Topological **M**ethods in **N**onlinear **A**nalysis
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REIDEMEISTER NUMBERS

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Abstract. In [5] we have conjectured that the Reidemeister number is infinite as long as an endomorphism of a discrete group is injective and the group has exponential growth. In the paper we prove this conjecture for any automorphism of a non-elementary, Gromov hyperbolic group. We also prove some generalisations of this result. The main results of the paper have topological counterparts.

1. Introduction

Let G be a finitely generated group and $\phi: G \to G$ an endomorphism. Two elements $\alpha, \alpha' \in G$ are said to be *ϕ-conjugate* if and only if there exists $\gamma \in G$ with

$$\alpha' = \gamma\alpha\phi(\gamma)^{-1}.$$

We shall write $\{x\}_\phi$ for the ϕ-conjugacy class of the element $x \in G$. The number of ϕ-conjugacy classes is called the *Reidemeister number* of an endomorphism ϕ, denoted by $R(\phi)$. If ϕ is the identity map then the ϕ-conjugacy classes are the usual conjugacy classes in the group G.

We note that $R(\phi)$ is infinite if group G is free Abelian and the action of ϕ on G has 1 as eigenvalue (see [3]).

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In [5] we have conjectured that the Reidemeister number is infinite as long as an endomorphism ϕ is injective and the group G has exponential growth.

In this paper we prove our conjecture for any automorphism of any non-elementary (i.e. not virtually cyclic), Gromov hyperbolic group. We also prove some generalisations of this result. The work [8] of G. Levitt and M. Lustig plays the key role in this new development of the subject.

Main results of this paper have their topological counterparts. Let X to be a connected, compact polyhedron and $f: X \to X$ to be a continuous map. Let $p: \widetilde{X} \to X$ be the universal cover of X and $\widetilde{f}: \widetilde{X} \to \widetilde{X}$ a lifting of f, i.e. $p \circ \widetilde{f} = f \circ p$. Two liftings $\widetilde{f}$ and $\widetilde{f}'$ are called *conjugate* if there is a elementγ in the deck transformation group $\Gamma \cong \pi_1(X)$ such that $\widetilde{f}' = \gamma \circ \widetilde{f} \circ \gamma^{-1}$. The subset $p(\mathrm{Fix}(\widetilde{f})) \subset \mathrm{Fix}(f)$ is called *the fixed point class of f determined by the lifting class* $[\widetilde{f}]$. Two fixed points x_0 and x_1 of f belong to the same fixed point class if and only if there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X. A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of f (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of f, denoted $R(f)$. This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$.The Nielsen number is always finite.

It follows immediately from main results of the paper that the topological Reidemeister number $R(f)$ is infinite for any homeomorphism f of a compact polyhedron X with a non-elementary, Gromov hyperbolic fundamental group.

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2. Twisted conjugacy classes and Reidemeister number of group endomorphism

Lemma 1. *If G is a group and ϕ is an endomorphism of G then an element $x \in G$ is always ϕ-conjugate to its image $\phi(x)$.*

Proof. Put $\gamma = x^{-1}$. Now x is ϕ-conjugate to $x^{-1}x\phi(x) = \phi(x)$. □

The mapping torus $M(\phi)$ of the group endomorphism $\phi: G \to G$ is obtained from group G by adding a new generator z and adding the relations $zgz^{-1} = \phi(g)$ for all $g \in G$. This means that $M(\phi)$ is a semi-direct product of G with Z.

Lemma 2. *Two elements x, y of G are ϕ-conjugate if and only if xz and yz are conjugate in the usual sense in $M(\phi)$. Therefore $R(\phi)$ is the number of usual conjugacy classes in the coset $G \cdot z$ of G in $M(\phi)$.*

Proof. If x and y are ϕ-conjugate, then there is a $\gamma \in G$ such that $\gamma x = y\phi(\gamma)$. This implies $\gamma x = yz\gamma z^{-1}$ and therefore $\gamma(xz) = (yz)\gamma$. So xz and yz are conjugate in the usual sense in $M(\phi)$. Conversely suppose xz and yz are conjugate in $M(\phi)$. Then there is a $\gamma z^n \in M(\phi)$ with $\gamma z^n xz = yz\gamma z^n$.

From the relation $zxz^{-1} = \phi(x)$, we obtain $\gamma\phi^n(x)z^{n+1} = y\phi(\gamma)z^{n+1}$ and therefore $\gamma\phi^n(x) = y\phi(\gamma)$. This shows that $\phi^n(x)$ and y are ϕ-conjugate. However, by Lemma 1, x and $\phi^n(x)$ are ϕ-conjugate, so x and y must be ϕ-conjugate. $\square$

Lemma 3 (T. Delzant). *Let J be a non-elementary Gromov hyperbolic group. Let K be a normal subgroup with abelian quotient. Then every coset C of $J \bmod K$ contains infinitely many conjugacy classes.*

Proof (see [8]). Fix u in the coset C under consideration. Suppose for a moment that we can find $c, d \in K$, generating a free group of rank 2, such that $uc^\infty \neq c^{-\infty}$ and $ud^\infty \neq d^{-\infty}$ (recall that we denote $g^{-\infty} = \lim_{n\to\infty} g^{-n}$ for g of infinite order).

Consider $x_k = c^k uc^k$ and $y_k = d^k ud^k$. For k large, the above inequalities imply that these two elements have infinite order, and do not generate a virtually cyclic group because ${x_k}^\infty$ and ${x_k}^{-\infty}$(respectively ${y_k}^\infty$ and ${y_k}^{-\infty}$) is close to c^∞ and $c^{-\infty}$ (respectively d^∞ and $d^{-\infty}$). Fix k, and consider the elements $z_n = {x_k}^{n+1}{y_k}^{-n}$. They belong to the coset C, because J/K is abelian, and their stable norm goes to infinity with n. Therefore C contains infinitely many conjugacy classes.

Let us now construct c, d as above. Choose $a, b \in K$ generating a free group of rank 2. We first explain how to get c. There is a problem only if $ua^\infty = a^{-\infty}$ and $ub^\infty = b^{-\infty}$. In that case there exists integers p, q with $ua^pu^{-1} = a^{-p}$ and $ub^qu^{-1} = b^{-q}$. We take $c = a^pb^q$, noting that $ucu^{-1} = a^{-p}b^{-q}$ is different from $c^{-1} = b^{-q}a^{-p}$.

Once we have c, we choose $c^* \in K$ with $\langle c, c^*\rangle$ free of rank 2, and we obtain d by applying the preceding argument using c^* and cc^* instead of a and b. The group $\langle c, d\rangle$ is free of rank 2 because d is a positive word in c^* and cc^*. $\square$

2.1. Automorphisms of Gromov hyperbolic groups. Let now ϕ be an automorphism of the Gromov hyperbolic group G and let $\|\cdot\|$ denote the word metric with respect to some finite generating set for G. The automorphism ϕ is called hyperbolic if there is an integer m and a number $\lambda > 1$ such that, for all $g \in G$ we have $\max(\|\phi^m(g)\|, \|\phi^{-m}(g)\|) \geq \lambda\|g\|$. For example a pseudo-Anosov homeomorphism of a closed surface of genus larger then one induces a hyperbolic automorphism on the level of fundamental group. Also, an automorphism of finitely generated free group with no nontrivial periodic conjugacy classes is hyperbolic.

Lemma 4 ([1]). *The mapping torus $M(\phi)$ of a hyperbolic automorphism ϕ is Gromov hyperbolic group.*

Theorem 5. *The Reidemeister number $R(\phi)$ is infinite if group G is Gromov hyperbolic, non-elementary, and ϕ is hyperbolic automorphism.*

Proof. The proof immediately folows from Lemmas 2–4. □

Corollary 6. *For pseudo-Anosov homeomorphisms of closed surfaces of genus larger then one the Reidemeister number is infinite.*

The following theorem, actually, was proved (implicitly) in the paper [2] of Cohen and Lustig in Proposition 5.4

Theorem 7. *The Reidemeister number $R(\phi)$ is infinite if group G is a free group F_n and an automorphism ϕ fixes a nontrivial conjugacy class in F_n.*

Proof. Let $D(\phi)$ be the graph with a vertex $V(v)$ for each $v \in F_n$ and an oriented edge x from $V(v)$ to $V(w)$ whenever $w = x^{-1}v\phi(x)$. The component of $D(\phi)$ containing vertex $V(v)$ is denoted $D_v(\phi)$. The graph $D(\phi)$ was introduced by Goldstein and Turner (see [6]). Since ϕ fixes a non-trivial conjugacy class in F_n we can choose a non-trivial word X such that $\phi(X) = v^{-1}Xv$ for some $v \in F_n$. Notice that $X^{-1}(X^n v)\phi(X) = X^n v$. Thus there is, for each $n \in Z$, a loop based at $V(X^n v)$ which reads off the word X. Such a loop is carried by a graph containing a simple closed curve. If infinitely many of the vertices $V(X^n v)$ were contained in the same component $D_w(\phi)$ then, since the lenghts of these loops are bounded (in fact equal to $\|X\|$), there would be an infinite family of pairwise disjoint simple closed curves in $D_w(\phi)$. This is impossible since $\mathrm{rank}(\pi_1(D_w(\phi)))$ is finite (see [2]). Hence there exist infinitely many components which contain non-trivial loops labelled X based at vertices of the form $V(X^n v)$. This means that the number of twisted conjugacy classes is also infinite. □

Let us now consider an outer automorphism $\Phi \in \mathrm{Out}\, G$ corresponding to automorphism of $\phi \in \mathrm{Aut}\, G$ and viewed as a collection of ordinary automorphisms $\alpha \in \mathrm{Aut}\, G$. We define $\alpha, \beta \in \Phi$ to be isogredient if $\beta = i_h \cdot \alpha \cdot {i_h}^{-1}$ for some $h \in G$, with $i_h(g) = hgh^{-1}$.

Lemma 8 ([8]). *The set $S(\Phi)$ of isogredience classes is infinite if group G is Gromov hyperbolic, non-elementary, and Φ has finite order in the group* $\mathrm{Out}\, G$.

Proof. Let J be the subgroup of $\mathrm{Aut}\, G$ consisting of all automorphisms whose image in $\mathrm{Out}\, G$ is a power of Φ. The exact sequence $1 \to K \to J \to$

$\langle\Phi\rangle \to 1$, with $K = G/\text{Center}$ and $\langle\Phi\rangle$ finite, shows that J is hyperbolic, non-elementary. The set of automorphisms $\alpha \in \Phi$ is a coset of $J \bmod K$. If $\alpha, \beta \in \Phi$ are isogredient they are conjugate in J. The proof is therefore concluded by applying Lemma 3. □

Theorem 9 ([8]). *For any $\Phi \in \operatorname{Out} G$, with G Gromov hyperbolic, non-elementary, the set $S(\Phi)$ of isogredience classes is infinite.*

Proof. We describe here main steps of the proof in [8]. By Lemma 8, we may assume that Φ has infinite order. By Paulin's theorem (see [9]) Φ preserves some R-tree T with nontrivial minimal small action of G (recall that an action of G is small if all ars stabilisers are virtually cyclic; the action of G on T is always irreducible (no global fixed point, no invariant line, no invariant end)). This means that there is an R-tree T equipped with an isometric action of G whose length function satisfies $l \cdot \Phi = \lambda l$ for some $\lambda \geq 1$.

Step 1. Suppose $\lambda = 1$. Then $S(\Phi)$ is infinite.

Step 2. Suppose $\lambda > 1$. Assume that arc stabilisers are finite, and there exists $N_0 \in N$ such that, for every $Q \in T$, the action of $\operatorname{Stab} Q$ on $\pi_o(T - Q)$ has at most N_0 orbits. Then $S(\Phi)$ is infinite.

Step 3. If $\lambda > 1$, then T has finite arc stabilisers. If $\lambda > 1$ then from work of Bestwina–Feighn (see [1]) it follows that there exists $N_0 \in N$ such that, for every $Q \in T$, the action of $\operatorname{Stab} Q$ on $\pi_o(T - (Q))$ has at most N_0 orbits. □

Theorem 10. *The Reidemeister number $R(\phi)$ is infinite if group G is Gromov hyperbolic, non-elementary, and ϕ is any automorphism of G.*

Proof. By definition, the automorphisms $\beta = i_m \cdot \alpha$ and $\gamma = i_n \cdot \alpha$ are isogredient if and only if there exists $g \in G$ with $\gamma = i_g \cdot \beta \cdot {i_g}^{-1}$, or equivalently $n = gm\alpha(g^{-1})c$ with c in center of G. So, the set $S(\Phi)$ of isogredience classes of automorphisms representing Φ may be identified to the set of twisted conjugacy classes of G mod its center.

If ϕ is automorphism of finite order in $\operatorname{Aut} G$, then the theorem immediately follows from Lemma 8.

If an automorphism ϕ has infinite order in $\operatorname{Aut} G$ then theorem follows from Theorem 9. □

2.2. Reduction to injective endomorphisms and the co-Hopf property. A group G is called *co-Hopf* if every monomorphism of G into itself is an isomorphism. It is fairly immediate to see that a freely decomposable group is not co-Hopf.

Lemma 11 ([10]). *Let G be a non-elementary, torsion-free, Gromov hyperbolic group. Then G is co-Hopf if and only if G is freely indecomposable.*

THEOREM 12. *The Reidemeister number $R(\phi)$ is infinite if group G is Gromov hyperbolic, non-elementary,torsion free, freely indecomposable and ϕ is any monomorphism of G into itself.*

PROOF. The proof follows from Lemma 11 and Theorem 10. □

Reduction to injective endomorphisms. Let G be a group and $\phi\colon G \to G$ an endomorphism. We shall call an element $x \in G$ *nilpotent* if there is an $n \in \mathbb{N}$ such that $\phi^n(x) = \mathrm{id}$. Let N be the set of all nilpotent elements of G.

LEMMA 13. *The set N is a normal subgroup of G. We have $\phi(N) \subset N$ and $\phi^{-1}(N) = N$. Thus ϕ induces an endomorphism $[\phi/N](xN) := \phi(x)N$. The endomorphism $[\phi/N]\colon G/N \to G/N$ is injective, and we have $R(\phi) = R([\phi/N])$.*

PROOF. (i) Let $x \in N$, $g \in G$. Then for some $n \in \mathbb{N}$ we have $\phi^n(x) = \mathrm{id}$. Therefore $\phi^n(gxg^{-1}) = \phi^n(gg^{-1}) = \mathrm{id}$. This shows that $gxg^{-1} \in N$ so N is a normal subgroup of G.

(ii) Let $x \in N$ and choose n such that $\phi^n(x) = \mathrm{id}$. Then $\phi^{n-1}(\phi(x)) = id$ so $\phi(x) \in N$. Therefore $\phi(N) \subset N$.

(iii) If $\phi(x) \in N$ then there is an n such that $\phi^n(\phi(x)) = \mathrm{id}$. Therefore $\phi^{n+1}(x) = \mathrm{id}$ so $x \in N$. This shows that $\phi^{-1}(N) \subset N$. The converse inclusion follows from (ii).

(iv) We shall write $\mathcal{R}(\phi)$ for the set of ϕ-conjugacy classes of elements of G. We shall now show that the map $x \to xN$ induces a bijection $\mathcal{R}(\phi) \to \mathcal{R}([\phi/N])$. Suppose $x, y \in G$ are ϕ-conjugate. Then there is a $g \in G$ with $gx = y\phi(g)$. Projecting to the quotient group G/N we have $gNxN = yN\phi(g)N$, so $gNxN = yN[\phi/N](gN)$. This means that xN and yN are $[\phi/N]$-conjugate in G/N.

Conversely suppose that xN and yN are $[\phi/N]$-conjugate in G/N. Then there is a $gN \in G/N$ such that $gNxN = yN[\phi/N](gN)$. In other words $(gx\phi(g)^{-1}y^{-1})^n = \mathrm{id}$. Therefore $\phi^n(g)\phi^n(x) = \phi^n(y)\phi^n(\phi(g))$.

This shows that $\phi^n(x)$ and $\phi^n(y)$ are ϕ-conjugate. However, by Lemma 1, x and $\phi^n(x)$ are ϕ-conjugate as are y and $\phi^n(y)$. Therefore x and y are ϕ-conjugate.

(v) We have shown that x and y are ϕ-conjugate if and only if xN and yN are ϕ/N-conjugate. From this it follows that $x \to xN$ induces a bijection from $\mathcal{R}(\phi)$ to $\mathcal{R}([\phi/N])$. Therefore $R(\phi) = R([\phi/N])$. □

THEOREM 14. *The Reidemeister number $R(\phi)$ is infinite if group G/N is Gromov hyperbolic, non-elementary,torsion free, freely indecomposable and ϕ is any endomorphism of G into itself.*

PROOF. The proof follows from Lemma 13 and Theorems 10 and 12. □

Corollary 15. *Let X to be a connected, compact polyhedron and $f: X \to X$ to be a continuous map. It is well known that the topological Reidemeister number $R(f) = R(f_*)$, where f_* is an induced endomorphism of the fundamental group of X. From Theorem* 4 *immediately follows that the topological Reidemeister number $R(f)$ is infinite for any homeomorphism f of a compact polyhedron X with a non-elementary, Gromov hyperbolic fundamental group.*

2.3. Reidemeister coincidence number. Let G be a finitely generated group and $\phi, \psi: G \to G$ two endomorphisms. Two elements $\alpha, \alpha' \in G$ are said to be (ϕ, ψ)-*conjugate* if and only if there exists $\gamma \in G$ with

$$\alpha' = \psi(\gamma)\alpha\phi(\gamma)^{-1}.$$

The number of (ϕ, ψ)-conjugacy classes is called the *Reidemeister coincidence number of endomorphisms ϕ and ψ*, denoted by $R(\phi, \psi)$. If ψ is the identity map then the (ϕ, id)-conjugacy classes are the ϕ-conjugacy classes in the group G. The Reidemeister coincidence number $R(\phi, \psi)$ has useful applications in Nielsen coincidence theory.

Lemma 16. *Let $\phi, \psi: G \to G$ are two automorphisms. Two elements x, y of G are $\psi^{-1}\phi$-conjugate if and only if elements $\psi(x)$ and $\psi(y)$ are (ψ, ϕ)-conjugate. Therefore the Reidemeister number $R(\psi^{-1}\phi)$ is equal to $R(\phi, \psi)$.*

Proof. If x and y are $\psi^{-1}\phi$-conjugate, then there is a $\gamma \in G$ such that $x = \gamma y \psi^{-1}\phi(\gamma^{-1})$. This implies $\psi(x) = \psi(\gamma)\psi(y)\phi(\gamma^{-1})$. So $\psi(x)$ and $\psi(y)$ are (ϕ, ψ)-conjugate. The converse statement follows if we move in opposite direction in previous implications. □

Theorem 17. *The Reidemeister number $R(\phi, \psi)$ is infinite if group G is Gromov hyperbolic, non-elementary, and ϕ, ψ are any automorphisms of G.*

Proof. From Theorem 10 it follows that the Reidemeister number $R(\psi^{-1}\phi)$ of an automorphism $\psi^{-1}\phi$ is infinite. The proof is therefore concluded by applying Lemma 16. □

Theorem 18. *The Reidemeister number $R(\phi, \psi)$ is infinite if group G is Gromov hyperbolic, non-elementary, torsion free, freely indecomposable and ϕ, ψ are any monomorphisms of G into itself.*

Proof. The proof follows from Lemma 11 and Theorems 12 and 13. □

Corollary 19. *Let $f, g: X \to X$ two homeomorphisms of compact, connected polyhedron X. The Reidemeister coincidence number of f and g, denoted by $R(f, g)$ is simply defined to be the Reidemeister number $R(\phi, \psi)$, where ϕ and ψ are induced fundamental groups automorphisms of f and g (see* [11]*). From Theorem* 17 *immediately follows that the Reidemeister number $R(f, g)$ is infinite if X has a non-elementary, Gromov hyperbolic fundamental group.*

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