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SOME MULTIPLICITY RESULTS FOR A SUPERLINEAR ELLIPTIC PROBLEM IN $\mathbb{R}^N$

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Abstract. In this paper we shall study the semilinear elliptic problem

$$\begin{cases} -\Delta u + \sigma(x)u = |u|^{p-2}u + f(x) & \text{in } \mathbb{R}^N, \\ u \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $\sigma(x) \to \infty$ as $|x| \to \infty$, $p > 2$ and $f \in L^2(\mathbb{R}^N)$. Thanks to a compact embedding of a suitable weigthed Sobolev space in $L^2(\mathbb{R}^N)$, a direct use of the Symmetric Mountain Pass Theorem (if $f = 0$) and of the fibering method (if $f \neq 0$) allows to extend some multiplicity results, already known in the case of bounded domains.

1. Introduction

Let us consider the semilinear elliptic problem of the form

$$\begin{cases} -\Delta u = \lambda u + |u|^{p-2}u + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open smooth bounded subset of $\mathbb{R}^N$, $N \geq 2$, $f\colon \Omega \to \mathbb{R}$ given function, $p > 2$ and $p < 2^*$, $2^* = 2N/(N-2)$ if $N \geq 3$ while $2^* = \infty$ if $N = 2$.

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This problem has been widely studied by many authors via variational methods (see e.g. [17] and the references therein). Indeed, it is well known that the solutions of (1.1) can be found as critical points of the functional

$$I_f^\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega |u|^2\,dx - \frac{1}{p}\int_\Omega |u|^p dx - \int_\Omega f u\,dx$$

defined on the Sobolev space $H_0^1(\Omega)$. If $f \equiv 0$, the symmetric Mountain Pass Theorem allows to prove the existence of infinitely many solutions of (1.1) (see [2]); on the contrary, if $f \neq 0$, the problem loses its Z_2-symmetry and therefore multiplicity results in general do not hold. In fact, the forcing term f destroys the structure of the Mountain Pass of the problem, so the classical techniques of critical point theory are not directly applicable in order to find multiple solutions of the problem. However, if $\lambda = 0$ and f is small enough, the "fibering method" introduced by Pohozaev permits to state some existence and multiplicity results of solutions for the problem (1.1) (see e.g. [11], [13]). A similar but weaker multiplicity result (available in the critical case, too) is contained also in [19]. The aim of this paper is to extend those results to a case in which $\Omega = \mathbb{R}^N$. In this situation variational methods cannot be directly applied since the Sobolev space $H^1(\mathbb{R}^N)$ is not compactly embedded in $L^2(\mathbb{R}^N)$, so the action functional I_f^λ in general does not verify the well known Palais–Smale condition. In order to overcome the lack of compactness, we will deal with the following problem

$$(\mathrm{P}_f^\lambda) \qquad \begin{cases} -\Delta u \ + \sigma(x)u = \lambda u + |u|^{p-2}u \ + f(x) & \text{in } \mathbb{R}^N, \\ u \to 0 \quad \text{as } |x| \to \infty, \end{cases}$$

where $\lambda \in \mathbb{R}$, $p \in \,]2, 2^*[$, $f \in L^2(\mathbb{R}^N)$ and $\sigma(x)$ verifies the following condition

$$(\mathrm{H}) \qquad \begin{cases} \sigma \in L^2_{\mathrm{loc}}(\mathbb{R}^N) \text{ is such that } \operatorname{infess}\sigma(x) > 0 \text{ and} \\ \displaystyle\int_{B(x)} \frac{1}{\sigma(y)}\,dy \to 0 \quad \text{if } |x| \to \infty, \end{cases}$$

where $B(x)$ is the unit ball of $\mathbb{R}^N$ centered at x.

Now, let us consider the space

$$L^2(\mathbb{R}^N, \sigma) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \sigma(x)|u(x)|^2\,dx < \infty \right\}$$

equipped with the inner product $\int_{\mathbb{R}^N} \sigma(x)u(x)v(x)\,dx$ and set

$$X = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, \sigma).$$

In [7] it has been proved that X is compactly embedded in $L^2(\mathbb{R}^N)$, then a direct use of the Symmetric Mountain Pass Theorem (if $f = 0$ and $\lambda \in \mathbb{R}$) and of the fibering method (if $f \neq 0$ and $\lambda = 0$) allows to state the following multiplicity results, already known in the case of bounded domains.

THEOREM 1.1. *Let $f = 0$ and σ verifying assumption* (H). *Then, for any $\lambda \in \mathbb{R}$, problem* (P_0^λ) *has an unbounded sequence of critical points $\{u_n\}$ in X such that as $n \to \infty$*

$$\frac{1}{2}\int_\Omega (|\nabla u_n|^2 + \sigma(x)|u_n(x)|^2)\,dx - \frac{\lambda}{2}\int_\Omega |u_n|^2\,dx - \frac{1}{p}\int_\Omega |u_n|^p\,dx \to \infty.$$

If, moreover, it is $\lambda = 0$, then, there exists a positive solution and a negative solution of the problem (P_0^0).

THEOREM 1.2. *Let $\lambda = 0$ and $\sigma(x)$ verifying assumption* (H). *Then, for any $f \in L^2(\mathbb{R}^N)$ with L^2-norm small enough, problem* (P_f^0) *has at least three solutions.*

REMARK 1.3. Let us point out that, in the first part of Theorem 1.1, since λ is any real number, assumption (H) can be replaced by the weakened assumption

$$\text{(H')}\qquad \begin{cases} \sigma \in L^2_{\mathrm{loc}}(\mathbb{R}^N) \text{ is such that } \operatorname{infess} \sigma(x) > -\infty \text{ and} \\ \displaystyle\int_{B(x)} \frac{1}{\sigma(y)}\,dy \to 0 \quad \text{if } |x| \to \infty. \end{cases}$$

Moreover, always the first part of Theorem 1.1 still holds (with small changes in the proof), even if the term $|u|^{p-2}u$ is replaced by a more general superlinear term $p(x,u)$ even in the variable u.

REMARK 1.4. It is easy to see that (H) (respectively, (H')) holds in particular if $\sigma(x)$ is a strictly positive continuous (respectively, only continuous) function on $\mathbb{R}^N$ which goes to infinity at infinity.

A similar assumption on $\sigma(x)$ has been introduced by Rabinowitz in [15] in order to prove the existence of two solutions, the first one positive, the second one negative, for a superlinear Schrödinger equation like (P_0^0) (see also [4] and [5] for later improvements).

REMARK 1.5. A different type of multiplicity results for asymmetric problems like (P_f^λ), with λ and f different from zero, can be stated by perturbative methods developped in the eighties by Bahri and Berestycki, Rabinowitz and Struwe. In [3], [14], [17] the existence of infinitely many solutions of (1.1) has been proved if the power p ($p > 2$) is close enough to 2, while a multiplicity result holds also for any subcritical p if f is "small enough" (cf. [1], [3]). An extension of some of these results to the problem (P_f^λ) is given in [16].

2. Variational tools and proof of Theorem 1.1

In the sequel we shall denote by $L^s(\mathbb{R}^N)$, $s \geq 1$, and $H^1(\mathbb{R}^N)$ the Banach spaces $L^s(\mathbb{R}^N, \mathbb{R})$ and $H^1(\mathbb{R}^N, \mathbb{R})$ with the usual norms

$$|u|_s = \bigg(\int_{\mathbb{R}^N} |u(x)|^s \, dx\bigg)^{1/s} \quad \text{and} \quad \|u\|_{H^1} = \bigg(\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \, dx\bigg)^{1/2}.$$

Moreover, if Ω is an open subset of $\mathbb{R}^N$, we shall denote by $|u|_{s,\Omega}$ the norm in $L^s(\Omega) = L^s(\Omega, \mathbb{R})$. Let us consider the Hilbert space

$$X = \bigg\{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + \sigma(x)u^2) \, dx < \infty\bigg\}$$

endowed with the inner product $(u, v) = \int_{\mathbb{R}^N} ((\nabla u, \nabla v) + \sigma(x)uv) \, dx$ and the corresponding norm

$$\|u\| = \bigg(\int_{\mathbb{R}^N} (|\nabla u|^2 + \sigma(x)u^2) \, dx\bigg)^{1/2}.$$

The space X is a particular case of weighted Sobolev spaces introduced in [6]. The choice of the space X is justified by the following result.

Proposition 2.1. *Let σ verify assumption* (H). *Then, the space X is embedded in L^s for any $s \in [2, 2^*]$ and the embedding is compact for any $s \in [2, 2^*[$. Moreover, the spectrum of the self-adjoint realization of $-\triangle + \sigma$ in $L^2(\mathbb{R}^N)$ is discrete, i.e. it consists of an increasing sequence $\{\lambda_n\}$ of eigenvalues of finite multiplicity such that $\lambda_n \to \infty$ as $n \to \infty$ and $L^2(\mathbb{R}^N) = \sum_n M_n$, $M_n \perp M_{n'}$ for $n \neq n'$, where M_n is the eigenspace corresponding to λ_n.*

Proof. Since σ verifies assumption (H), by [7, Theorem 3.1] it follows that X is compactly embedded in $L^2(\mathbb{R}^N)$. Now, by the Sobolev embeddings it follows that

$$X \hookrightarrow H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N) \quad \text{for any } s \in [2, 2^*].$$

Then, by the interpolation inequality (see e.g. [8]) for any $u \in L^2(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$ it is $u \in L^s(\mathbb{R}^N)$ and

$$|u|_s \leq |u|_2^{1-a} |u|_{2^*}^a \quad \text{with } \frac{1}{s} = \frac{1-a}{2} + \frac{a}{2^*}, \quad 0 \leq a \leq 1.$$

Hence, the embedding of X in $L^s(\mathbb{R}^N)$ is compact for any $s \in [2, 2^*[$.

The second part of the proposition has been proved in [7, Theorem 4.1]. $\square$

In the following let us still denote by I_f^λ the action functional on X associated to the problem (P_f^λ), i.e.

$$I_f^\lambda(u) = \frac{1}{2}\int_{R^N} (|\nabla u|^2 + \sigma(x)u^2)\,dx - \frac{\lambda}{2}\int_{R^N} |u|^2\,dx - \frac{1}{p}\int_{R^N} |u|^p\,dx - \int_{R^N} fu\,dx.$$

Standard arguments prove that $I_f^\lambda \in C^1(X,\mathbb{R})$ and that for all $u, v \in X$ it is

$$[(I_f^\lambda)'(u)](v) = \int_{R^N} (\nabla u \cdot \nabla v + \sigma(x)uv)\,dx - \int_{R^N} (\lambda uv + |u|^{p-2}uv + fv)\,dx,$$

where "$\cdot$" denotes the inner product in $\mathbb{R}^N$. Hence, the critical points of f are weak solutions of the equation contained in (P_f^λ). Moreover, by the Harnack inequality (see [10, Theorem 8.17]) they satisfy the asymptotic boundary condition $u \to 0$ as $|x| \to \infty$.

First, let us consider the case $f = 0$. In order to find the critical points of I_0^λ, the following lemma will be crucial. In the sequel, we will denote by c_i some suitable positive constants.

Lemma 2.2. *The functional I_0^λ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$, i.e. any sequence $\{u_n\}$ in X such that*

$$I_0^\lambda(u_n) \to c \quad \textit{and} \quad (I_0^\lambda)'(u_n) \to 0 \quad \textit{as } n \to \infty \tag{2.1}$$

has a converging subsequence.

Proof. Let $\{u_n\}$ be a sequence in X verifying (2.1), that is there exist two infinitesimal sequences $\{\varepsilon_{1,n}\}$ and $\{\varepsilon_{2,n}\}$ such that

$$\frac{1}{2}\|u_n\|^2 - \frac{\lambda}{2}|u_n|_2^2 - \frac{1}{p}|u_n|_p^p = c + \varepsilon_{1,n} \tag{2.2}$$

and

$$\|u_n\|^2 - \lambda|u_n|_2^2 - |u_n|_p^p = \varepsilon_{2,n}\|u_n\|. \tag{2.3}$$

Multiplying (2.3) by $-1/p$ and adding to (2.2), we have

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 - \lambda\left(\frac{1}{2} - \frac{1}{p}\right)|u_n|_2^2 = c + \varepsilon_{1,n} - \frac{\varepsilon_{2,n}}{p}\|u_n\|.$$

If $\lambda < \lambda_1$, λ_1 being the first eigenvalue of $-\triangle+\sigma$, it follows that $\|u_n\|$ is bounded. Now, assume $\lambda \geq \lambda_1$. Multiplying (2.3) by $-1/2$ and adding to (2.2), we obtain

$$|u_n|_p^p \leq c_1(1 + \|u_n\|). \tag{2.4}$$

Following the Proposition 2.1, denote by $\{\varphi_n\}$ the eigenfunctions corresponding to the eigenvalues $\{\lambda_n\}$ of $-\triangle+\sigma$ in $L^2(\mathbb{R}^N)$. If n_0 is the first integer such that $\lambda_{n_0} > \lambda$, consider

$$V_{n_0}^+ = \operatorname{span}\{\varphi_n : n \geq n_0\} \quad \text{and} \quad V_{n_0}^- = \operatorname{span}\{\varphi_n : n < n_0\}.$$

Clearly, it is $X = V_{n_0}^+ \oplus V_{n_0}^-$ and there exists $\lambda_+ > 0$ such that for any $u \in V_{n_0}^+$ it results

$$\|u^+\|^2 - \lambda |u^+|_2^2 \geq \lambda_+ \|u^+\|^2.$$

Hence, for all $n \in \mathbb{N}$ we can write $u_n = u_n^- + u_n^+$. Since by (2.1) it is

$$[(I_0^\lambda)'(u_n)](u_n^+) \leq \varepsilon_{2,n} \|u_n^+\|,$$

Hölder inequality and (2.4) imply

$$\lambda_+ \|u_n^+\|^2 \leq \int_{R^N} |u_n|^{p-1} u_n^+ \, dx + \varepsilon_{2,n} \|u_n^+\| \leq c_2 (1 + \|u_n\|)^{1-1/p} \|u_n^+\|$$

and therefore

$$\|u_n^+\| \leq c_3 (1 + \|u_n\|)^{1-1/p}. \tag{2.5}$$

Moreover, as X^- has finite dimension, by (2.4) and (2.5) we have

$$\|u_n^-\| \leq c_4 |u_n^-|_p \leq c_4 (|u_n|_p + |u_n^+|_p) \leq c_5 (1 + \|u_n\|)^{1-1/p}.$$

Then, it follows that $\|u_n\|$ is bounded even in the case $\lambda > \lambda_1$. Hence, passing to a subsequence, $u_n \rightharpoonup u$ in X. Since X is compactly embedded in $L^s(\mathbb{R}^N)$, $s \in [2, 2^*[$,

$$u_n \to u \quad \text{in } L^2(\mathbb{R}^N) \text{ and in } L^p(\mathbb{R}^N).$$

As the sequence

$$[(I_0^\lambda)'(u_n) - (I_0^\lambda)'(u)](u_n - u) = \|u_n - u\|^2 - \lambda |u_n - u|_2^2 - |u_n - u|_p^p$$

tends to zero as $n \to \infty$, we conclude that $u_n \to u$ in X. $\square$

Proof of Theorem 1.1. Our aim is to prove that the functional I_0^λ satisfies all the assumptions of the symmetric Mountain Pass Theorem in [2]. Let us suppose $\lambda \geq \lambda_1$ (otherwise, the proof is simpler). Clearly, I_0^λ is even, $I_0^\lambda(0) = 0$, and, by Lemma 2.2, it verifies the Palais–Smale condition. Now, if $V_{n_0}^+$ is the vector space introduced in Lemma 2.2, it results

$$I_0^\lambda(u) \geq \frac{1}{2} \lambda_+ \|u\|^2 - \frac{1}{p} |u|_p^p \geq \frac{1}{2} \lambda_+ \|u\|^2 - c_5 \|u\|^p,$$

and therefore there exist $\rho > 0$, $\alpha > 0$ such that, for $u \in V_{n_0}^+$, $\|u\| = \rho$, it is $I_0^\lambda(u) \geq \alpha$. Let us point out that if $\lambda < \lambda_1$ it is $X^- = \{0\}$ and $X^+ = X$, then the previous inequality hold for any $u \in X$ with $\|u\| = \rho$ small enough.

Moreover, for any finite dimensional subspace $W \subset X$ it is

$$I_0^\lambda(u) \leq \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} |u|_2^2 - \frac{1}{p} |u|_p^p \to -\infty \quad \text{as } \|u\| \to \infty,$$

hence, there exists $R = R(W)$ such that $I_0(u) \leq 0$ for $u \in W$, $\|u\| \geq R$. Then, by the symmetric Mountain Pass Theorem I_0^λ possesses an unbounded sequence of critical values. Clearly, the form of I_0^λ implies that also $\|u_n\| \to \infty$.

Finally, assume $\lambda = 0 = f$. Using the homogeneity of the problem, it is easy to prove that a solution of (P_0^0) can be obtained also by minimizing the functional

$$J(u) = \frac{1}{2} \int\limits_{R^N} (|\nabla u|^2 + \sigma(x) u^2)\, dx$$

on the set

$$M = \bigg\{ u \in X : \int\limits_{R^N} |u|^p \, dx = 1 \bigg\}.$$

By Proposition 2.1, M is a weakly closed subset of X then, by the Weierstrass Theorem, J attains its minimum at a point $\overline{u} \in M$. Since $J(\overline{u}) = J(|\overline{u}|)$, we can assume $\overline{u} \geq 0$. Obviously, $-\overline{u}$ gives a negative solution of the problem. □

3. The fibering method and an alternative proof of Teorem 1.1

We recall the fibering method introduced by Pohozaev (see [11]–[13]), more precisely we present the particular case of the "spherical fibering" we shall use in the sequel.

Let Y be a real Banach space with a norm which is differentiable for $w \neq 0$, and let E be a functional on Y of class $C^1(Y/\{0\})$. We associate with E a functional $\widetilde{E}$ defined on $\mathbb{R} \times Y$ by

$$\widetilde{E}(t, v) = E(tv).$$

Denoted by S the unit sphere in Y, the following result holds (see [13, Theorem 1.2.1]):

Theorem 3.1. *Let Y be a real Banach space with norm differentiable on $Y/\{0\}$, and let $(t, v) \in (\mathbb{R}/\{0\}) \times S$ be a conditionally stationary point of the functional $\widetilde{E}$ on $\mathbb{R} \times S$. Then, the vector $u = tv$ is a stationary point of the functional E, that is $E'(u) = 0$.*

The previous theorem says that any critical point (t, v) of $\widetilde{E}$ restricted on $(\mathbb{R}/\{0\}) \times S$ generates the critical point $u = tv$ of E and vice-versa, that is the equation $\widetilde{E}'(u) = 0$, $u \neq 0$ is equivalent to the system

$$\begin{cases} \widetilde{E}_t'(t, v) = 0, \\ \widetilde{E}_v'(t, v) = 0, \end{cases}$$

for $\|v\| = 1$.

In the following we will call the first scalar equation of the previous system the "bifurcation equation".

Now, we will apply this theorem in order to give an alternative and simpler proof of Theorem 1.1 if $\lambda = 0$. Moreover, we shall examine the case $f \neq 0$ (and $\lambda = 0$) (see [11]–[13] for similar results in bounded domains).

Alternative proof of Theorem 1.1 in the case $\lambda = 0$. According to the spherical fibering method, we look for the critical points $u \in X$ of the even functional I_0^0 in the form

$$u = tv \quad \text{where } t \in \mathbb{R} \text{ and } v \in X, \|v\| = 1.$$

Then, the functional I_0^0 can be extended to the space $\mathbb{R} \times X$ as follows

$$\widetilde{I}_0^0(t,v) = \frac{t^2}{2} \int_{R^N} (|\nabla v|^2 + \sigma(x)v^2)\,dx - \frac{|t|^p}{p} \int_{R^N} |v|^p\,dx.$$

Choosen $\|v\| = \int_{R^N} (|\nabla v|^2 + \sigma(x)v^2)\,dx = 1$, the functional $\widetilde{I}_0^0$ becomes

$$\widetilde{I}_0^0(t,v) = \frac{t^2}{2} - \frac{|t|^p}{p} \int_{R^N} |v|^p\,dx.$$

By the bifurcation equation

$$\frac{\partial \widetilde{I}_0^0}{\partial t}(t,v) = t - |t|^{p-2}t|v|_p^p = 0$$

we obtain two nontrivial solutions

$$t_{\pm}(v) = \pm\biggl(\int_{R^N} |v|^p\,dx\biggr)^{-1/(p-2)},$$

then, the functional $\widehat{I}_0^0(v) = \widetilde{I}_0^0(t(v),v)$ on the unit sphere S of X reduces to

$$\widehat{I}_0^0(v) = \frac{p-2}{2p}\biggl(\int_{R^N} |v|^p\,dx\biggr)^{-2/(p-2)}.$$

Since $\widehat{I}_0^0$ is bounded from below and weakly continuous on $B - \{0\}$, $B = \{v \in X : \|v\| \leq 1\}$, clearly it attains its minimum on $B - \{0\}$ at a point $\overline{v} \in S$. As

$$\widehat{I}_0^0(|\overline{v}|) = \widehat{I}_0^0(\overline{v}),$$

according to the fibering method it results that

$$u_+ = t_+(|\overline{v}|)|\overline{v}| \quad u_- = t_-(|\overline{v}|)|\overline{v}|,$$

are a positive and a negative solution of the problem (P_0^0).

Moreover, by applying the classical Lusternik–Schnirelmann theory we prove that $\widehat{I}_0^0$ has a countable set of geometrical different conditionnally critical points $v_1,\dots,v_n,\dots$ on S with $\widehat{I}_0^0(v_n) \to \infty$ (and therefore $|v_n|_p \to 0$) as $n \to \infty$). Hence, the Theorem 3.1 implies that the problem (P_0^0) has a countable set of geometrically different solutions $\pm u_1,\dots,\pm u_n,\dots$ with

$$u_n(x) = \frac{v_n(x)}{(\int_{R^N} |v_n|^p\,dx)^{1/(p-2)}} \quad \text{and} \quad \|u_n\| \to \infty \text{ as } n \to \infty. \qquad \square$$

Remark 3.2. The previous arguments work in the same way if we assume $\lambda \neq 0$ but small enough, more precisely $\lambda < \lambda_1$, λ_1 denoting the first eigenvalue of $-\triangle + \sigma$ in $L^2(\mathbb{R}^N)$.

4. The case without symmetry

Proof of Theorem 1.2. Let us consider the case $f \neq 0$ and $\lambda = 0$. Then, the functional I_f^0 extended to the space $\mathbb{R} \times X$ becomes

$$\widetilde{I}_f^0(t,v) = \frac{t^2}{2}\|v\|^2 - \frac{|t|^p}{p}|v|_p^p - t\int\limits_{R^N} fv\,dx$$

and its restriction to the unit sphere S in X is

$$\widetilde{I}_f^0(t,v) = \frac{t^2}{2} - \frac{|t|^p}{p}|v|_p^p - t\int\limits_{R^N} fv\,dx.$$

Hence, the bifurcation equation

$$\frac{\partial \widetilde{I}_f^0}{\partial t}(t,v) = t - |t|^{p-2}t|v|_p^p - \int\limits_{R^N} fv\,dx = 0 \tag{4.1}$$

has at least three different rooths $t_i(v)$, $i = 1,2,3$, if f is small enough. Indeed, setting $\varphi(t) = t - |t|^{p-2}t|v|_p^p$, direct calculations give that

$$\varphi'(t) = 0 \text{ if and only if } t = \pm((p-1)|v|_p^p)^{-1/(p-2)}$$

and φ has a local maximum M and a local minimum $m = -M$ with

$$M = \varphi((p-1)|v|_p^p)^{-1/(p-2)} = (p-2)(p-1)^{-(p-1)/(p-2)}|v|_p^{-p/(p-2)}.$$

Clearly, the equation (4.1) has three distinct rooths if

$$\bigg|\int\limits_{R^N} fv\,dx\bigg| < M,$$

and therefore, assumed

$$\sup_{v\in S}\bigg\{\bigg|\int\limits_{R^N} fv\,dx\bigg|\bigg(\int_{R^N}|v|^p\,dx\bigg)^{1/(p-2)}\bigg\} < (p-2)(p-1)^{-(p-1)/(p-2)}, \tag{4.2}$$

the bifurcation equation possesses three isolated smooth branches of solutions $t_i = t_i(v)$, $i = 1,2,3$. Hence, we obtain three induced functionals

$$\widehat{I}_{f,i}^0(v) = \widetilde{I}_f^0(t_i(v),v) = \frac{1}{2}t_i^2(v) - \frac{|t_i(v)|^p}{p}|v|_p^p - t_i(v)\int\limits_{R^N} fv\,dx,$$

which are distinct and defined on $B - \{0\}$.

Now, we will prove that, for $i = 1, 2, 3$, $\widehat{I}^0_{f,i}$ attains its minimum at a point $\overline{v}_i \in S$ such that $t_i(\overline{v}_i) \neq 0$. Indeed, given a minimizing sequence $\{v_{n,i}\} \subset S$ for the functional $\widehat{I}^0_{f,i}$ on S, there exists $\overline{v}_i \in B$ such that, passing to a subsequence,

$$v_{n,i} \rightharpoonup \overline{v}_i \quad \text{in } X.$$

Since $\widehat{I}^0_{f,i}(v)$ is weakly continuous in B, we deduce

$$\widehat{I}^0_{f,i}(\overline{v}_i) = \inf_{v \in S} \widehat{I}^0_{f,i}(v) < 0$$

and therefore $t_i(\overline{v}_i) \neq 0$. In order to prove that $\overline{v}_i \in S$, we calculate

$$\begin{aligned} \frac{d}{d\theta}\widehat{I}^0_{f,i}(\theta v) = &\bigg(t_i(\theta v) - |t_i(\theta v)|^{p-2} t_i(\theta v)|\theta v|^p_p - \int\limits_{R^N} f\theta v\, dx\bigg)\frac{d}{d\theta} t_i(\theta v) \\ &- \frac{1}{\theta}|t_i(\theta v)|^p|\theta v|^p_p - t_i(\theta v)\int\limits_{R^N} f v\, dx \end{aligned}$$

and, using the bifurcation equation, we deduce

$$\frac{d}{d\theta}\widehat{I}^0_{f,i}(\theta v) = -\frac{1}{\theta} t_i^2(\theta v) < 0 \quad \text{for any } \theta > 0.$$

Then, if $\|\overline{v}_i\| < 1$, by choosing $\overline{\theta}_i = \|\overline{v}_i\|^{-1/p} > 1$, it follows that $\overline{\theta}_i\overline{v}_i \in S$ and we have the contradiction

$$\widehat{I}^0_{f,i}(\overline{\theta}_i\overline{v}_i) < \widehat{I}^0_{f,i}(\overline{v}_i) = \inf_{v \in S} \widehat{I}^0_{f,i}(v).$$

By the Theorem 3.1 it follows that the original action functional I^0_f has at least three critical points of the form

$$\overline{u}_1(x) = t_1\overline{v}_1(x), \quad \overline{u}_2(x) = t_2\overline{v}_2(x), \quad \overline{u}_3(x) = t_3\overline{v}_3(x).$$

Moreover, as the sign of $t_i(\overline{v}_i)$ depends on the sign of $\int\limits_{R^N} f\overline{v}_i dx$, it is

$$\int\limits_{R^N} f\overline{v}_1 dx \le 0, \quad \int\limits_{R^N} f\overline{v}_2 dx \ge 0, \quad \int\limits_{R^N} f\overline{v}_3 dx \ge 0. \qquad \square$$

Remark 4.1. Let us point out that a direct application of the Hölder inequality and Proposition 2.1 implies that the inequality (4.2) is satisfied if $|f|_2$ is small enough, the estimate for $|f|_2$ depending on p and on the constants of the imbeddings of X in $L^2(\mathbb{R}^N)$ and in $L^p(\mathbb{R}^N)$. A similar estimate on f has been found in [19] in order to prove the existence of two solutions in bounded domains both in the subcritical and in the critical case.

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