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SOME PAIRS OF MANIFOLDS WITH INFINITE UNCOUNTABLE φ-CATEGORY

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Abstract. In this paper we will improve some results previously obtained, showing that the so called φ-category of a pair of manifolds is infinte uncountable under certain topological conditions on the two given manifolds.

1. Introduction

The *φ-category* of a pair (M,N) of differentiable (smooth) manifolds and the *algebraic φ-category* of a pair (G,H) of groups are defined as

$$\varphi(M,N) = \min\{\#C(f) \mid f \in C^\infty(M,N)\},$$
$$\varphi_{alg}(G,H) = \min\{[H : \operatorname{Im} f] \mid f \in \operatorname{Hom}(G,H)\},$$

where $C(f)$ is the critical set of $f \in C^\infty(M,N)$. Observe that the set of regular points $R(f) = M \setminus C(f)$ of f is open, which means that the critical set $C(f)$ is closed. Therefore when f is closed, the bifurcation set $B(f) = f(C(f))$ of f is also closed. Using the Lusternik–Schnirelman multiplicity theorem (see for instance [2, p. 190]) it follows, for a compact manifold M, that $\varphi(M,\mathbb{R}) \geq \operatorname{cat}(M)$ where $\operatorname{cat}(M)$ is the Lusternik–Schnirelman category of M.

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In the previous papers [3], [4] we have shown that the φ-category of the pair (M,N) of differentiable manifolds is infinite, that is any differentiable mapping $f\colon M\to N$ has infinitely many critical points, under one of the following conditions:

(i) $\dim M=\dim N\geq 3$ and $\pi_1(M)$ cannot be embedded as a subgroup in $\pi_1(N)$,

(ii) $\dim M = \dim N \geq 4$ and $\pi_q(M) \not\preceq \pi_q(N)$ for some $q\in\{2,\dots,\dim M-2\}$,

(iii) $\dim M\geq\dim N\geq 2$ and $\varphi_{alg}(\pi_1(M),\pi_1(N))\geq\aleph_0$.

In this paper we will improve these results proving that in all these cases the φ-category is infinite uncountable, that is any differentiable mapping $f\colon M\to N$ has an infinite uncountable number of critical points.

Finally, in the last section, we observe that the φ-category of the pair (M^m,S^m) is a lower bound for the minimum number of points of zero Gauss–Kronecker curvature of an orientable manifold M^m immersible in $\mathbb{R}^{m+1}$, with respect to all of these immersions.

2. A useful homotopy associated to a finite family of charts

For $r>0$ and $n\in\mathbb{N}^*$ denote by D^n_r and S^{n-1}_r the open disk and the sphere respectively, both of them having the center at the origin of the space $\mathbb{R}^n$ and radius r. D^n_1 and S^{n-1}_1 will be simply denoted by D^n and S^{n-1} respectively.

For $x_0\in D^n$, consider the mapping $h_{x_0}\colon\mathbb{R}^n\setminus\{x_0\}\to\mathbb{R}^n\setminus\{x_0\}$ given by

$$h_{x_0}(x)=\begin{cases} x & \text{if } x\in\mathbb{R}^n\setminus D^n,\\ x_0+\alpha(x)(x-x_0) & \text{if } x\in\overline{D}^n\setminus\{x_0\},\end{cases}$$

where

$$\alpha(x)=\left\langle\frac{x_0}{\|x-x_0\|},\frac{x_0-x}{\|x-x_0\|}\right\rangle+\sqrt{\left\langle\frac{x_0}{\|x-x_0\|},\frac{x_0-x}{\|x-x_0\|}\right\rangle^2+\frac{1-\|x_0\|^2}{\|x-x_0\|^2}}.$$

Let us show that h_{x_0} is well defined and continuous. To prove that, it's enough to show that $x_0+\alpha(x)(x-x_0)=x$, for all $x\in S^{n-1}$.

First of all let us mention that $\alpha(x)$ is the positive solution of the equation

$$\|x_0+t(x-x_0)\|^2=1.$$

This means that for $x\in S^{n-1}$ we have successively

$$\begin{aligned}\|x_0+\alpha(x)(x-x_0)\|^2=1 &\Leftrightarrow \|(1-\alpha(x))x_0+\alpha(x)x\|^2=1\\ &\Leftrightarrow (1-\alpha(x))^2\|x_0\|^2+2\alpha(x)(1-\alpha(x))\langle x_0,x\rangle+\alpha(x)^2\|x\|^2=1\\ &\Leftrightarrow (1-\alpha(x))^2\|x_0\|^2+2\alpha(x)(1-\alpha(x))\langle x_0,x\rangle=(1-\alpha(x))(1+\alpha(x))\\ &\Leftrightarrow (1-\alpha(x))\big[(1-\alpha(x))\|x_0\|^2+2\alpha(x)\langle x_0,x\rangle-1-\alpha(x)\big]=0.\end{aligned}$$

But

$$(1-\alpha(x))\|x_0\|^2 + 2\alpha(x)\langle x_0, x\rangle - 1 - \alpha(x) = \|x_0\|^2 - 1 - \alpha(x)\|x - x_0\|^2 < 0.$$

Therefore $\alpha(x) = 1$, namely $h_{x_0} := x_0 + \alpha(x)(x - x_0) = x$.

The mapping h_{x_0} acts on $\mathbb{R}^n \setminus \{x_0\}$ like in Figure 1.

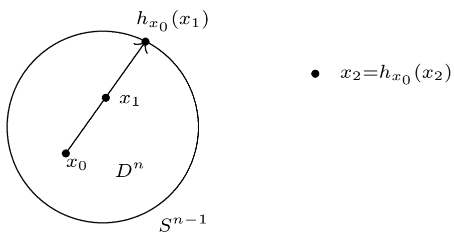

FIGURE 1

Let M be an n-dimensional manifold and $c = (U, \varphi)$ be a local chart of M such that $\overline{D}^n \subseteq \varphi(U)$. Denote by D_φ and by S_φ the sets $\varphi^{-1}(D^n)$ and $\varphi^{-1}(S^{n-1})$ respectively. For $x_0 \in D_\varphi$ the mapping $h_{c,x_0}\colon M \setminus \{x_0\} \to M \setminus \{x_0\}$ given by

$$h_{c,x_0}(x) = \begin{cases} x & \text{if } x \in M \setminus D_\varphi, \\ \varphi^{-1}(h_{\varphi(x_0)}(\varphi(x))) & \text{if } x \in U \setminus \{x_0\}. \end{cases}$$

is well defined and continuous.

PROPOSITION 2.1.

(i) $h_{x_0}(x) = x$, *for all* $x \in S^{n-1}$ *and* $h_{x_0}(D^n \setminus \{x_0\}) = S^{n-1}$.

(ii) $h_{x_0} \simeq_{H_{x_0}} \mathrm{id}_{\mathbb{R}^n \setminus \{x_0\}} (\mathrm{rel}\, \mathbb{R}^n \setminus D^n)$, *where* $H_{x_0}\colon \mathbb{R}^n \setminus \{x_0\} \times [0,1] \to \mathbb{R}^n \setminus \{x_0\}$ *is given by* $H_{x_0}(x,t) = (1-t)x + t h_{x_0}(x)$.

(iii) $h_{c,x_0}(D_\varphi \setminus \{x_0\}) = S_\varphi$ *and* $h_{c,x_0}(x) = x$, *for all* $x \in S_\varphi$.

(iv) $h_{c,x_0} \simeq_{H_c^{x_0}} \mathrm{id}_{M \setminus \{x_0\}}$, *where* $H_c^{x_0}\colon (M \setminus \{x_0\}) \times [0,1] \to M\{x_0\}$,

$$H_c^{x_0}(x,t) = \begin{cases} x & \text{if } x \in M \setminus D_\varphi, \\ \varphi^{-1}(H_{\varphi(x_0)}(\varphi(x), t)) & \text{if } x \in \overline{D}_\varphi \setminus \{x_0\}. \end{cases}$$

(v) *Let* $c_1 = (U_1, \varphi_1), \ldots, c_k = (U_k, \varphi_k)$ *be local charts in* M *such that* $i \neq j$ *implies* $U_i \cap U_j = \emptyset$ *and* $D^n \subseteq \bigcap_{i=1}^k \varphi_i(U_i)$. *If* $x_i \in D_{\varphi_i}$, $i \in \{1, \ldots, k\}$, *then*

$$g_{c_1,x_1} \circ \ldots \circ g_{c_k,x_k} = g_{c_{\pi(1)},x_{\pi(1)}} \circ \ldots \circ g_{c_{\pi(k)},x_{\pi(k)}},$$

where π *is an arbitrary element of the group* S_k *of permutations of the set* $\{1, \ldots, k\}$ *and* $g_{c_i x_i}\colon M \setminus \{x_1, \ldots, x_k\} \to M \setminus \{x_1, \ldots, x_k\}$ *are given by* $g_{c_i,x_i}(x) = h_{c_i,x_i}(x)$, *for all* $i \in \{1, \ldots, k\}$.

(vi) *Under the conditions of the statement* (v), *it is also true that*

$$g_{c_1,x_1} \circ \dots \circ g_{c_k,x_k} \simeq_{G^{x_1\dots x_k}_{c_1\dots c_k}} \mathrm{id}_{M\setminus\{x_1,\dots,x_k\}} \left(\mathrm{rel}\, M \setminus \bigcup_{i=1}^{k} D_{\varphi_i}\right)$$

where $G^{x_1\dots x_k}_{c_1\dots c_k}\colon (M \setminus \{x_1,\dots,x_k\}) \times [0,1] \to M \setminus \{x_1,\dots,x_k\}$ *is given by*

$$G^{x_1\dots x_k}_{c_1\dots c_k}(x,t) = \begin{cases} x & \text{if } x \in M \setminus \bigcup_{i=1}^{k} D_{\varphi_i}, \\ \varphi_1^{-1}(H_{\varphi_1(x_1)}(\varphi_1(x),t)) & \text{if } x \in \overline{D}_{\varphi_1} \setminus \{x_1\}, \\ \dots & \\ \varphi_k^{-1}(H_{\varphi_k(x_k)}(\varphi_k(x),t)) & \text{if } x \in \overline{D}_{\varphi_k} \setminus \{x_k\}. \end{cases}$$

PROOF. (i) The equality $h_{x_0}(x) = x$ for $x \in S^{n-1}$ follows immediately from the definition of h_{x_0}. To prove the equality $h_{x_0}(D^n \setminus \{x_0\}) = S^{n-1}$ take $x \in D^n \setminus \{x_0\}$ and observe that $\|h_{x_0}(x)\| = \|x_0 + \alpha(x)(x - x_0)\| = 1$, because $\alpha(x)$ is a solution of the equation $\|x_0 + t(x - x_0)\|^2 = 1$. Consequently the inclusion $h_{x_0}(D^n \setminus \{x_0\}) \subseteq S^{n-1}$ is proved. To prove the inclusion $S^{n-1} \subseteq h_{x_0}(D^n \setminus \{x_0\})$ we will firstly show that, for $x \in S^{n-1}$ and $t \in (0,1)$, $x_0 + t(x - x_0) \in D^n \setminus \{x_0\}$. Indeed, on the one hand $x_0 + t(x - x_0)$ cannot be equal to x_0 and on the other hand the function $\beta\colon \mathbb{R} \to \mathbb{R}$, $\beta(t) = \|x_0 + t(x - x_0)\|^2$ is convex because $\beta''(t) = 2\|x - x_0\|^2 > 0$, for all $t \in \mathbb{R}$. Therefore, for any $t \in (0,1)$ we have successively:

$$\beta(t) = \beta((1-t)\cdot 0 + t \cdot 1) \le (1-t)\beta(0) + t\beta(1) = (1-t)\|x_0\|^2 + t\|x\|^2 < 1.$$

Let us observe that the arguments presented before are also working for $x \in \overline{D}^n$, that is $\|x_0 + t(x - x_0)\| < 1$ for any $x \in \overline{D}^n$ and any $t \in (0,1)$.

Making very easy computations one can deduce that $\alpha(x_0 + t(x - x_0)) = \alpha(x)/t$ which leads us to the conclusion that $h_{x_0}(x_0 + t(x - x_0)) = h_{x_0}(x) = x$, and the inclusion $S^{n-1} \subseteq h_{x_0}(D^n \setminus \{x_0\})$ is completely proved.

(ii) It is enough to prove that the homotopy

$$H_{x_0}\colon (\mathbb{R}^n \setminus \{x_0\}) \times [0,1] \to \mathbb{R}^n \setminus \{x_0\}, \quad H_{x_0}(x,t) = (1-t)x + th_{x_0}(x)$$

is well defined, namely $H_{x_0}(x,t) \ne x_0$ for any $t \in [0,1]$ and any $x \in \mathbb{R}^n \setminus \{x_0\}$. If $x \in \mathbb{R}^n \setminus D^n$, then $H_{x_0}(x,t) = x \in \mathbb{R}^n \setminus \{x_0\}$ for any $t \in [0,1]$. Otherwise, assuming that there exists $x \in D^n \setminus \{x_0\}$ and $t \in [0,1]$ such that $H_{x_0}(x,t) = x_0$ one can be easily seen that

$$t(1 - \alpha(x)) = 1, \tag{1}$$

meaning that t cannot be zero. Because $\alpha(x) > 0$ it implies on the one hand that t cannot be one, and on the other hand that $\alpha(x) \ge 1$ for any $x \in D^n \setminus \{x_0\}$, since we have alredy seen that $\|x_0+t(x-x_0)\|^2<1$ for $x\in D^n\setminus\{x_0\}$ and $t\in(0,1)$, such that the relation (1) fails to be true.

(iii) Follows immediately from the definition of h_{c,x_0} and from (i).

(iv) It is enough to prove that $H_c^{x_0}$ is well defined, namely $H_{x_0}(x,t)\in\overline{D}^n\setminus\{x_0\}$ for all $x\in\overline{D}^n\setminus\{x_0\}$ and all $t\in[0,1]$. For this purpose, let us firstly observe that

$$\|x_0+t(x-x_0)\|\le 1 \quad \text{for all } t\in[\alpha'(x),\alpha(x)],$$

where $\alpha'(x)$ is the negative solution of the equation $\|x_0+t(x-x_0)\|^2=1$. Because $H_{x_0}(x,t)=x_0+[1+t(\alpha(x)-1)](x-x_0)$ and

$$0\le 1+t(\alpha(x)-1)\le\alpha(x) \quad \text{for all } t\in[0,1] \text{ and all } x\in\overline{D}^n\setminus\{x_0\},$$

we conclude that $\|H_{x_0}(x,t)\|\le 1$, for all $(x,t)\in(\overline{D}^n\setminus\{x_0\})\times[0,1]$.

(v) If $x\in M\setminus\bigcup_{i=1}^k D_{\varphi_i}$, then x is a fixed point both for $g_{c_1,x_1}\circ\ldots\circ g_{c_k,x_k}$ and $g_{c_{\pi(1)}x_{\pi(1)}}\circ\ldots\circ g_{c_{\pi(k)},x_{\pi(k)}}$ simply because x is a fixed point for each of g_{c_i,x_i}. Otherwise there exists a unique $i_0\in\{1,\ldots,k\}$ such that $x\in D_{\varphi_{i_0}}\setminus\{x_{i_0}\}$. Because x is a fixed point for

$$g_{c_{i_0+1},x_{i_0+1}},\ldots,g_{c_k,x_k},g_{c_{\pi(\pi^{-1}(i_0)+1)},x_{\pi(\pi^{-1}(i_0)+1)}},\ldots,g_{c_{\pi(k)},x_{\pi(k)}}$$

and $g_{c_{i_0},x_{i_0}}$ is a fixed point for

$$g_{c_{i_0-1},x_{i_0-1}},\ldots,g_{c_1,x_1},g_{c_{\pi(\pi^{-1}(i_0)-1)},x_{\pi(\pi^{-1}(i_0)-1)}},\ldots,g_{c_{\pi(1)},x_{\pi(1)}},$$

it follows that

$$\begin{aligned}(g_{c_1,x_1}\circ\ldots\circ g_{c_k,x_k})(x)&=(g_{c_1,x_1}\circ\ldots\circ g_{c_{i_0},x_{i_0}})(x)=g_{c_{i_0},x_{i_0}}(x)\\&=(g_{c_{\pi(1)},x_{\pi(1)}}\circ\cdots\circ g_{c_{i_0},x_{i_0}})(x)\\&=(g_{c_{\pi(1)},x_{\pi(1)}}\circ\cdots\circ g_{c_{\pi(k)},x_{\pi(k)}})(x).\end{aligned}$$

(vi) It is enough to observe that $G_{c_1\ldots c_k}^{x_1\ldots x_k}$ is well defind, the fact that it is a homotopy (rel $M\setminus\bigcup_{i=1}^k D_{\varphi_i}$) between $g_{c_1,x_1}\circ\ldots\circ g_{c_k,x_k}$ and $\mathrm{id}_{M\setminus\{x_1,\ldots,x_k\}}$ being obvious. We also observe that $(G_{c_1\ldots c_k}^{x_1\ldots x_k})_t=(G_{c_1}^{x_1})_t\circ\ldots\circ(G_{c_k}^{x_k})_t=(G_{c_{\pi(1)}}^{x_{\pi(1)}})_t\circ\ldots\circ(G_{c_{\pi(k)}}^{x_{\pi(k)}})_t$, for all $t\in[0,1]$ and all $\pi\in S_k$, where $G_{c_i}^{x_i}\colon(M\setminus\{x_1,\ldots,x_k\})\times[0,1]\to M\setminus\{x_1,\ldots,x_k\}$, $G_{c_i}^{x_i}(x,t)=H_{c_i}^{x_i}(x,t)$. $\square$

If we give up the condition $U_i\cap U_j=\emptyset$ for $i\ne j$ of (v), we can ask ourselves if the mappings $g_{c_1,x_1}\circ\ldots\circ g_{c_k,x_k}$ $\mathrm{id}_{M\setminus\{x_1,\ldots,x_k\}}$ are still homotopic. Of course the best candidate for a homotopy between the two of them whould be $(G_{c_1}^{x_1})_t\circ\ldots\circ(G_{c_k}^{x_k})_t$, $t\in[0,1]$. Althought the intersections $U_i\cap U_j$, $i\ne j$ are not empty anymore we are forced to work with families of charts $c_1=(U_1,\varphi_1),\ldots,c_1=(U_k,\varphi_k)$ and points $x_1\in D_{\varphi_1},\ldots,x_k\in D_{\varphi_k}$ such that $x_i\notin\overline{D}_{\varphi_j}$ for $j\ne i$, just to be sure that the mappings $g_{c_1,x_1}\circ\ldots\circ g_{c_k,x_k}$ and $(G_{c_1}^{x_1})_t\circ\ldots\circ(G_{c_k}^{x_k})_t$,

$t \in [0,1]$ are well defined. With such a choice of the charts $c_1, \dots, c_k$ and the points $x_1, \dots, x_k$, the homotopy

$$(G^{x_1\dots x_k}_{c_1\dots c_k})_t : M \setminus \{x_1 \dots x_k\} \to M \setminus \{x_1, \dots, x_k\},$$
$$(G^{x_1\dots x_k}_{c_1\dots c_k})_t = (G^{x_1}_{c_1})_t \circ \dots \circ (G^{x_k}_{c_k})_t, \quad t \in [0,1]$$

is well defined and it joins indeed $g_{c_1,x_1} \circ \dots \circ g_{c_k,x_k}$ with $\mathrm{id}_{M\setminus\{x_1,\dots,x_k\}}$.

The difference between the situations of empty and nonempty intersections is that any two mappings $g_{c_1,x_1} \circ \dots \circ g_{c_k,x_k}$, $g_{c_{\pi(1)}x_{\pi(1)}} \circ \dots \circ g_{c_{\pi(k)},x_{\pi(k)}}$ are equal and

$$(G^{x_1\dots x_k}_{c_1\dots c_k})_t = (G^{x_1}_{c_1})_t \circ \dots \circ (G^{x_k}_{c_k})_t, \quad (G^{x_{\pi(1)}\dots x_{\pi(k)}}_{c_{\pi(1)}\dots c_{\pi(k)}})_t = (G^{x_{\pi(1)}}_{c_{\pi(1)}})_t \circ \dots \circ (G^{x_{\pi(k)}}_{c_{\pi(k)}})_t$$

are also equal for all $t \in [0,1]$ if the open sets $U_1, \dots, U_k$ are distincte to each other, not being the case otherwise. But in [4] we never used neither the commutativity of the mappings g_{c_i,x_i} nor that of the homotopies $(G^{x_i}_{c_i})_t$, such that we can fortunately give up to the condition of empty intersections of certain charts domains.

3. Improved results

In this section the results of [3], [4] will be improved by justifing the existence of an infinite uncountable number of critical points for all the mappings acting between the manifolds of all the pairs appearing in the papers [3], [4].

Theorem 3.1. *Let M be an n-dimensional differentiable manifold ($\partial M = \emptyset$) and A be a closed countable subset of M. If P is a compact differentiable k-dimensional manifold ($k < n$, $\partial P \neq \emptyset$) and $f : P \to M$ is a continuous map such that $f(\partial P) \subseteq M \setminus A$, then there exists a continuous map $g : P \to M$ such that $g(P) \subseteq M \setminus A$, $g|_{\partial P} = f|_{\partial P}$ and $f \simeq g(\mathrm{rel}\,\partial P)$. If M is connected, then one particularly gets, using the particular case $P = [0,1]$, that $M \setminus A$ is also connected.*

Proof. According to [4, Lemma 2.1] there exists a homotopy $H : P \times [0,1] \to M$ such that $f = H_0$, $g_1 = H_1$ is a differentiable mapping and $H_t(\partial P) \subseteq M \setminus A$ for all $t \in [0,1]$.

Because $g_1(P) \cap A$ is compact and A is countable, it follows that there exist the local charts $c_1 = (U_1, \varphi_1), \dots, c_k = (U_l, \varphi_l)$ such that $\overline{D}^n \subseteq \bigcap_{i=1}^{l} \varphi_i(U_i)$, $g_1(P) \cap A \subseteq \bigcup_{i=1}^{l} D_{\varphi_i}$ and $S_{\varphi_i} \cap A = \emptyset$, $S_{\varphi_i} \cap g_1 \partial P = \emptyset$, for all $i \in \{1, \dots, l\}$. We can assume that $i \neq j \Rightarrow D_{\varphi_i} \setminus D_{\varphi_j} \neq \emptyset$. In these conditions it can be easily seen that $D_{\varphi_i} \setminus \overline{D}_{\varphi_j} \neq \emptyset$ for all $i, j \in \{1, \dots, l\}$, $i \neq j$.

Using Sard theorem (see for instance [5, Theorem VII.28, p. 263]) for the mapppings $g_1|_{\mathrm{int}\,P} : \mathrm{int}\,P \to M$, $g_1|_{\partial P} : \partial P \to M$ we get that $g_1(\mathrm{int}\,P)$ and $g_1(\partial P)$

have measure zero, namely $g_1(P) = g_1(\operatorname{int} P \cup \partial P) = g_1(\operatorname{int} P) \cup g_1(\partial P)$ has also measure zero.

Because A is countable it obviously has measure zero, meaning that $A \cup g_1(P)$ has also measure zero. We conclude that, for each $i \in \{1, \dots, l\}$, the set

$$D_{\varphi_i} \setminus \Big(A \cup g_1(P) \cup \bigcup_{\substack{j=1 \\ j \neq i}}^{l} \overline{D}_{\varphi_j} \Big) = \Big(D_{\varphi_i} \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{l} \overline{D}_{\varphi_j} \Big) \setminus (A \cup g_1(P))$$

is not empty, because $A \cup g_1(P)$ has measure zero and the non-empty set $D_{\varphi_i} \setminus \bigcup_{\substack{j=1 \\ j \neq i}}^{l} \overline{D}_{\varphi_j}$ doesn't, being open. For $i \in \{1, \dots, l\}$, let us consider

$$y_i \in D_{\varphi_i} \setminus \Big(A \cup g_1(P) \cup \bigcup_{\substack{j=1 \\ j \neq i}}^{l} \overline{D}_{\varphi_j} \Big)$$

and the mappings

$$g_{c_i, y_i} \colon M \setminus \{y_1, \dots, y_l\} \to M \setminus \{y_1, \dots, y_l\}$$
$$h \colon P \to M,\ h(x) = (j \circ g_{c_1, y_1} \circ \ldots \circ g_{c_l, y_l})(g(x))$$

where $j \colon M \setminus \{y_1, \dots, y_l\} \to M$ is the inclusion. Obviously, $h(P) \subseteq M \setminus A$, $h|_{\partial P} = g_1|_{\partial P}$ and $h \simeq_{G'} g_1$ $(\operatorname{rel} \partial P)$, where $G' \colon P \times [0,1] \to M$ is given by $G'(x,t) = G^{y_1 \cdots y_l}_{c_1 \ldots c_l}(g_1(x), t)$.

By the transitivity of the relation "$\simeq$", one can conclude that $f \simeq_{H'} h$ where

$$H' \colon P \times [0,1] \to M,\ H'(x,t) = \begin{cases} H(x, 2t) & 0 \leq t \leq 1/2, \\ G'(x, 2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

Consider the following two homotopies $\psi \colon P \times [0,1] \to P$ and $G \colon P \times [0,1] \to M$ given by:

$$\psi(x,t) = \begin{cases} x & \text{if } x \in P \setminus Q(\partial O \times [0,2)), \\ Q((\pi_1 \circ Q^{-1})(x), (2/(2-t)) & \\ \quad \cdot (\pi_1 \circ Q^{-1})(x) + 2t/(t-2)) & \text{if } x \in Q(\partial O \times [0,2)), \\ (\pi_1 \circ Q^{-1})(x) & \text{if } x \in Q(\partial O \times [0,2)), \end{cases}$$

$$G(x,t) = \begin{cases} H'(\psi(x,t), t) & \text{if } x \in P \setminus Q(\partial P \times [0,t)), \\ H'(Q^{-1}(x)) & \text{if } x \in Q(\partial P \times [0,2)), \end{cases}$$

where $Q \colon \partial P \times [0, \infty) \to U \subset P$ is a collar neighbourhood of ∂P and

$$\pi_1 \colon \partial P \times [0, \infty) \to \partial P, \quad \pi_2 \colon \partial P \times [0, \infty) \to \partial P$$

are the projections.

Denoting $G(\cdot,1)$ by g and observing that $G(\cdot,0)=f$ it can be easily seen that

$$f \simeq_G g(\operatorname{rel}\partial P) \quad \text{and that} \quad g(P) \subseteq M \setminus A,$$

and the theorem is completely proved. □

Corollary 3.2. *Let M, N be connected differentiable manifolds such that $\dim M \geq \dim N$. If $f\colon M \to N$ is a non-surjective closed differentiable mapping, then either $C(f)=M$ or f has an infinite uncountable number of critical values. Therefore, in any case, f has an infinite uncountable number of critical points. If M is compact and N is non-compact, then one particularly gets that $\varphi(M,N)=\aleph_1$.*

Proof. Let us firstly prove that $f^{-1}(\partial \operatorname{Im} f) \subseteq C(f)$, which implies that $\partial \operatorname{Im} f \subseteq B(f)$. Indeed, otherwise $f^{-1}(\partial \operatorname{Im} f) \cap R(f) \neq \emptyset$ and f is locally open around any point of the set $f^{-1}(\partial \operatorname{Im} f) \cap R(f)$. If $x \in f^{-1}(\partial \operatorname{Im} f) \cap R(f)$ is a fixed point and U is an open neighbourhood of x such that the restriction $f|_U\colon U \to N$ is open, then $f(U)$ is particularly open. But this is a contradiction with the fact that $f(x) \in \partial \operatorname{Im} f$.

If $C(f) \neq M$ it follows, by Sard's theorem, that $\operatorname{Im} f \setminus B(f) \neq \emptyset$. In what follows we shall show that $N \setminus B(f)$ is not connected. Indeed, if $y \in \operatorname{Im} f \setminus B(f)$, $y' \in N \setminus \operatorname{Im} f$ then obviously $y, y' \in N \setminus B(f)$. Consider $\gamma\colon [0,1] \to N$ a continuous path joining y to y'. Because $y \in \operatorname{Im} f$ and $y' \in N \setminus \operatorname{Im} f$ it follows that $\gamma([0,1])$ intersects the border $\partial \operatorname{Im} f$ and hence the set $B(f)$. Consequently $B(f)$ cannot be finite or infinite countable because in both cases $N \setminus B(f)$ would be connected. Therefore $B(f)$ must be infinite uncountable meaning that $C(f)$ is also infinite uncountable. □

Further on, using Theorem 3.1 and the homotopy sequence of the pair $(M, M \setminus A)$ we have, in a completely similar manner with the proof of [4, Proposition 2.3], the following:

Corollary 3.3. *Let M be an n-dimensional differentiable manifold ($n \geq 2$, $\partial M = \emptyset$) and A be a closed countable subset of M. If M is connected, then $M \setminus A$ is also connected, and the inclusion $i\colon M \setminus A \to M$ is $(n-1)$-connected, that is, the homomorphism induced by inclusion $i_q\colon \pi_q(M \setminus A) \to \pi_q(M)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$.*

Theorem 3.4. *Let M, N be compact connected differentiable manifolds of the same dimension m.*

(i) *If $m \geq 3$ and $\pi_1(M)$ can't be embedded as a subgroup in $\pi_1(N)$, then $\varphi(M,N)=\aleph_1$.*
(ii) *If $\pi_q(M) \not\simeq \pi_1(N)$ for some $q \in \{2,\dots,m-2\}$, then $\varphi(M,N)=\aleph_1$.*

Proof. Assume that $\varphi(M,N)=\aleph_0$, that is there exists a mapping $f\colon M\to N$ with an infinite countable number of critical points. Because $C(f)$ is also closed it follows that it will be enough to treat only the case when f is surjective. In this case the restriction

$$M\setminus f^{-1}(B(f))\xrightarrow{g} N\setminus B(f),\quad p\mapsto f(p)$$

is proper and has not critical points. This means that for $q\in N\setminus B(f)$ the pre-image $g^{-1}(q)$ is compact and discrete, that is finite. It can be easily seen, following a similar argument to that one from the proof of [4, Theorem 1.2], that $\#g^{-1}(q)$ doesn't depend on $q\in N\setminus B(f)$, that is g is a finite-fold covering mapping. This means that

$$g_1\colon \pi_1(M\setminus f^{-1}(B(f)))\to \pi_1(N\setminus B(f))$$

is a monomorphism and

$$g_q\colon \pi_q(M\setminus f^{-1}(B(f)))\to \pi_q(N\setminus B(f))$$

is isomorphism for all $q\geq 2$. On the other hand the set $f^{-1}(B(f))$ can be represented as the union $C(f)\cup(f^{-1}(B(f))\cap R(f))$, where $R(f)\subseteq M$ is the set of regular points of f. For any $q\in N$ the set $f^{-1}(q)=(f^{-1}(q)\cap C(f))\cap(f^{-1}(q)\cap R(f))$ is countable at most because both $f^{-1}(q)\cap C(f)$ and $f^{-1}(q)\cap R(f)$ are countable at most, the last one being like this because it is discrete. In particular the closed set $f^{-1}(B(f))=\bigcup_{q\in B(f)}f^{-1}(q)$ is also countable at most, being a finite or countable union of countable sets at most. Because the sets $M\setminus f^{-1}(B(f))$, $N\setminus B(f)$ are closed and countable at most it follows that the inclusions $i\colon M\setminus f^{-1}(B(f))\hookrightarrow M$, $j\colon N\setminus B(f)\hookrightarrow N$ induces the isomorphisms $i_q\colon\pi_q(M\setminus f^{-1}(B(f)))\to\pi_q(M)$, $j_q\colon\pi_q(N\setminus B(f))\to\pi_q(N)$ for all $q\in\{0,\dots,m-2\}$. From the commutative diagram

$$\begin{array}{ccc} M\setminus f^{-1}(B(f)) & \xrightarrow{g} & N\setminus B(f) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ M & \xrightarrow[f]{} & N \end{array}\qquad j\circ g=f\circ i$$

we get the following commutative diagram

$$\begin{array}{ccc} \pi_q(M\setminus f^{-1}(B(f))) & \xrightarrow{g_q} & \pi_q(N\setminus B(f)) \\ {\scriptstyle i_q}\big\downarrow & & \big\downarrow{\scriptstyle j_q} \\ \pi_q(M) & \xrightarrow[f_q]{} & \pi_q(N) \end{array}\qquad j_q\circ g_q=f_q\circ i_q$$

(i) For $q=1$, because $f_1\circ i_1=j_1\circ g_1$ and i_1, j_1 are isomorphisms, it follows that $f_1=j_1\circ g_1\circ i_1^{-1}$ is a monomorphism, that is a contradiction with the hypothesis of statement (i).

(ii) For $q \in \{2, \dots , m-2\}$, i_q, j_q, g_q are isomorphisms which combined with the equality $j_q \circ g_q = f_q \circ i_q$ one can deduce that $f_q = j_q \circ g_q \circ i_q^{-1}$ are isomorphisms for all $q \in \{2, \dots , m-2\}$, which is a contradiction with the hypothesis of the statement (ii). $\square$

Theorem 3.5. *Let M^m, N^n be compact connected differentiable manifolds such that $m \geq n \geq 2$. If $\varphi_{alg}(M, N) \geq \aleph_0$, then $\varphi(M, N) = \aleph_1$.*

The proof of Theorem 3.5 uses Corollary 3.2 and it is the same with that of [3, Theorem 3.3], except that in any point of the proof where some critical set is infinite there, here it will be infinite uncountable.

Corollary 3.6.

(i) *If m, n, k are natural numbers such that $1 < k < m$ and $k+n \geq m \geq 2$, then $\varphi(T^k \times S^n, T^m) = \aleph_1$.*

(ii) *If Σ_g is a compact connected orientable surface of genus g and $g < g'$, then $\varphi(\Sigma_g, \Sigma_{g'}) = \aleph_1$.*

(iii) *If P_g is a compact connected surface having the same topological type with a connected sum of g projective spaces, and $g < g'$, then $\varphi(P_g, P_{g'}) = \aleph_1$.*

The proof follows easily using Theorem 3.5 and the fact that the algebraic φ-category φ_{alg} of each mentioned pair is infinite as it was argued in the proof [3, Proposition 4.1].

4. Application

Let M^m be an orientable manifold immersible in $\mathbb{R}^{m+1}$, $f\colon M \to \mathbb{R}^{m+1}$ be an immersion and $N_f\colon M \to S^m$ its associated Gauss mapping. The Gauss–Kronecker curvature of f is defined as $K_f(p) = \det(dN_f)_p$. Consequently $K_f(p) = 0$ if and only if $p \in C(N_f)$, that is

$$C(N_f) = \{p \in M \mid K_f(p) = 0\}.$$

Therefore if we define the *G-category* of M as

$$G(M) = \min\{\#C(N_f) \mid f \in \mathrm{Imm}\,(M, \mathbb{R}^{m+1})\},$$

where $\mathrm{Imm}\,(M, \mathbb{R}^{m+1})$ is the set of all immersions of M into $\mathbb{R}^{m+1}$, observe that

$$\varphi(M, S^m) \leq G(M). \tag{2}$$

According to Theorem 3.1 and the inequality (2) we have:

Proposition 4.1. *If M is an m dimensional manifold immersible in $\mathbb{R}^{m+1}$, then we have:*

(i) *If $m \geq 3$ and M is not simply connected, then $G(M) = \aleph_1$.*

(ii) *If $m \geq 4$ and $\pi_q(M)$ is not trivial for some $q \in \{2, \dots, m-2\}$, then $G(M) = \aleph_1$.*

COROLLARY 4.2. *If $k, n_1, \dots, n_k$ are natural numbers such that $k \geq 2$ and $n_1 + \ldots + n_k \geq 3$, then $S^{n_1} \times \ldots \times S^{n_k}$ is obviously orientable and immersible in $\mathbb{R}^{n_1+\ldots+n_k+1}$ and*

$$G(S^{n_1} \times \ldots \times S^{n_k}) = \aleph_1.$$

In other words any immersion $f: S^{n_1} \times \ldots \times S^{n_k} \to \mathbb{R}^{n_1+\ldots+n_k+1}$ has an infinite uncountable number of points of zero Gauss–Kronecker curvature.

PROOF. If $1 \in \{n_1, \dots, n_k\}$, that is $n_i = 1$ for some $i \in \{1, \dots, k\}$, then

$$\pi_1(S^{n_1} \times \ldots \times S^{n_k}) \simeq \pi_1(S^{n_1}) \times \ldots \times \pi_1(S^{n_i}) \times \ldots \times \pi_1(S^{n_k})$$

has the infinite cyclic subgroup $\pi_1(S^{n_i}) = \pi_1(S^1) \simeq \mathbb{Z}$. Therefore, according to Proposition 4.1(i) we have $G(S^{n_1} \times \ldots \times S^{n_k}) = \aleph_1$. If $n_1, \dots, n_k \geq 2$ it follows that

$$\pi_{n_i}(S^{n_1} \times \ldots \times S^{n_k}) \simeq \pi_{n_i}(S^{n_1}) \times \ldots \times \pi_{n_i}(S^{n_i}) \times \ldots \times \pi_{n_i}(S^{n_k})$$

has the infinite cyclic subgroup $\pi_{n_i}(S^{n_i}) \simeq \mathbb{Z}$ for all $i \in \{1, \dots, k\}$. Therefore, according to Proposition 4.1(ii) we have $G(S^{n_1} \times \ldots \times S^{n_k}) = \aleph_1$. $\square$

So far we didn't obtain any information on the φ-category of the pair (M_g, S^2), where M_g is a compact orientable surface of genus $g \geq 1$. Consequently we will study it and the G-category of M_g, in what follows.

Let us first observe that any differentiable mapping from M_g to S^2 has one critical point at least. Indeed, if $f: M_g \to S^2$ would be a mapping without critical points, then f would be a covering mapping and $f_1: \pi_1(M_g) \to \pi_1(S^2)$ a monomorphism. But $\pi_1(S^2)$ is trivial and $\pi_1(M_g)$ is certainly non trivial, M_g having the same topological type like the connected sum of g tori. This argument shows that $\varphi(M_g, S^2) \geq 1$.

THEOREM 4.3. $\varphi(M_g, S^2) \geq 3$.

PROOF. We will treat firstly the case $g \geq 2$. Assuming that $\varphi(M_g, S^2) < 3$ it follows, taking into account the inequality $\varphi(M_g, S^2) \geq 1$, that $\varphi(M_g, S^2) \in \{1, 2\}$. Therefore there exists a smooth mapping $f: M_g \to S^2$ such that $\#C(f) \in \{1, 2\}$ and of course $\#B(f) \in \{1, 2\}$. The mapping

$$M_g \setminus f^{-1}(B(f)) \xrightarrow{h} S^2 \setminus B(f), \quad p \mapsto f(p)$$

is obviously a finite-fold covering mapping, meaning that

$$h_1: \pi_1(M_g \setminus f^{-1}(B(f))) \to \pi_1(S^2 \setminus B(f))$$

is injective and finaly that the fundamental group $\pi_1(M_g \setminus f^{-1}(B(f)))$ is isomorphic with a subgroup of $\pi_1(S^2 \setminus B(f))$. If $\#B(f) = 1$ or $\#B(f) = 2$, then $S^2 \setminus B(f)$ is topologically equivalent with $\mathbb{R}^2$ or with $\mathbb{R}^2 \setminus \{\text{one point}\}$, respectively. In the first case $\pi_1(S^2 \setminus B(f))$ is trivial and in the second case $\pi_1(S^2 \setminus B(f))$ is isomorphic with $\mathbb{Z}$. We can therefore conclude that $\pi_1(M_g \setminus f^{-1}(B(f)))$ is isomorphic with a subgroup of $\mathbb{Z}$. On the other hand, according to [4, Theorem 1.2] and a particular case of [4, Proposition 2.3], the group homomorphism

$$i_1 \colon \pi_1(M_g \setminus f^{-1}(B(f))) \to \pi_1(M_g)$$

is surjecive. Because the canonical projection

$$p \colon \pi_1(M_g) \to \pi_1(M_g)/[\pi_1(M_g), \pi_1(M_g)]$$

is also surjective it follows that the homomorphism

$$p \circ i_1 \colon \pi_1(M_g \setminus f^{-1}(B(f))) \to \pi_1(M_g)/[\pi_1(M_g), \pi_1(M_g)]$$

is surjective too. Because, according to [1, p. 135], $\pi_1(M_g)/[\pi_1(M_g), \pi_1(M_g)]$ is a free abelian group with $2g$ generators it implies that $p \circ i_1$ is a surjective group homomorphism from a subgroup of $\mathbb{Z}$ to $\mathbb{Z}^{2g}$ which is impossible because such a homomorphism doesn't exists. Therefore our first assumption is false so that $\#C(f) \geq 3$. The case $g = 1$ can be treated in a completely analogous manner, following the same steps, and we will finaly get a surjective group homomorphism of type $p \circ i_1$ from a subgroup of $\mathbb{Z}$ to $\mathbb{Z}^2$ which doesn't again exists. □

Let S be a regular surface embedded in $\mathbb{R}^3$ and $N \colon S \to S^2$ its Gauss mapping. Recall that a point $p \in S$ is called *parabolic* if $\det(dN)_p = 0$ but $(dN)_p \neq 0$ and $p \in S$ is called *planar* if $(dN)_p = 0$.

COROLLARY 4.4. $G(M_g) \geq 3$. *In particular if M_g is embedded in $\mathbb{R}^3$ and P_1, P_2 are the sets of parabolic and planar points of M_g then $\#(P_1 \cup P_2) \geq 3$. Therefore if M_g has not planar/parabolic points, then it has three parabolic/planar points at least.*

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