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OBSTRUCTION THEORY AND MINIMAL NUMBER OF COINCIDENCES FOR MAPS FROM A COMPLEX INTO A MANIFOLD

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Abstract. The Nielsen coincidence theory is well understood for a pair of maps between n-dimensional compact manifolds for n greater than or equal to three. We consider coincidence theory of a pair $(f, g)\colon K \to \mathbb{N}^n$, where K is a finite simplicial complex of the same dimension as the manifold $\mathbb{N}^n$. We construct an algorithm to find the minimal number of coincidences in the homotopy class of the pair based on the obstruction to deform the pair to coincidence free. Some particular cases are analyzed including the one where the target is simply connected.

1. Introduction

Let K be a finite simplicial complex of dimension n and let $f, g\colon K \to \mathbb{N}^n$ be maps, where $\mathbb{N}^n$ is an n-dimensional manifold. The purpose of this work is to define a sharper invariant than the coincidence Nielsen number to study the minimal number of coincidences in the homotopy class of the pair (f, g). This invariant is based on algebraic and geometric features of the pair (f, g) and of the complex K. The case where K is a manifold has been treated by H. Shirmer, in [10], for K and $\mathbb{N}^n$ orientable, while the general case has been done by R. Dobreńko and J. Jezierski, in [3], and by D. L. Gonçalves in [6]. In [1], we study the case where K is the union of two subcomplexes K_1, K_2 each being a

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closed manifold and $K_i - K_1 \cap K_2$ being by-passing in K_i. On trying to drop the by-passing condition in this particular case, it became clear that the geometry of K was very relevant, and the difficulties to find a Nielsen type number to describe $MC[f,g]$, the minimal number of coincidences in the homotopy class of the pair (f,g), were basically the same as if we consider a general complex K of dimension n. So we treat in this work the general problem. The invariants defined here are homotopy invariants with respect to the pair (f,g) but are not invariants with respect to the homotopy type of K as one can see in examples in [6, Section 4].

The article is divided into four sections, besides this one. In Section 2 we show that it suffices to work with complexes that are homogeneous and with no $(n-1)$-simplices facing only one n-simplex. Section 3 is devoted to the definition of a homotopy invariant, in terms of the obstruction cocycles representing the obstruction class to deform the pair (f,g) to coincidence free. In Section 4 we show that this invariant coincides with the minimum number of coincidences, under mild conditions. Finally, in Section 5 we analyze two special cases. One of them is when the target $\mathbb{N}^n$ is simply connected. The other case is when K is a finite union of manifolds without boundary. In these examples we estimate the difference between the minimal number of coincidence points and the usual coincidence Nielsen number.

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2. The geometry of the complex K and the minimal number of coincidences

In this section we show that to solve the coincidence problem it suffices to consider homogeneous simplicial complexes K with the property that every $(n-1)$-simplex faces at least two n-simplices. Recall that an n-simplicial complex K is homogeneous if every maximal simplex has dimension n. We denote by $MC[f,g]$ the minimum number of coincidences in the homotopy class of the pair (f,g).

By [6, Proposition 2.7] we know that $MC[f,g] = MC[f',g']$, for given $f,g\colon K \to \mathbb{N}^n$, where f',g' are the restrictions of f, g to the subcomplex $K\langle n\rangle \subset K$, where $K\langle n\rangle$ is the smallest subcomplex which contains all n-simplices of K. So we assume that K is homogeneous and consider the n-simplices containing a $(n-1)$-face which is not contained in another n-simplex. Define $\widehat{K} \subset K$ *the subcomplex of* K obtained from K by removing these n-simplices as well as its $(n-1)$-faces which are not contained in another n-simplex.

Lemma 2.1. *Given* $f, g\colon K \to \mathbb{N}^n$ *then* $MC[f,g] = MC[\widehat{f},\widehat{g}]$, *where* $\widehat{f}$, $\widehat{g}$ *are the restrictions to* $\widehat{K}$ *of* f, g, *respectively.*

Proof. Clearly $MC[f,g] \geq MC[\widehat{f},\widehat{g}]$. To show the converse let K' be a subcomplex obtained from K by removing one n-simplex, Δ^n, together with one of its $(n-1)$ faces, Δ_0^{n-1}, which faces only this n-simplex. We will show that $MC[f',g'] \geq MC[f,g]$ where f', g' are the restrictions to K' of f and g, respectively. Since $MC[f',g']$ is finite let us consider a pair of maps (f',g') such that $\operatorname{coin}(f',g')$ is finite. We will construct a pair of maps (f_1,g_1) in the homotopy class of the pair (f,g) such that its restriction to K' is homotopic to (f',g') and such that $\operatorname{coin}(f_1,g_1) = \operatorname{coin}(f',g')$. Observe that this implies the result for $\widehat{K}$ because $\widehat{K} = K'\langle n\rangle$.

The finite set $\operatorname{coin}(f',g') \cap \Delta^n$ is in the complement of the interior of the $(n-1)$-face Δ_0^{n-1} in the boundary of Δ^n. Each coincidence point x_i belongs to the interior of a k-simplex, $\{v_{i_1},\dots,v_{i_k}\}$, for $k \leq n-1$.

For each point x_i in $\operatorname{coin}(f',g')$ we consider a small n-simplex Δ_i^n, with faces parallel to the faces of Δ^n and containing x_i as an interior point of the k-face which is parallel to $\{v_{i_1},\dots,v_{i_k}\}$. Assume this simplex is small enough, so that its image by f and g lie in a coordinate neighbourhood of $f'(x_i) = g'(x_i) = y_i$.

We divide the points of $\operatorname{coin}(f',g') \cap \Delta^n$ in two types according to whether the simplex $\{v_{i_1},\dots,v_{i_k}\}$ is or is not contained in Δ_0^{n-1}.

Let x_i be of first type. Then necessarily x_i is a point in the interior of a k-simplex for $k < n-1$. Let $a_0,\dots,a_n$ be the vertices of Δ_i^n and suppose $\langle a_0,\dots,a_{n-1}\rangle$ is the $(n-1)$-face of Δ_i^n parallel to Δ_0^{n-1} and that x_i belongs to the interior of $\langle a_0,\dots,a_k\rangle$. In this case the functions f', g' are defined in the star of $\langle a_0,\dots,a_k,a_n\rangle$ as a subcomplex of the boundary of Δ_i^n. Now we construct a retraction $r\colon \Delta_i^n \to \Delta_i^n$ on the star of $\langle a_0,\dots,a_k,a_n\rangle$ as a subcomplex of $\partial\Delta_i^n$. Introducing a new vertex b in the interior of the $(n-k-2)$-face $\langle a_{k+1},\dots,a_{n-1}\rangle$ we can define r as the simplicial map given by $r(a_i) = a_i$ for $i = 0,\dots,n$ and $r(b) = a_n$.

Let x_i be of the second type and consider x_i in Δ_i^n. Call Δ_i^k the k-face of Δ_i^n having x_i in its interior, and let Δ_i^{n-k-1} be the face determined by the vertices in Δ_i^n which are not in Δ_i^k. Observe that the star of Δ_i^k and of Δ_i^{n-k-1} as subcomplexes of $\partial\Delta_i^n$ are homeomorphic to $(n-1)$- discs, their union is the hole $\partial\Delta_i^n$, and their intersection is the boundary of each, which is homeomorphic to an $(n-2)$-sphere. We assumed Δ_i^n small enough, so that the images of the star of Δ_i^k as a subcomplex of Δ_i^n by f' and g' lie in a coordinate neighbourhood of $f'(x_i) = g'(x_i) = y_i$. After composing the maps with a chart, we take $f' - g'$ restrict to the boundary of the star of Δ_i^k, which can be viewed as a map from the $(n-2)$-sphere to $R^n - \{0\}$. Since this map is homotopic to a constant, we can extend it to the star of Δ_i^{n-k-1} as a subcomplex of $\partial\Delta_i^n$, without introducing coincidences. We need now to extend it to the interior of Δ_i^n. For this take a homeomorphism of the n-disk to Δ_i^n that takes boundary to boundary, and the south pole of the boundary of the disc to x_i. Extend now, to the interior of the n-disk, the composition of $f' - g'$ with the homeomorphism above, linearly in each segment from the south pole to a point in the boundary.

We succeed extending the maps f', g' to $K' \cup \Delta_1^n \cup \ldots \cup \Delta_s^n$ without introducing new coincidence points. In order to extend it to the hole Δ^n, we observe first that there is a well known retraction from Δ^n onto its $(n-1)$-faces, except the interior of one. Denote by $st_L(\Delta^k)$ $(\mathrm{ost}_L(\Delta^k))$ the star (the open star) of the simplex Δ^k as a subcomplex of the complex L. It suffices to define a homeomorphism of pairs from

$$\bigg(\Delta^n - \bigcup_i \mathrm{ost}_{\Delta_i^n}(\Delta_i^k), \bigg(\partial\Delta^n - \bigg(\mathrm{int}\,\Delta_0^{n-1} \bigcup_i \mathrm{ost}_{\partial\Delta_i^n}(\Delta_i^k)\bigg) \bigcup_i \mathrm{st}_{\partial\Delta_i^n}(\Delta_i^{n-k-1})\bigg) \\ \to (\Delta^n, \partial\Delta^n - \mathrm{int}\,\Delta_0^{n-1}).$$

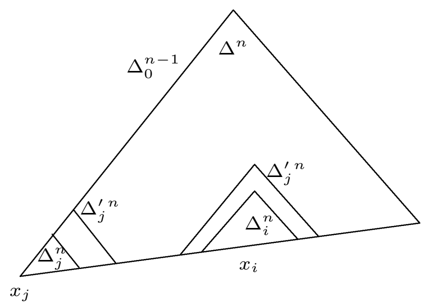

Figure 1

For this consider closed neighbourhoods $\Delta_i'^n$ of Δ_i^n respecting the parallelism with Δ^n, see Figure 1.

Define the homeomorphism to be the identity in $\Delta^n - \bigcup_i \mathrm{ost}_{\Delta_i'^n}(\Delta_i'^k)$ and sending the part of the face Δ_0^{n-1} in $\Delta^n - \bigcup_i \mathrm{ost}_{\Delta_i^n}(\Delta_i^k)$ homeomorphically onto Δ_0^{n-1}. □

Given an n-dimensional simplicial complex K we can iterate the following two operations: the first is to consider the homogeneous subcomplex $K\langle n\rangle \subset K$ of dimension n (see [6]) and the second is the operation, defined in the beginning of the section, which consists of eliminating the n-simplices containing faces that do not face any other n-simplex, together with these faces. This process of iterating the two operations will stop after a finite number of steps, and we make the following

Definition 2.2. For an n-dimensional simplicial complex K, we define the *soul* of K, denoted by $s(K)$, to be the subcomplex obtained at the end of the process indicated above.

Observe that it does not matter in which order we perform these operations on a simplicial complex K. At some moment the process will become stable and the resulting subcomplex will be the soul of K. To see this, denote by $\theta_1(K)$ ($\theta_2(K)$) the subcomplex obtained from K after applying the first (second) operation. It is not hard to see that these operations preserve inclusions, that is, if L is a subcomplex of M, then $\theta_i(L)$ is a subcomplex of $\theta_i(M)$, $i = 1, 2$. Therefore, $\theta_1(\theta_2(K)) \subset \theta_1(K)$ and since $(\theta_1(\theta_2(K)) \subset \theta_2(K)$, we have that any subcomplex $\theta_1^{k_1}\theta_2^{l_1}\dots\theta_1^{k_r}\theta_2^{l_r}(K)$ will contain a subcomplex of the form $(\theta_1\theta_2)^m(K)$.

Now consider an integer ℓ so that $(\theta_1\theta_2)^\ell(K) = (\theta_1\theta_2)^{\ell+1}(K)$ and any other subcomplex, $\theta_1^{k_1}\theta_2^{l_1}\dots\theta_1^{k_r}\theta_2^{l_r}(K)$, obtained from a process that has become stable. We have

$$(\theta_1\theta_2)^m(K) \subset \theta_1^{k_1}\theta_2^{l_1}\dots\theta_1^{k_r}\theta_2^{l_r}(K) \subset K.$$

Applying $(\theta_1\theta_2)^\ell$ to these inclusions we obtain:

$$(\theta_1\theta_2)^{\ell+m}(K) \subset \theta_1^{k_1}\theta_2^{l_1}\dots\theta_1^{k_r}\theta_2^{l_r}(K) \subset (\theta_1\theta_2)^\ell(K).$$

Therefore we have $\theta_1^{k_1}\theta_2^{l_1}\dots\theta_1^{k_r}\theta_2^{l_r}(K) = (\theta_1\theta_2)^\ell(K)$.

Proposition 2.3. *Given $f, g\colon K \to \mathbb{N}^n$ then $MC[f,g] = MC[f',g']$ where f' and g' are the restriction of the f and g, respectively, to $s(K)$.*

Proof. The proof follows from Lemma 2.1 and [6, Proposition 2.7]. □

The figure below shows a complex and its soul.

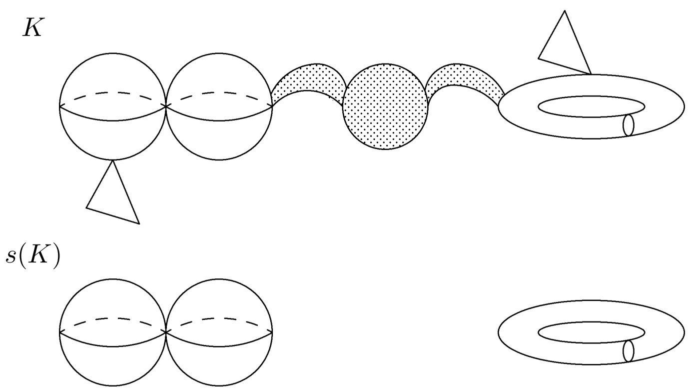

Figure 2

Remark 2.4. A typical example of an n-complex which coincides with its soul is the union of n-manifolds without boundary. The converse is not true, though. To see this consider three disjoint spheres joint to a 2-simplex so that each of its faces belongs to one of the spheres.

3. Local coincidence index and the number $NO(f,g;K)$

In this section we define a homotopy invariant which will coincide, under mild conditions, with the minimal number of coincidences in the homotopy class of the pair $f,g\colon K \to \mathbb{N}^n$. This invariant is constructed in terms of the primary obstruction to deform a pair of maps to coincidence free as well as in terms of the geometry of the complex K. We will start by reviewing the notion of local index as formulated by E. Fadell and S. Husseini in [5] where we adapted the terminology to the coincidence case.

Let U be an open set of K and $(f,g)\colon U \to \mathbb{N}^n$ be a pair of maps where the set of coincidence points is compact.

As in [5], we consider the diagonal $\triangle$ in $\mathbb{N}^n \times \mathbb{N}^n$ and replace the inclusion $\mathbb{N}^n \times \mathbb{N}^n - \triangle \hookrightarrow \mathbb{N}^n \times \mathbb{N}^n$ by a fiber map $p\colon E \to \mathbb{N}^n \times \mathbb{N}^n$, where

$$E = \{(\alpha,\beta) \mid \alpha,\beta : [0,1] \to \mathbb{N}^n,\ \alpha(0) \neq \beta(0)\},$$

and $p(\alpha,\beta) = (\alpha(1),\beta(1))$. For $b = (x,y)$ in $\mathbb{N}^n \times \mathbb{N}^n$ and $F_b = p^{-1}(b)$, $\pi_{n-1}(F_b)$ is a local system of coefficients on $\mathbb{N}^n \times \mathbb{N}^n$. There is an isomorphism of local systems on $\mathbb{N}^n \times \mathbb{N}^n$

$$\zeta\colon \pi_{n-1}(F_b,b) \to \mathcal{Z}[\pi],$$

where $\pi = \pi_1(\mathbb{N}^n,x)$ and the action of $\pi \times \pi$ on $\mathcal{Z}[\pi]$ is given by

$$\alpha \cdot (\sigma,\tau) = sgn\sigma\sigma^{-1} \cdot \alpha \cdot \tau$$

Here, since σ is an element of the fundamental group of a manifold, $sgn\sigma$ is ± 1, according to whether it preserves or reverses local orientation. We wil refer to this system as $\mathcal{B}$.

Let the local system on U be the one induced from $\mathcal{B}$ by $f \times g\colon U \to \mathbb{N}^n \times \mathbb{N}^n$ and denote it by $\mathcal{B}(f \times g)$. Consider the fiber space $E(f,g)$ obtained by pulling back $p\colon E \to \mathbb{N}^n \times \mathbb{N}^n$ over U by $f \times g$.

The obstruction to deform the pair (f,g) to a coincidence free pair is related to the obstruction to extend sections of the fiber map $E(f,g) \to U$.

Following the steps in [5] and making the usual adaptations to the coincidence case, we end up with:

Definition 3.1. For an open set U of K, the *coincidence index of* $(f,g)\colon U \to \mathbb{N}^n$ is the cohomology class $i(f,g)$ in $H^n_c(U;\mathcal{B}(f \times g))$ given by the obstruction to deform (f,g), by a compact homotopy, to a coincidence free pair. In case U coincides with K, we denote this class by $O^n(f,g)$ and, since K is compact, it lies in $H^n(K,Z[\pi])$.

Consider now F an isolated set of coincidences of (f,g) and let V be an open set of U such that $F = V \cap \operatorname{coin}(f,g)$. Consider the composition

$$H^n(V,V-F;\mathcal{B}(f \times g)) \xrightarrow{j^{*-1}} H^n(U,U-F;\mathcal{B}(f \times g)) \xrightarrow{k^*} H^n_c(U;\mathcal{B}(f \times g))$$

where the first arrow is the inverse of the excision isomorphism and the second is the composition of the homomorphism, induced by the inclusion,

$$H^n(U,U-F;\mathcal{B}(f\times g)) \xrightarrow{i^*} H^n(U,U-\operatorname{coin}(f,g);\mathcal{B}(f\times g)),$$

with the natural homomorphism to $H^n_c(U;\mathcal{B}(f\times g))$.

Recall that $H^n_c(U;\mathcal{B}(f\times g))$ is the direct limit of $H^n(U,U-C;\mathcal{B}(f\times g))$, taken over all compact subsets C of U.

DEFINITION 3.2. The *local coincidence index of* F, denoted by $i(f,g;F)$, is the element in $H^n_c(U;\mathcal{B}(f\times g))$ given by $k^*(j^*)^{-1}(\alpha)$, where $\alpha\in H^n(V,V-F;\mathcal{B}(f\times g))$ corresponds to the coincidence index of $(f,g)\colon V\to\mathbb{N}^n$.

Let us consider the group $H^n(K,A)$, the n-th simplicial cohomology group of K with local coefficients, where A is a free abelian group and identified with the direct sum of $Z's$ indexed by some set J. We call a cochain $c_n\in C^n(K,A)$ elementary if c_n is nonzero in only one n-simplex, called its support, and has value in one summand Z of A indexed by $j\in J$. So we can associate to each elementary cochain a pair (Δ^n,j), where Δ^n is its support and j is the index of the summand $Z\subset A$ where the cochain assumes its value. Two elementary cochains are *disjoint* if the pairs (Δ^n,j), (Δ'^n,j') are not equal. Given an arbitrary cocycle (or cochain) $c_n\in C^n(K,A)$ we define an integer, $\ell(c_n)$, as follows: The cocycle c_n can be uniquely written as a sum of disjoint elementary cocycles i.e. $c_n = c_{n,1}+\ldots+c_{n,r}$, where each $c_{n,i}$ is elementary.

DEFINITION 3.3. A cocycle is essential if it represents a nonzero cohomology class.

DEFINITION 3.4. A partial sum $c_{n,i_1}+\ldots+c_{n,i_s}$ of the decomposition of c_n is said to be *combinable* if the intersection of the supports of all elementary summands is nonempty and they have values in the same summand Z of A. Define $\ell(c_n)$ to be the minimal number of combinable partial summands among all decompositions of c_n.

Now let $f,g\colon K\to\mathbb{N}^n$ be maps.

DEFINITION 3.5. The number $NO(f,g;K)$ is defined as the minimum of the numbers $\ell(c_n)$, where c_n runs over the set of all cocycles representing the obstruction $O^n(f,g)\in H^n(K,Z[\pi])$ to deform (f,g) to coincidence free.

THEOREM 3.6. *$NO(f,g;K)$ is a homotopy invariant.*

PROOF. The result follows from the fact that $O^n(f,g)=O^n(f_1,g_1)$, for (f_1,g_1) homotopic to (f,g). $\square$

4. The minimal number of coincidences and the realization of the number $NO(f,g;K)$

We will now prove that the number $NO(f,g;K)$ coincides with the minimal number of coincidences in the homotopy class of the pair (f,g). The tecnicques applied are based on works by H. Schirmer ([10]), X. Zhao ([12]), L. D. Borsari and D. L. Gonçalves ([1]) and D. L. Gonçalves ([6]). From what we have seen before, we may assume that K coincides with its soul.

We will define a decomposition of K in terms of a simplicial structure of K, although it can be shown that this decomposition does not depend on the particular simplicial structure. For each maximal simplex Δ^n let $C(\Delta^n)$ be the smallest subcomplex which contains all n-simplices Δ'^n such that there is a sequence of n-simplices starting at Δ^n and ending at Δ'^n so that the intersection of two consecutive ones is a $(n-1)$-simplex which faces only these two n-simplices. This defines a *covering* of K by homogeneous simplicial subcomplexes of dimension n which we denote by $\{K_1,\dots,K_r\}$. These subcomplexes happen to be, in many situations, manifolds but not necessarily. Take for example, K to be the n-sphere with its poles identified. Associated to this covering we have the subcomplex $K_0=\bigcup_{i\neq j} K_i\cap K_j$. Observe that the points of K_0 are characterized by the property that they are not locally Euclidean in K.

Theorem 4.1. *Let $(f,g)\colon K\to \mathbb{N}^n$ be a pair of maps where K and $\mathbb{N}^n$ have dimension bigger than or equal to three. Assume every component of K_0 and of all intersections of any number of K_i are of non-zero dimension. Then the minimum number of coincidences in the homotopy class of the pair (f,g) is given by $NO(f,g;K)$.*

Proof. The process of deforming the pair (f,g) to (f_1,g_1) having all coincidences lying in the interior of n-simplices is based on [10] and it guarantees that the cocycle associated to this new pair, $c_n=c_n(f_1,g_1)$, satisfies $\ell(c_n)\leq \operatorname{coin}(f,g)$. Therefore $NO(f,g;K)\leq \operatorname{coin}(f,g)$, and being a homotopy invariant, it becomes a lower bound for the minimum number of coincidences in the homotopy class of the pair (f,g). It remains to prove that $NO(f,g;K)$ can be realized and this is done in what follows.

Let c_n be a n-cocycle representing the obstruction to deform (f,g) to coincidence free and such that $\ell(c_n)=NO(f,g;K)$. Consider (f',g') a pair homotopic to (f,g) so that $c_n(f',g')=c_n$. This means that (f',g') has coincidences appearing in the simplices that are support for each elementary cocycle in the decomposition of c_n. Each combinable partial sum of c_n will correspond to a set of n-simplices having non-empty intersection.

In fact, we may also assume that there will be only one coincidence in each of those n-simplices. To see this, consider the group $\pi_n(M\times M, M\times M-\Delta)$ which is isomorphic to $Z[\pi]$ as in [4, p. 62]. Take $\alpha \in \pi_n(M \times M, M \times M - \Delta)$ of the form $l.1_\beta$ where $\beta \in \pi_1(M)$ and 1_β is a generator of the copy of Z indexed by β. We will show that α can be represented by a pair of maps $(\overline{f}, \overline{g})\colon (\Delta^n, \partial\Delta^n) \to (M\times M, M\times M-\Delta)$ with only one coincidence point. To see this, we perform the definition of the action of β in one element. Namely, regard Δ^n as the unit ball and consider its boundary, an $(n-1)$- sphere, as the quotient of the disk D^{n-1} by its boundary. We define the map in D^{n-1} as a composition of a map into $\widetilde{M}$, the universal covering of M, with the projection $\widetilde{M} \to M$. Let $\widetilde{g}\colon D^{n-1} \to \widetilde{M}$ be the map sending the sphere of radius r into the point $\widetilde{\beta}(2-2r)$ as r runs from 1 to 1/2, where $\widetilde{\beta}$ is a lifting of the loop β. The ball of radius 1/2 is mapped into the $(n-1)$- sphere of radius ε around the point $\widetilde{y}_1$ in the pre-image of y, as a map of degree l. Now, extend the map $\widetilde{g}$ to the interior of D^n by sending the origin to $\widetilde{y}_1$, and each segment from the origin to a point x in the sphere will perform either a radial segment from $\widetilde{y}_1$ to $g(x)$, or a radial segment from $\widetilde{y}_1$ to $\widetilde{\beta}(1)$ followed by the part of the path $\widetilde{\beta}^{-1}$ ending at $g(x)$. Now given the pair $(f|_{\Delta^n}, g|_{\Delta^n})\colon (\Delta^n, \partial\Delta^n) \to (M\times M, M\times M - \Delta)$ it represents, by hypothesis, an element of the form $l.1_\beta$. By the homotopy sequence of the pair, there is a homotopy H between the pairs $(f|_{\partial\Delta^n}, g|_{\partial\Delta^n})\colon (\partial\Delta^n) \to M\times M - \Delta$ and $(\overline{f}|_{\partial\Delta^n}, \overline{g}|_{\partial\Delta^n})\colon (\partial\Delta^N) \to M\times M - \Delta$. We are ready to extend (f', g') to the interior of Δ^n with only one coincidence. First, we identify the simplex Δ^n with the unit ball of dimension n. In the annulus with radius varying from 1 to 1/2 we define the map as the homotopy H. In the ball of radius 1/2 we make use of the model constructed above, having only one coincidence.

Consider the subspace given by the union of the n-simplices appearing in a combinable partial sum of c_n. It is contractible and therefore the local system induced over it is trivial. For each coincidence point a_i, lying in the interior of one of the i-th maximal simplices consider the segment, denoted by α_i, from a_i to a. Take two points, namely a_1 and a_i. The local index at the point a_i represents an element which is an integer multiplied by the generator of Z at the local group indexed by the element 1. The assumption that the index lies in the same summand (with respect to the trivialization of the bundle over the subspace) is the same as saying that if we transport the local index at $(f(a_1), g(a_1))$ to the point $(f(a_i), g(a_i))$ along the path $(f(\alpha_1\alpha_i^{-1}), g(\alpha_1\alpha_i^{-1}))$, we get an element of the summand Z indexed by the neutral element. Now, by Proposition 3.6 in [4] it follows that $f(\alpha_1\alpha_i^{-1}) \cong g(\alpha_1\alpha_i^{-1})$, for all i. These conditions, together with the tecnicques developed in [1], allow us to deform the pair (f', g') so that all these coincidences coalesce to a. Repeating this procedure to all others combinable sets we end up with $\ell(c_n) = NO(f, g; K)$ coincidence points. $\square$

Remark 4.2. In the case where some, if not all, components of K_0 or some components of the intersections of a certain number of $K_i's$ have zero dimension, it could happen that two or more combinable partial sums have the intersection of their supports being only one point. In this case, only one set of coincidences, arising from the combinable partial sums, would be joint to this point. Therefore, we would have to add to the number $\ell(c_n)$ the number of elements of all, except the biggest, combinable partial sums for which the intersection of supports is the same single point. Then, the minimum of these numbers, as c_n runs through all possible cocycles representing the obstruction class, will give us the minimum number of coincidences in the homotopy class of the pair (f, g).

As an application of the above result, let $K' \subset K$ be any subcomplex such that the homomorphism $i^*: H^n(K, Z[\pi]) \to H^n(K', Z[\pi])$, induced by the inclusion map, is a cohomology isomorphism with local coefficients where $\pi = \pi_1(\mathbb{N}^n)$. Observe that if two subcomplexes have this property then their intersection does too. Hence, we may always consider the minimal one, namely, the intersection of all subcomplexes satisfying the above condition.

Theorem 4.3. *Given $f, g: K \to \mathbb{N}^n$ then $MC[f, g] = MC[f', g']$, where f', g' are the restrictions of f, g, respectively, to K'.*

Proof. Given (f, g) consider its restriction (f', g') to K', and let (f'', g'') a pair of maps in K' homotopic to (f', g'). Take any cocycle c_n representing the obstruction $O^n(f', g')$. Since K' reflects all cohomology of K, this cocycle also represents the obstruction $O^n(f, g)$. Therefore there exists (f_1, g_1) homotopic to (f, g) such that $c^n(f_1, g_1)$ is the cocycle c_n . Observe that any combinable partial sum of c_n in K', is also combinable in K. Therefore, from Theorem 4.1, it follows that $MC[f, g] \leq MC[f', g']$. The other inequality is clear and the result follows. □

5. Some special cases

Before analyzing the special cases, let us observe that in the context we are working, we do not expect the minimum number of coincidences to coincide with the Nielsen number. This can be seen in [1] even in the case where the target is simply connected. The coincidence Nielsen number can be defined as in [2] or in [6], since, in our context, the given definitions are equivalent.

In the fixed point case, in dimension two, it is well known that the *Wecken property does not hold*, i.e. the minimal number of fixed points does not coincide with the Nielsen number. It was observed in [8] that for maps $f: P \to P$, where P is the pantalon, the disk with two holes, the difference $MC[f] - N(f)$ can become arbitrarily large as we vary over the homotopy classes of self-maps on P. Inspired on these facts we set

Definition 5.1. Let $W\{K, \mathbb{N}^n\}$ be the maximum of all $MC[f, g] - N(f, g)$, where $[f, g]$ runs over all homotopy classes of pairs of maps from K to $\mathbb{N}^n$.

It is well known that this number is zero in the fixed point case when the complex K has no local cut points, it is not a surface, and it has dimension greater than or equal to two, see [2] . This is also the case in the coincidence context, where the spaces are orientable manifolds of dimension greater than or equal to three, see [10]. In our context, we will be seeing that $W\{K,\mathbb{N}^n\}$ can be either finite or infinity.

5.1. The case where $\mathbb{N}^n$ is simply connected. Let $f,g\colon K \to \mathbb{N}^n$ be a pair of maps. Since $\mathbb{N}^n$ is simply connected we have only one Nielsen class. We will show that in general the number $MC[f,g]$ is bounded for a fixed K but it goes to infinity as we vary K. Many examples can be construct having $MC[f,g] > 1$, and a upper bound for the maximum of $MC[f,g]$, among all pairs (f,g), is given.

Let $C = \{K_{i_1},\dots,K_{i_r}\}$ be the covering of K defined in the previous section, and assume that all components of K_0 have nonzero dimension.

Definition 5.2. A subset $\{K_{i_1},\dots,K_{i_s}\}$ of the covering $C = \{K_1,\dots,K_r\}$ is called admissible if the intersection $K_{i_1}\cap\dots\cap K_{i_r} \neq \emptyset$. Let $\ell(C)$ be the minimal number of admissible subsets which cover C. For the purpose of computing $\ell(C)$ we can assume, without loss of generality, that the admissible sets are maximal in the sense that for any $K_j \neq K_{i_t}$, $t = 1,\dots,r$, we have $K_j\cap K_{i_1}\cap\dots\cap K_{i_r} = \emptyset$.

Proposition 5.3. *Given $f,g\colon K \to \mathbb{N}^n$ then $MC[f,g] \leq \ell(C)$. In particular $W\{K,\mathbb{N}^n\}$ is finite.*

Proof. Let c^n be any cocycle representing the obstruction to deform the pair (f,g) to coincidence free. For each admissible covering of K we obtain a decomposition of c^n into combinable partial sums. The number of elements in this decomposition is less than or equal to the number of elements of the covering. So it follows that $MC[f,g] \leq \ell(C)$. □

Remark 5.4. Observe that in this context, where the target is simply connected, we may replace the complex K by any subcomplex K' of K so that the inclusion $i\colon K' \to K$ induces an isomorphism of the n-th cohomology group with coefficients in Z, not twisted. The minimal of these complexes has been considered in [6]. Therefore $W\{K,\mathbb{N}^n\}$ can be computed as $W\{K',\mathbb{N}^n\}$, where K' varies over a larger family of subcomplexes of K, than the family considered in Section 4.

5.2. Examples. Consider the simplicial complex K obtained from a collection of six tori joint by tubes. Observe that $K = K_1 \cup \dots \cup K_{21}$, the tori are $K_1 \cup K_2$, $K_3 \cup K_4,\dots,K_{11} \cup K_{12}$, and the tubes $K_{13},\dots,K_{21}$ either have empty intersection or intercept in a boundary circle, see Figure 3 for an immersed

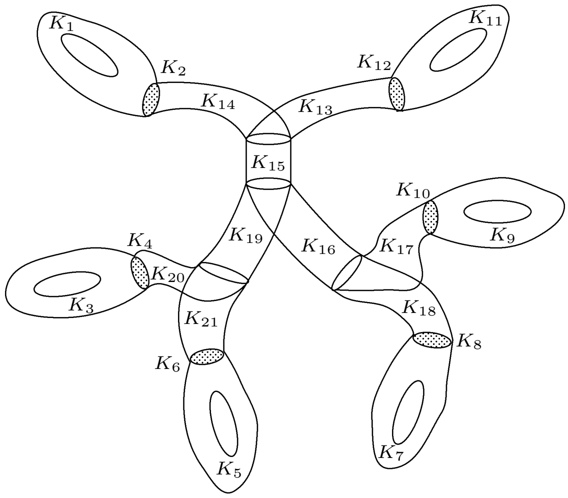

Figure 3

model of K. Notice also that the decomposition of K as the union of the K_i, $i = 1, \dots, 21$, is the one defined in the preceeding section.

It is not hard to see that for $C = \{K_1, \dots K_{21}\}$, the minimal number of admissible subsets that cover C, $\ell(C)$, is 7 and this covering is realized by the subsets:

$$\{K_1, K_2, K_{14}\}, \quad \{K_3, K_4, K_{20}\}, \quad \{K_5, K_6, K_{21}\}, \quad \{K_7, K_8, K_{18}\},$$
$$\{K_9, K_{10}, K_{17}\}, \quad \{K_{11}, K_{12}, K_{13}\}, \quad \{K_{15}, K_{16}, K_{19}\}.$$

Consider now a pair of maps $(f, g): K \to S^2$. Since S^2 is simply connected, we know that the obstruction to deform (f, g) to coincidence free can be represented by a sum of elementary cocycles so that no two of them have supports in the same K_i. It is not hard to see that with techniques developed by H. Schirmer in [10] and in [1] we may, by adding suitable coboundaries to this cocycle, assume that the cocycle is composed by elementary ones with supports lying in some, if not all, of the complexes $K_1, \dots, K_{12}$. Therefore (f, g) can be made homotopic to a pair with at most 12 coincidences lying in different $K_i's$, $i = 1, \dots 12$. Since any two coincidences lying in the tori $K_i \cup K_{i+1}$, $i = 1, 3, 5, 7, 9, 11$ can be joint to one, we end up with $MC[f, g] \leq 6$. It is also clear that we can construct a pair (f, g) such that $MC[f, g]$ is, in fact, 6.

We have therefore an example where

$$MC[f, g] \leq 6 < 7 = \ell(C) \quad \text{and} \quad W\{K, \mathbb{N}^n\} = 6.$$

Let us represent the complex K by the graph in Figure 4. So each tube is represented by a segment and each torus by a circle.

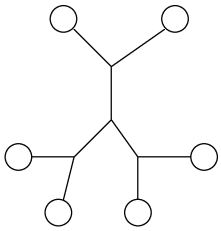

Figure 4

With this in mind, we can produce more examples as represented in Figure 5.

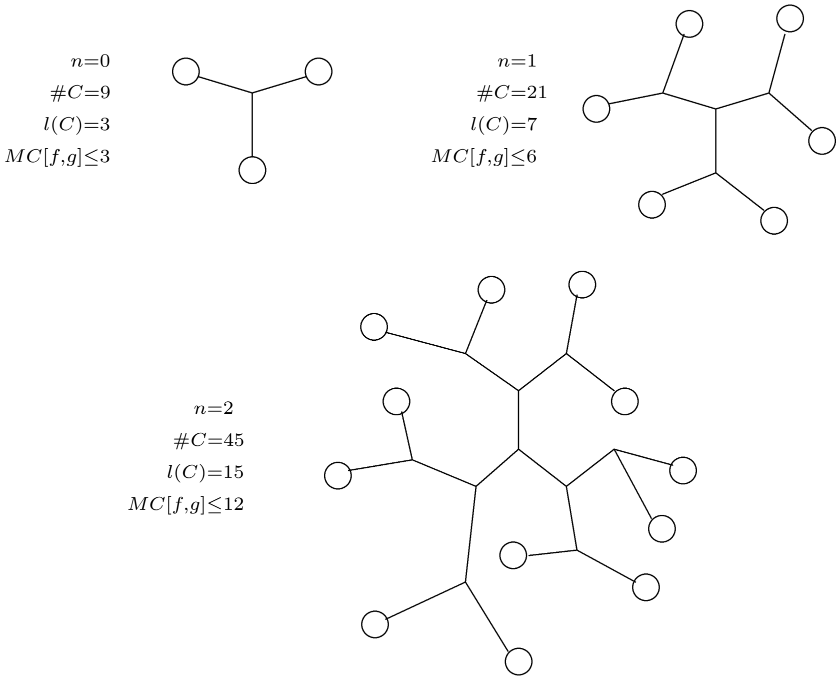

Figure 5

In general we have, for the n-th step, the following: cardinality of C is $3.2^0 + \ldots + 3.2^n$,

$$\ell(C) = \begin{cases} 3.2^n + 3.2^{n-2} + \ldots + 3.2^2 + 3.2^0 & \text{for } n \text{ even,} \\ 3.2^n + 3.2^{n-2} + \ldots + 3.2^1 + 1 & \text{for } n \text{ odd.} \end{cases}$$

$MC[f,g] \leq 3.2^n$, for all $(f,g)\colon K \to S^2$, and we have examples of pairs of maps for which $MC[f,g] = 3.2^n$.

Hence, we have that $W\{K,\mathbb{N}^n\} = 3.2^n - 1$ which is strictly less than $\ell(C)$. Also, as K varies with n, both $\ell(C)$ and $\ell(C) - W\{K,\mathbb{N}^n\}$ go to infinity.

5.4. The case where K is a union of closed manifolds. Let us consider the minimizing problem for $(f,g)\colon K \to \mathbb{N}^n$, where K is a union of n-dimensional manifolds $M_1, \dots, M_r$ with no boundary. We can always assume that this covering has no proper subcovering. We make the assumption that $M_0 = \bigcup_{i\neq j} M_i \cap M_j$ is by-passing in each M_i. This notion of by-passing does not depend on the decomposition of K as a union of closed manifolds. The results obtained here are a generalization of what we have done in [1].

For a pair of maps $(f,g)\colon K \to \mathbb{N}^n$, consider $(f_i,g_i)\colon M_i \to \mathbb{N}^n$ be the restriction of (f,g) to M_i. For each (f_i,g_i) we consider the usual essential Nielsen classes.

Let $F_1, \dots, F_r$ be a set of essential Nielsen classes so that each F_i belongs to a different manifold M_j. We say that this set is combinable if the following hold:

(a) The intersection of the manifolds M_j, to where the classes F_i belong to, is non-empty.

(b) There is a point a in the intersection mentioned above, and paths α_i in M_i from any point in F_i to a so that $f(\alpha_1 * \alpha_i^{-1}) \cong g(\alpha_1 * \alpha_i^{-1})$, see Figure 6.

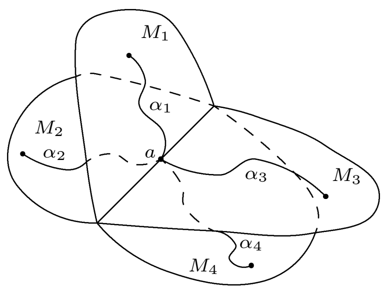

Figure 6

This relation enable us to divide the set of Nielsen classes of $(f_i,g_i)\colon M_i \to \mathbb{N}^n$, for $i = 1, \dots, r$ into combinable subsets. These can be done in various manners. We are interested in doing it in a way that we end up with the least number of combinable subsets. Therefore we consider all possible coverings of the set of Nielsen classes by combinable subsets and we set

Definition 5.5. The number $N(f,g;K)$ corresponds to the smallest cardinality of combinable subsets among all possible coverings.

Theorem 5.6. *Let $(f,g)\colon K \to \mathbb{N}^n$ be a pair of maps, where K is a union of n-dimensional closed manifolds $M_1, \dots, M_r$. Let K_0 be the union of $M_i \cap M_j$, $i \neq j$, and assume all of its components, as well as all components of the intersections* *of any number of $M_i's$, are of positive dimension. Suppose $M_i \cap M_j$ is properly contained in both M_i and M_j and $K_0 \cap M_i$ is by-passing in M_i, for all i. Then $N(f, g; K)$ is the minimum number of coincidences in the homotopy class of the pair (f, g) and therefore $N(f, g; K) = NO(f, g; K)$.*

PROOF. Observe that the subcomplexes in the decomposition of K, defined in the beginning of Section 4, are exactly the manifolds M_i. Let c_n be an n-cocycle representing the obstruction to deform (f, g) to coincidence free and such that $\ell(c_n) = NO(f, g; K)$. Consider (f', g') a pair homotopic to (f, g) so that $c_n(f', g') = c_n$. As before, this means that (f', g') has coincidences appearing in the simplices that are support for each elementary cocycle in the decomposition of c_n. Each combinable partial sum of c_n will correspond to a set of n-simplices having non-empty intersection, each of them belonging to some M_i and containing only one coincidence. It follows from the proof of Theorem 4.1 that this set of coincidence points lying in these n-simplices will be combinable. Hence $N(f, g; K) \leq NO(f, g; K)$ and therefore it suffices to prove that the number $N(f, g; K)$ can be realized. The same tecnicques developed in [1] to join coincidence points in two different manifolds can be applied step by step in case more manifolds are involved, provided the by-passing condition is imposed. □

REMARK 5.7. If some of the above mentioned components have zero dimension, to obtain the minimum we should add to $N(f, g; K)$ the number of coincidences lying in each combinable subset of Nielsen classes, except for the biggest, having the intersection of the manifolds to where their Nielsen classes belong to, being the same one point.

Finally, we exhibit an example where $W\{K, \mathbb{N}^n\}$ is infinity. Let K be the union of two tori $T^n = (S^1)^{n-1} \times S^1$ glued by the subcomplex $(S^1)^{n-1} \times \{1\}$, and $\mathbb{N}^n = T^n$. Consider the sequence of pairs of maps (f_m, g_m), where g_m is the constant map for all m, and f_m is defined by its restrictions, f_{m_1} and f_{m_2}, to each torus, where f_{m_1} is given by $f_{m_1}(z_1, \ldots, z_n) = (z_1^m, \ldots, z_n)$, and f_{m_2} is the identity map for all m. By Theorem 4.1 in [1] we have that $MC[f_m, g_m] = m$ and $N(f_m, g_m) = 1$. Therefore $W\{K, \mathbb{N}^n\} = \infty$

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