpha_2(b, L) < \widehat{\lambda}$  such that if  $a \in (\alpha_2, \widehat{\lambda}]$  then*

$$(3.9) \quad Lu + bu^+ - au^- = 0, \quad u \in D(L)$$

has only the trivial solution  $u = 0$ .

PROOF. For fixed  $b$  we consider the functional

$$J_a(u) = \frac{1}{2} (\|P_+u\|^2 - \|P_-u\|^2) + \frac{1}{2} \int (b|u^+|^2 + a|u^-|^2), \quad u \in E$$

whose critical points are the solutions of (3.9) (cf. Lemma 2.2). By negation, let us assume there exist sequences  $(u_n), (a_n)$  with  $u_n \neq 0, a_n \rightarrow \widehat{\lambda}, a_n < \widehat{\lambda}$  and

$$(3.10) \quad Lu_n + bu_n^+ - a_nu_n^- = 0,$$

that is,  $J'_{a_n}(u_n) = 0$ . Then, writing  $u_n = P_+u_n + P_-u_n \in E_+ \oplus E_-$ , and recalling that

$$E_- = E_1 \oplus E_2 = [\chi_{(-\infty, -b)}(L)](E) \oplus [\chi_{[-b, 0)}(L)](E), \quad E_+ = [\chi_{[0, \infty)}(L)](E)$$

where  $E_2 = [\chi_{[-b,0)}(L)](E) = \ker(L + \widehat{\lambda})$ , we obtain from (3.10) that

$$(3.11) \quad \langle (P_+ - P_-)u_n, h \rangle + b \int u_n^+ h - a_n \int u_n^- h = 0 \quad \text{for all } h \in E,$$

$$(3.12) \quad \|P_+ u_n\|^2 - \|P_- u_n\|^2 + b \int |u_n^+|^2 + a_n \int |u_n^-|^2 = 0.$$

From (3.12) it follows that  $\|P_+ u_n\|^2 \leq \|P_- u_n\|^2$ , so that  $\|P_- u_n\| \neq 0$  since  $u_n \neq 0$ . Let  $z_n = u_n / \|P_- u_n\|$ . Then  $1 \leq \|z_n\|^2 \leq 2$  and (passing to a subsequence, if necessary) we obtain that  $z_n \rightharpoonup z$  weakly in  $E$ .

Next, we note that the embedding  $E_- \hookrightarrow H$  is compact in view of hypotheses  $(L_2)$ ,  $(L_4)$  and the comments following Remark 3.2. Therefore,

$$(3.13) \quad P_- z_n \rightarrow P_- z \quad \text{strongly in } H.$$

Now, dividing (3.11) by  $\|P_- u_n\|$  and letting  $h = P_- z_n - P_- z$ , we obtain

$$-\|P_- z_n\|^2 + \langle P_- z_n, P_- z \rangle + b \int z_n^+ (P_- z_n - P_- z) - a_n \int z_n^- (P_- z_n - P_- z) = 0,$$

and, by taking limits, it follows that  $\lim \|P_- z_n\|^2 = \|P_- z\|^2$ , hence

$$(3.14) \quad P_- z_n \rightarrow P_- z \quad \text{strongly in } E.$$

Next, we note that  $J_a(u) = (\|P_+ u\|^2 - \|P_- u\|^2)/2 + \Psi_a(u)$ , where  $\Psi_a(u) = (1/2) \int (b|u^+|^2 + a|u^-|^2)$  is a *convex* functional. Therefore, recalling that  $0 < a_n < \widehat{\lambda}$  and  $a_n \rightarrow \widehat{\lambda}$ , we obtain

$$\begin{aligned} \Psi_{\widehat{\lambda}}(z) - \Psi_{\widehat{\lambda}}(z_n) &\geq \Psi_{a_n}(z) - \Psi_{a_n}(z_n) - |a_n - \widehat{\lambda}| \|z_n\|_2^2 \\ &\geq \Psi'_{a_n}(z_n) \cdot (z - z_n) - |a_n - \widehat{\lambda}| \|z_n\|_2^2 \\ &= J'_{a_n}(z_n) \cdot (z - z_n) - \langle (P_+ - P_-)z_n, z - z_n \rangle - |a_n - \widehat{\lambda}| \|z_n\|_2^2 \\ &= \|P_+ z_n\|^2 - \langle P_+ z_n, z \rangle - \|P_- z_n\|^2 + \langle P_- z_n, z \rangle - |a_n - \widehat{\lambda}| \|z_n\|_2^2. \end{aligned}$$

Taking lim sup in the above estimate and using (3.14) together with the fact that  $\Psi_{\widehat{\lambda}}$  is *weakly lower semicontinuous*, we get

$$0 \geq \Psi_{\widehat{\lambda}}(z) - \liminf \Psi_{\widehat{\lambda}}(z_n) \geq \limsup \|P_+ z_n\|^2 - \|P_+ z\|^2$$

which implies  $\lim \|P_+ z_n\|^2 = \|P_+ z\|^2$ , hence

$$P_+ z_n \rightarrow P_+ z \quad \text{strongly in } E.$$

Finally, dividing (3.11) by  $\|P_- u_n\|$  and passing to the limit gives

$$\langle (P_+ - P_-)z, h \rangle + b \int z^+ h - \widehat{\lambda} \int z^- h = 0 \quad \text{for all } h \in E.$$

As in the proof of Lemma 2.2, this shows that  $z \in D(L)$  and

$$Lz + bz^+ - \widehat{\lambda}z^- = 0.$$

By assumption it follows that  $z = 0$ , hence  $z_n \rightarrow 0$  strongly in  $E$ , which contradicts the fact that  $1 \leq \|z_n\|^2 \leq 2$ . The proof is complete.  $\square$

REMARK 3.8. Given the selfadjoint operator  $L: D(L) \subset H \rightarrow H$ , let us call the *Fučík spectrum of  $L$*  the subset  $\Sigma_F(L) \subset \mathbb{R}^2$  given by

$$\Sigma_F(L) = \{(a, b) \in \mathbb{R}^2 \mid Lz + bz^+ - az^- = 0 \text{ has a nonzero solution}\}.$$

An inspection of the proof of Lemma 3.7 shows that, under conditions (L<sub>1</sub>)–(L<sub>4</sub>), for any given  $b > 0$  the *horizontal slice*  $(\mathbb{R} \times \{b\}) \cap \Sigma_F(L)$  is a *closed set*. In fact, our next lemma shows that, under some weaker assumptions, the compactness condition (PS)<sub>c</sub><sup>\*</sup> is intimately connected to the *Fučík spectrum*.

LEMMA 3.10. *Let  $L: D(L) \subset H \rightarrow H$  be a selfadjoint operator satisfying conditions (L<sub>1</sub>), (L<sub>2</sub>), (L<sub>4</sub>), and assume (G<sub>1</sub>)–(G<sub>3</sub>). If  $(a, b) \notin \Sigma_F(L)$  then the functional*

$$I(u) = \frac{1}{2} (\|P_+u\|^2 - \|P_-u\|^2) + \int G(x, u) - \int fu, \quad u \in E,$$

satisfies (PS)<sub>c</sub><sup>\*</sup> w.r.t.  $\{W_n\}$  for all  $c \in \mathbb{R}$ .

PROOF. Let  $(u_n)$  be a (PS)<sub>c</sub><sup>\*</sup> sequence w.r.t.  $\{W_n\}$ . As before, we write  $u_n = P_+u_n + P_-u_n = u_{n,1} + u_{n,2} \in E_+ \oplus E_-$ . We recall that

$$V = E_+ \oplus E_2, \quad W = E_1 = \text{cl} \left( \bigcup_{n=1}^{\infty} W_n \right).$$

By assumption, there exists  $k_n \rightarrow \infty$  such that

$$P_W(u_n) \in W_{k_n}, \quad I(u_n) \rightarrow c \quad \text{and} \quad \|I'(u_n)|_{V \oplus W_{k_n}}\| \rightarrow 0$$

as  $n \rightarrow \infty$ . And we must show that  $(u_n)$  possesses a subsequence that converges to a critical point of  $I$ . For that, we first show that  $(u_n)$  is bounded.

Indeed, note that from condition  $\|I'(u_n)|_{V \oplus W_{k_n}}\| \rightarrow 0$  we have

$$\begin{aligned} I'(u_n) \cdot u_n &= \|u_{n,1}\|^2 - \|u_{n,2}\|^2 + b \int |u_n^+|^2 + a \int |u_n^-|^2 \\ &\quad + \int g_0(x, u_n)u_n - \int fu_n = o(1)\|u_n\| \end{aligned}$$

and, in view of hypothesis (G<sub>2</sub>), it follows that

$$\|u_{n,1}\|^2 + \int |u_n^+|^2 + \int |u_n^-|^2 \leq C(1 + \|u_{n,2}\|^2),$$

hence

$$(3.15) \quad \|u_n\|^2 \leq C(1 + \|u_{n,2}\|^2).$$

Therefore, in order to show that  $(u_n)$  is bounded, it suffices to show that  $(u_{n,2})$  is bounded.

Arguing by contradiction, assume that  $t_n = \|u_{n,2}\| \rightarrow \infty$  and define  $z_n = u_n/\|u_{n,2}\|$ . Then, (3.15) implies that  $\|z_n\|$  is bounded, so that (passing to a subsequence, if necessary) we obtain  $z_n \rightharpoonup z$  weakly in  $E$ . As in the previous lemma, we first show that  $P_- z_n \rightarrow P_- z$ . For that, let us introduce the two families of projections

$$P_-^k: E \rightarrow E_2 \oplus W_k, \quad P^k: E \rightarrow E_+ \oplus E_2 \oplus W_k \quad (k \geq 1).$$

We have

$$\begin{aligned} & \frac{I'(u_n)}{\|u_{n,2}\|} \cdot (P_- z_n - P_-^{k_n} z) \\ &= -\langle P_- z_n, P_- z_n - P_-^{k_n} z \rangle + b \int z_n^+ (P_- z_n - P_-^{k_n} z) - a \int z_n^- (P_- z_n - P_-^{k_n} z) \\ & \quad + \int \frac{g_0(x, u_n)}{\|u_{n,2}\|} (P_- z_n - P_-^{k_n} z) - \int \frac{f}{\|u_{n,2}\|} (P_- z_n - P_-^{k_n} z). \end{aligned}$$

Since  $P_-^{k_n} z \rightarrow P_- z$  strongly in  $E$ , if we take limits as  $n \rightarrow \infty$  and use  $(G_2)$ , we obtain  $\lim \|P_- z_n\|^2 = \|P_- z\|^2$ , hence  $P_- z_n \rightarrow P_- z$  strongly in  $E$ . Again, using the fact that the functional  $\Psi(u) = (1/2) \int (b|u^+|^2 + a|u^-|^2)$  is convex, we get

$$\begin{aligned} \Psi(P^{k_n} z) - \Psi(z_n) &\geq \Psi'(z_n) \cdot (P^{k_n} z - z_n) \\ &= \frac{I'(u_n)}{\|u_{n,2}\|} \cdot (P^{k_n} z - z_n) - \langle (P_+ - P_-)z_n, P^{k_n} z - z_n \rangle \\ & \quad - \int \frac{g_0(x, u_n)}{\|u_{n,2}\|} (P^{k_n} z - z_n) + \int \frac{f}{\|u_{n,2}\|} (P^{k_n} z - z_n) \\ &\geq \|P_+ z_n\|^2 - \langle P_+ z_n, P^{k_n} z \rangle + \langle P_- z_n, P^{k_n} z \rangle - \|P_- z_n\|^2 + o(1). \end{aligned}$$

Since  $P^{k_n} z \rightarrow z$  then, by taking limsup in the above expression as before, we obtain  $\lim \|P_+ z_n\|^2 = \|P_+ z\|^2$ , hence  $z_n \rightarrow z$  strongly in  $E$ . It follows that  $z$  satisfies

$$\langle (P_+ - P_-)z, \phi \rangle + b \int z^+ \phi - a \int z^- \phi = 0 \quad \text{for all } \phi \in E.$$

As before, this yields  $z \in D(L)$  and

$$Lz + bz^+ - az^- = 0.$$

Since  $(a, b) \notin \Sigma_F(L)$  by assumption, it follows that  $z = 0$ , so that

$$z_n = \frac{u_n}{\|u_{n,2}\|} \rightarrow 0 \quad \text{strongly in } E.$$

which contradicts the fact that  $\|z_n\| \geq 1$ . We have shown that any  $(PS)_c^*$  sequence w.r.t.  $\{W_n\}$  is necessarily bounded, so that we may assume that  $u_n \rightharpoonup u$  weakly in  $E$ .

The rest of the proof follows by first showing that  $P_- u_n \rightarrow P_- u$  in  $E$ . Then, using the fact that the functional  $u \mapsto \int G(x, u)dx$  is convex by  $(G_3)$ , we argue



as before to conclude that  $P_+u_n \rightarrow P_+u$  in  $E$ , and finally that  $u_n \rightarrow u$  in  $E$ . This completes the proof of Lemma 3.10.  $\square$

REMARK 3.10. It is worth noticing that the above proof of boundedness of  $(PS)_c^*$  sequences did not use the convexity assumption  $(G_3)$ . In fact, as the following applications show, in some situations the strong convergence of bounded  $(PS)_c^*$  sequences follows from  $(G_1)$ ,  $(G_2)$  alone.

PROOF OF THEOREM 2. Lemmas 3.3, 3.6, 3.7 and 3.9 show that the conditions of Theorem 3.1 are satisfied for our functional  $I(u) = I_a(u)$  provided that  $a \in (\alpha, \widehat{\lambda}]$ , where  $\alpha := \min\{\alpha_1, \alpha_2\}$ . Therefore the functional  $I(u)$  has a critical point  $\widehat{u} \in E$ . And, by Lemma 2.2, it follows that  $\widehat{u} \in D(L)$  and it is a solution of problem (P).  $\square$

### 4. Applications

4.1. **The beam equation.** Given continuous functions  $g(x, t, u)$ ,  $f(x, t)$  which are  $2\pi$ -periodic in  $t$ , we consider the following problem for the *nonlinear beam equation*

$$(NBE) \quad \begin{cases} u_{tt} + u_{xxxx} + g(x, t, u) = f(x, t) & \text{for } (x, t) \in (0, \pi) \times \mathbb{R}, \\ u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0 & \text{for } t \in \mathbb{R}, \\ u(x, t + 2\pi) = u(x, t) & \text{for } (x, t) \in (0, \pi) \times \mathbb{R}, \end{cases}$$

that is, we seek time-periodic solutions with period  $2\pi$  for the above nonlinear beam equation in the interval  $(0, \pi)$  under Navier boundary conditions.

The corresponding eigenvalue problem for the *beam operator*  $L = \partial_t^2 + \partial_x^4$ ,

$$\begin{cases} Lu = \lambda u & \text{for } (x, t) \in (0, \pi) \times \mathbb{R}, \\ u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \\ u(x, t + 2\pi) = u(x, t), \end{cases}$$

has infinitely many eigenvalues  $\lambda_{mn}$  and eigenfunctions  $\varphi_{mn}, \psi_{mn}$  given by

$$\begin{cases} \lambda_{mn} = m^4 - n^2, & m \in \mathbb{N}, n \in \{0\} \cup \mathbb{N}, \\ \varphi_{mn}(x, t) = \sin mx \sin nt & \text{for } m, n \in \mathbb{N}, \\ \psi_{mn}(x, t) = \sin mx \cos nt & \text{for } m \in \mathbb{N}, n \in \{0\} \cup \mathbb{N}. \end{cases}$$

(Similarly to  $\varphi_{mn}, \psi_{mn}$ , note that the eigenfunctions  $\varphi_{m'n'}, \psi_{m'n'}$  correspond to the eigenvalue  $\lambda_{mn}$  whenever  $m'^4 - n'^2 = m^4 - n^2$ .) Letting  $\Omega = (0, \pi) \times (0, 2\pi)$ , we also note that  $\{\varphi_{mn}, \psi_{mn}\}$  is a complete orthogonal system for  $H = L^2(\Omega)$ ,

and the operator  $L: D(L) \subset H \rightarrow H$  defined by

$$D(L) := \left\{ u = \sum (c_{mn}\varphi_{mn} + d_{mn}\psi_{mn}) \in H \mid \sum \lambda_{mn}(c_{mn}\varphi_{mn} + d_{mn}\psi_{mn}) \in H \right\},$$

$$Lu := \sum \lambda_{mn}(c_{mn}\varphi_{mn} + d_{mn}\psi_{mn})$$

is a selfadjoint operator with *pure point spectrum*  $\sigma(L) = \{\lambda_{mn}\}$ . Moreover, except for  $\lambda = 0$ , which is an eigenvalue of infinite multiplicity, all other eigenvalues  $\lambda_{mn} \neq 0$  have finite multiplicity. In particular, it follows that assumption  $(L_4)$  holds true for any  $b > 0$ .

Next, let us assume that  $g(x, t, s): [0, \pi] \times [0, 2\pi] \times \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function satisfying

- (g<sub>1</sub>)  $0 < \lim_{s \rightarrow -\infty} \frac{g(x, t, s)}{s} = a < b = \lim_{s \rightarrow \infty} \frac{g(x, t, s)}{s}$  uniformly for  $(x, t) \in \Omega$ .
- (g<sub>2</sub>)  $g(x, t, \cdot)$  is *nondecreasing* for  $(x, t) \in \Omega$ .

When applying Theorem 1.2 to (NBE) we plan to use a symmetry result of Lazer–McKenna [16]. For that, we must restrict ourselves to crossing a positive eigenvalue of  $-L$  whose corresponding eigenfunctions  $\varphi_{mn}(x, t), \psi_{mn}(x, t)$  (for each fixed  $t \geq 0$ ) do not change sign for  $x \in (0, \pi)$ , a situation that only occurs when  $m = 1$ . Therefore, we shall assume that

- (1<sub>1</sub>)  $\sigma(-L) \cap [-\lambda_{1k}, b] = -\lambda_{1k} = k^2 - 1$  for some  $k \geq 2$ , and the equation  $m^4 - n^2 = 1 - k^2$  has the *unique* solution  $(m, n) = (1, k)$ .

Then, it follows that  $\lambda_{mn} \neq \lambda_{1k}$  for  $(m, n) \neq (1, k)$ , hence  $\ker(L - \lambda_{1k}) = \text{span}\{\sin x \cos kt, \sin x \sin kt\}$ . Let us denote by  $\mu_1$  the largest eigenvalue of  $-L$  that is smaller than  $-\lambda_{1k} = k^2 - 1$ , and by  $\mu_2$  the smallest eigenvalue of  $-L$  that is larger than  $-\lambda_{1k}$ : clearly, we have  $\mu_1 < k^2 - 1 < b < \mu_2$ .

Now, by a *symmetry result* of Lazer–McKenna ([16]), the solutions of the PDE

$$(4.1) \quad Lu + bu^+ - (k^2 - 1)u^- = 0$$

are of the form

$$u(x, t) = (\sin x)y(t)$$

where  $y(t)$  is a  $2\pi$ -periodic solution of the ODE

$$(4.2) \quad y'' + (b + 1)y^+ - k^2y^- = 0.$$

However, (4.2) has a  $2\pi$ -periodic solution if and only if (see [11])

$$(4.3) \quad \frac{\pi}{\sqrt{b+1}} + \frac{\pi}{k} = \frac{2\pi}{l}$$

for some  $l = 1, 2, \dots$ , which gives  $b = k^2 l^2 / (2k - l)^2 - 1$ . Since  $k^2 - 1 < b$ , it follows that  $k < l < 2k$ . Therefore, the smallest value of  $b$  for which we have a solution of (4.3) is attained when  $l = k + 1$ , that is,

$$b = k^2 \frac{(k + 1)^2}{(k - 1)^2} - 1.$$

Now, it is not difficult to see that  $\mu_2 < k^2(k + 1)^2 / (k - 1)^2 - 1$  (indeed, we have  $\mu_2 \leq -\lambda_{1,k+1} = (k + 1)^2 - 1$ , and a simple calculation shows that  $(k + 1)^2 - 1 < k^2(k + 1)^2 / (k - 1)^2 - 1$  for any  $k \geq 2$ ). Therefore, as long as  $b \in (k^2 - 1, \mu_2)$ , the equation

$$Lu + bu^+ - (k^2 - 1)u^- = 0$$

has *no* nontrivial solution, so that  $L$  satisfies assumption (L<sub>3</sub>) with  $\widehat{\lambda} = k^2 - 1$ . Furthermore, since  $\psi_{10} = \sin x$  is the only eigenfunction that does not change sign, and the corresponding eigenvalue is  $\lambda_{10} = 1$  (a simple eigenvalue of  $L$ ), it is clear that  $L$  satisfies assumption (L<sub>2</sub>) for any eigenvalue  $\lambda_{mn} \neq 1$ , in particular for  $\lambda_{1k} = 1 - k^2$ ,  $k \geq 2$ . We can now state:

**THEOREM 4.** *Assume condition (1<sub>1</sub>) with  $-\lambda_{1k} = k^2 - 1 < b < \mu_2$ . Then there exists  $\alpha = \alpha(b)$  with  $\mu_1 < \alpha < k^2 - 1$  such that, for  $a \in (\alpha, k^2 - 1]$  and  $g(x, t, s)$  satisfying (g<sub>1</sub>), (g<sub>2</sub>), equation (NBE) possesses a solution for any given  $f \in L^2(\Omega)$ .*

**REMARK 4.1.** (a) We exhibit below points of  $\sigma(L)$  in the interval  $[-20, 12]$

$$\begin{aligned} \dots < \lambda_{26} = -20 < \lambda_{310} = -19 < \lambda_{14} = -15 < \lambda_{23} = -9 < \lambda_{13} = -8 \\ < \lambda_{12} = -3 < \lambda_{11} = 0 < \lambda_{10} = 1 < \lambda_{23} = 7 < \lambda_{22} = 12 < \dots \end{aligned}$$

(b) Simple calculations show that, for example,  $\lambda_{12} = -3$ ,  $\lambda_{13} = -8$ ,  $\lambda_{14} = -15$ ,  $\lambda_{15} = -24$ ,  $\lambda_{16} = -35$  all satisfy the conditions of the above theorem.

**4.2. The wave equation.** Next we look for time-periodic solutions with period  $2\pi$  for the nonlinear wave equation in the interval  $(0, \pi)$  under Dirichlet boundary conditions:

$$(NWE) \quad \begin{cases} u_{tt} - u_{xx} + g(x, t, u) = f(x, t) & \text{for } (x, t) \in (0, \pi) \times \mathbb{R}, \\ u(0, t) = u(\pi, t) = 0 & \text{for } t \in \mathbb{R}, \\ u(x, t + 2\pi) = u(x, t) & \text{for } (x, t) \in (0, \pi) \times \mathbb{R}, \end{cases}$$

The corresponding eigenvalue problem for the *wave operator*  $L = \partial_t^2 - \partial_x^2$  has infinitely many eigenvalues  $\lambda_{mn}$  and eigenfunctions  $\varphi_{mn}, \psi_{mn}$  given by

$$\begin{aligned} \lambda_{mn} &= m^2 - n^2, \quad m \in \mathbb{N}, n \in \{0\} \cup \mathbb{N}, \\ \begin{cases} \varphi_{mn}(x, t) = \sin mx \sin nt & \text{for } m, n \in \mathbb{N}, \\ \psi_{mn}(x, t) = \sin mx \cos nt & \text{for } m \in \mathbb{N}, n \in \{0\} \cup \mathbb{N}. \end{cases} \end{aligned}$$

Similarly to the beam equation, letting  $\Omega = (0, \pi) \times (0, 2\pi)$ , we note that  $\{\varphi_{mn}, \psi_{mn}\}$  is a complete orthogonal system for  $H = L^2(\Omega)$ , and the operator  $L: D(L) \subset H \rightarrow H$  defined by

$$D(L) := \left\{ u = \sum (c_{mn}\varphi_{mn} + d_{mn}\psi_{mn}) \in H \mid \sum \lambda_{mn}(c_{mn}\varphi_{mn} + d_{mn}\psi_{mn}) \in H \right\},$$

$$Lu := \sum \lambda_{mn}(c_{mn}\varphi_{mn} + d_{mn}\psi_{mn})$$

is a selfadjoint operator with *pure point spectrum*  $\sigma(L) = \{\lambda_{mn}\}$ . Again, except for  $\lambda = 0$ , which is an eigenvalue of infinite multiplicity, all other eigenvalues  $\lambda_{mn} \neq 0$  have finite multiplicity.

Using similar arguments to those for the beam equation, we can now state

**THEOREM 5.** *Assume that  $k \geq 2$  is an integer for which the only solution  $(m, n)$  of the equation*

$$m^2 - n^2 = 1 - k^2, \quad (m, n) \in \mathbb{N} \times (\{0\} \cup \mathbb{N})$$

*is  $(m, n) = (1, k)$ . In addition, assume that  $-\lambda_{1k} = k^2 - 1 < b < \mu_2$ . Then there exists  $\alpha = \alpha(b)$  with  $\mu_1 < \alpha < k^2 - 1$  such that, for  $a \in (\alpha, k^2 - 1]$  and  $g(x, t, s)$  satisfying conditions  $(g_1), (g_2)$ , equation (NWE) possesses a solution for any given  $f \in L^2(\Omega)$ .*

**REMARK 4.2.** For other recent results on asymptotically linear wave and beam equations we refer the reader to [4] and references therein.

**4.3. The Schrödinger equation.** For our last application, we consider the *nonlinear Schrödinger equation*

$$(NSE) \quad -\Delta u + V(x)u + g(x, u) = f(x), \quad x \in \mathbb{R}^N.$$

Since we would like to give a simple application, we are not going to consider the most possible general situation. We assume that  $g: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  satisfies conditions  $(G_1), (G_2)$  in the introduction, and that  $V: \mathbb{R}^N \rightarrow \mathbb{R}$  satisfies:

- (V<sub>1</sub>)  $V \in C(\mathbb{R}^N)$  and  $V(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ ,
- (V<sub>2</sub>) there exists  $\varphi \in C_0^\infty(\mathbb{R}^N)$  such that  $\int (|\nabla \varphi|^2 + V(x)\varphi^2) < 0$ ,
- (V<sub>3</sub>)  $\widehat{\lambda} > 0$  is an isolated point of  $\sigma(\Delta - V(x))$  with  $\sigma(\Delta - V(x)) \cap [a, b] = \{\widehat{\lambda}\}$ .

**REMARK 4.3.** We will recall a few basic result in the theory of Schrödinger operators which are relevant to our discussion:

(a) Given  $V \in L^\infty(\mathbb{R}^N)$ , let the operator  $L: D(L) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$  be defined by

$$Lu = -\Delta u + V(x)u, \quad u \in D(L) = H^2(\mathbb{R}^N).$$

In addition, assume that  $\liminf_{|x| \rightarrow \infty} V(x) \geq \gamma$  for some  $\gamma \in \mathbb{R}$ . Then one has  $\sigma_{\text{ess}}(L) \subset [\gamma, \infty)$ . In particular, if  $\lim_{|x| \rightarrow \infty} V(x) = 0$  then  $\sigma_{\text{ess}}(L) = \sigma_{\text{ess}}(-\Delta) = [0, \infty)$ .

(b) The bottom of the spectrum  $\sigma(L)$  of the operator  $L$  is given by

$$\lambda_0 = \inf_{0 \neq u \in H^2(\mathbb{R}^N)} \frac{(Lu, u)_2}{|u|_2^2} = \inf_{0 \neq u \in H^2(\mathbb{R}^N)} \frac{\int |\nabla u|^2 + V(x)u^2}{\int u^2}.$$

Furthermore, if  $(V_2)$  is satisfied then  $\lambda_0 < 0$  and, using the Concentration Compactness Principle of P. L. Lions, one shows that  $\lambda_0$  is the principal eigenvalue of  $L$  with a positive eigenfunction  $\Phi_0$ :

$$\begin{cases} L\Phi = \lambda_0\Phi_0, \\ \Phi_0 \in H^2(\mathbb{R}^N), \quad \Phi_0 > 0. \end{cases}$$

In fact, by elliptic regularity theory, it follows that  $\Phi_0 \in C^\infty(\mathbb{R}^N)$ .

(c) The spectrum of  $L$  in  $(-\infty, \gamma)$ , namely  $\sigma(L) \cap (-\infty, \gamma)$  is at most a countable set, which we denote by  $\lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$  where each  $\lambda_k < \gamma$  is an isolated eigenvalue of  $L$  of finite multiplicity (counted as often as its multiplicity) and characterized by the minimax formula

$$\lambda_k = \inf_{F \in \mathcal{F}_k} \sup_{0 \neq u \in F} \frac{(Lu, u)_2}{|u|_2^2},$$

where  $\mathcal{F}_k$  denotes the collection of all  $k$ -dimensional subspaces of  $H^2(\mathbb{R}^N)$ .

So, if the potential  $V(x)$  verifies  $(V_1)$  and  $(V_2)$  then the selfadjoint operator  $L: D(L) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$  given by  $Lu = -\Delta u + V(x)u$ ,  $u \in D(L) = H^2(\mathbb{R}^N)$ , satisfies conditions  $(L_1)$ ,  $(L_2)$  and  $(L_4)$  in Theorem 2 with  $0 < a \leq \widehat{\lambda} = |\lambda_{k_0}| < b$ , for some  $k_0 \geq 1$ , where we are denoting by  $\lambda_0 < \lambda_1 \leq \dots < 0$  the eigenvalues of  $L$  which make up the spectrum of  $L$  in  $(-\infty, 0)$  (cf. remark above).

In fact, if we denote by  $\mu_1$  the largest eigenvalue of  $-L$  that is smaller than  $|\lambda_{k_0}|$ , and by  $\mu_2$  the smallest eigenvalue of  $-L$  that is larger than  $|\lambda_{k_0}|$ , then we have  $\mu_1 < a < |\lambda_{k_0}| < b < \mu_2$  and we can prove that the following result holds:

LEMMA 4.4. *If  $b \in (|\lambda_{k_0}|, \mu_2)$  then the equation*

$$(4.4) \quad -\Delta u + V(x)u + bu^+ - |\lambda_{k_0}|u^- = 0, \quad u \in H^2(\mathbb{R}^N)$$

*has no nonzero solution.*

PROOF. If (4.4) has a nonzero solution  $u \in H^2(\mathbb{R}^N)$ , then  $u$  must necessarily change sign in  $\mathbb{R}^N$ . Indeed, since  $b \notin \sigma(\Delta u - V(x))$  we cannot have  $u$  positive and, since  $\widehat{\lambda} = |\lambda_{k_0}|$ ,  $k_0 \geq 1$ , we also cannot have  $u$  negative. Therefore,  $u$  must change sign and we can rewrite (4.4) as

$$(4.5) \quad -\Delta u + [V(x) + |\lambda_{k_0}| + (b - |\lambda_{k_0}|)\chi_\Omega]u = 0, \quad 0 \neq u \in H^2(\mathbb{R}^N),$$

where  $\emptyset \neq \Omega = \{x \in \mathbb{R}^N | u(x) > 0\} \subsetneq \mathbb{R}^N$  and  $\chi_\Omega$  denotes the characteristic function of the set  $\Omega$ .

Now, if we set  $\tilde{V}(x) = V(x) + |\lambda_{k_0}| + (b - |\lambda_{k_0}|)\chi_\Omega$  then (4.5) shows that  $0 \in \sigma_p(\tilde{L})$ , where  $\tilde{L} = -\Delta + \tilde{V}(x)$ . Clearly, we have  $\tilde{V}(x) \in L^\infty(\mathbb{R}^N)$  and  $\liminf_{|x| \rightarrow \infty} \tilde{V}(x) \geq |\lambda_{k_0}| = \hat{\lambda} > 0$ . So, if we denote by  $\xi_0 < \xi_1 \leq \dots < 0$  the eigenvalues of  $\tilde{L}$  which make up the spectrum of  $\tilde{L}$  in  $(-\infty, \hat{\lambda})$ , then, by using the minimax characterization of the eigenvalues  $\lambda_k, \xi_k$  of  $L$  and  $\tilde{L}$ , respectively, we can write

$$\lambda_k = \inf_{\dim(F)=k} \sup_{0 \neq u \in F} \frac{\int (|\nabla u|^2 + V(x)u^2)}{\int u^2},$$

$$\xi_k = \inf_{\dim(F)=k} \sup_{0 \neq u \in F} \frac{\int (|\nabla u|^2 + \tilde{V}(x)u^2)}{\int u^2}.$$

But then, since  $V(x) + |\lambda_{k_0}| \leq \tilde{V}(x) \leq V(x) + b$ , with both inequalities being strict on sets of positive measure, we conclude that

$$\lambda_k - \lambda_{k_0} < \xi_k < \lambda_k + b, \quad k = 0, 1, \dots$$

Without loss of generality we assume  $\lambda_{k_0-1} < \lambda_{k_0}$ . Then, we have

$$\xi_{k_0-1} < \lambda_{k_0-1} + b = -|\lambda_{k_0-1}| + b < 0 = -\lambda_{k_0} + \lambda_{k_0} < \xi_{k_0},$$

hence  $\xi_{k_0-1} < 0 < \xi_{k_0}$ , which shows that  $0 \notin \sigma_p(\tilde{L})$ , a contradiction. □

We can now state:

**THEOREM 6.** *Assume  $V(x)$  satisfies conditions (V<sub>1</sub>)–(V<sub>3</sub>) and let  $L: D(L) \subset L^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$  be the selfadjoint operator given by  $Lu = -\Delta u + V(x)u$ ,  $u \in D(L) = H^2(\mathbb{R}^N)$ , so that  $\sigma(L) = \{\lambda_0, \lambda_1, \dots\} \cup [0, \infty)$ , where  $\lambda_j < \lambda_{j+1} < 0$  denote the distinct eigenvalues of  $L$ . Let  $\hat{\lambda} := -\lambda_k = |\lambda_k| > 0$  for some  $k \geq 1$  and  $|\lambda_k| < b < |\lambda_{k-1}|$  be given. Then there exists  $\alpha = \alpha(b)$  with  $|\lambda_{k+1}| < \alpha < |\lambda_k|$  (or  $0 < \alpha < |\lambda_k|$  if  $\lambda_k$  is the largest negative eigenvalue of  $L$ ) such that, for  $a \in (\alpha, |\lambda_k|]$  and  $g: \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  satisfying conditions (G<sub>1</sub>), (G<sub>2</sub>), equation*

$$-\Delta u + V(x)u + g(x, u) = f(x), \quad u \in H^2(\mathbb{R}^N)$$

*has a solution for any given  $f \in L^2(\mathbb{R}^N)$ .*

**PROOF.** In order to apply Theorem 1, since we do not have the convexity assumption (G<sub>3</sub>), we must show that bounded (PS)<sub>c</sub><sup>\*</sup> sequences are actually precompact (see Remark 3.11). In fact, if  $(u_n)$  is our bounded (PS)<sub>c</sub><sup>\*</sup> sequence with  $u_n \rightarrow u$  weakly in  $E = H^1(\mathbb{R}^N)$ , then the proof of Lemma 3.10 shows that  $P_- u_n \rightarrow P_- u$  strongly in  $H^1(\mathbb{R}^N)$ . Then, using the fact

that the nonlinearity  $g(x, \cdot)$  is nondecreasing at infinity, so that the functional  $\Psi(u) = (1/2) \int (b|u^+|^2 + a|u^-|^2)$  is convex, we have

$$\begin{aligned} \Psi(u) - \Psi(u_n) &\geq \Psi'(u_n) \cdot (u - u_n) \\ &= I'(u_n) \cdot (u - u_n) - \langle (P_+ - P_-)u_n, u - u_n \rangle \\ &\quad - \int g_0(x, u_n)(u - u_n) + \int f(u - u_n) \\ &\geq \|P_+u_n\|^2 - \langle P_+u_n, P_+u \rangle - \int g_0(x, u_n)(u - u_n) + o(1). \end{aligned}$$

Therefore, in order to show that  $P_+u_n \rightarrow P_+u$ , it is enough to verify that  $\int g_0(x, u_n)(u - u_n) = o(1)$ . In the present case, this follows from the estimate  $(G_2)$  and the fact that the embedding  $H_{\text{loc}}^1(\mathbb{R}^N) \hookrightarrow L_{\text{loc}}^2(\mathbb{R}^N)$  is compact. The proof of Theorem 4 is complete.  $\square$

#### REFERENCES

- [1] H. AMANN AND P. HESS, *A multiplicity result for a class of elliptic boundary value problems*, Proc. Roy. Soc. Edinburgh Sect. A **84** (1979), 145–151.
- [2] A. AMBROSETTI AND G. PRODI, *On the inversion of some differential mappings with singularities between banach spaces*, Ann. Mat. Pura Appl. **93** (1973), 231–247.
- [3] M. ARIAS, J. CAMPOS, M. CUESTA AND J.-P. GOSSEZ, *Asymmetric elliptic problems with indefinite weights*, Ann. Inst. H. Poincaré Anal. Non Linéaire (to appear).
- [4] T. BARTSCH AND Y. H. DING, *Critical point theory with applications to asymptotically linear wave and beam equations*, Differential Integral Equations **13** (2000), 973–1000.
- [5] A. K. BEN-NAOUM, C. FABRY AND D. SMETS, *Resonance with respect to the Fučík spectrum*, Electron. J. Differential Equations **37** (2000), 1–21.
- [6] ———, *Structure of the Fučík spectrum and existence of solutions for equations with asymmetric nonlinearities*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), 241–265.
- [7] M. BERGER AND E. PODOLAK, *On the solutions of a nonlinear Dirichlet problem*, Indiana Univ. Math. J. **24** (1975), 837–846.
- [8] H. BRÉZIS AND L. NIRENBERG, *Nonlinear Anal.* (to appear).
- [9] E. N. DANCER, *On the Dirichlet problem for weakly nonlinear elliptic partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **76** (1977), 283–300.
- [10] D. G. DE FIGUEIREDO, *On the superlinear Ambrosetti–Prodi problem*, Nonlinear Anal. **8** (1984), 655–665.
- [11] S. FUČIK, *Boundary value problems with jumping nonlinearities*, Cas. Pest. Mat. **101** (1976), 69–87.
- [12] ———, *Solvability of Nonlinear Equations and Boundary Value Problems*, D. Reidel Publishing Company, Dordrecht, Holland, 1980.
- [13] T. GALLOUET AND O. KAVIAN, *Résultats d'Existence et de Non Existence pour Certains Problèmes Demi-linéaires à l'Infini*, Ann. Fac. Sci. Toulouse Math. **3** (1981), 201–246.
- [14] J. L. KAZDAN AND F. W. WARNER, *Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567–597.
- [15] A. C. LAZER AND P. J. MCKENNA, *Critical point theory and Boundary value problems with nonlinearities crossing multiple eigenvalues*, Comm. Partial Differential Equations **10** (1985), 107–150.

- [16] ———, *A symmetry theorem and applications to nonlinear partial differential equations*, J. Differential Equations **72** (1988), 95–106.
- [17] ———, *Large-amplitude Periodic oscillations in suspension bridges: some new connections with nonlinear analysis*, SIAM Rev. **32** (1990), 537–578.
- [18] C. A. MAGALHÃES, *Semilinear elliptic problems with crossing of multiple eigenvalues*, Comm. Partial Differential Equations **15** (1990), 1265–1292.
- [19] B. RUF, *On nonlinear elliptic problems with jumping nonlinearities*, Ann. Mat. Pura Appl. (4) **CXXVIII** (1981), 133–151.
- [20] M. SCHECHTER, *The Fučik spectrum*, Indiana Univ. Math. J. **43** (1994), 1139–1157.
- [21] E. A. SILVA, *Critical point theorems and applications to differential equations*, Ph.D. Thesis, University of Wisconsin, Madison (1988).
- [22] S. SOLIMINI, *Some remarks on the number of solutions of some nonlinear elliptic problems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **2** (1985), 143–156.
- [23] M. M. VAINBERG, *Variational Methods for the Study of Nonlinear Operators*, Holden-Day, San Francisco, 1964.

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