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$SO(3) \times S^1$-EQUIVARIANT DEGREE WITH APPLICATIONS TO SYMMETRIC BIFURCATION PROBLEMS: THE CASE OF ONE FREE PARAMETER

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Abstract. Abstract The reduced equivariant degree for $G = SO(3) \times S^1$ is introduced and studied in the case of one free parameter equivariant maps. Computational and multiplication tables for the reduced $SO(3)\times S^1$-equivariant degree are presented together with an application to an $SO(3)$-symmetric Hopf bifurcation problem. A method for classification of $SO(3)$-symmetric bifurcations is established.

1. Introduction

1.1. Goal. The goal of this paper is to study the equivariant degree introduced by J. Ize et al. (cf. [13]–[17]) in the case of maps with one free parameter and respecting $SO(3) \times S^1$-symmetry.

Roughly speaking, the equivariant degree "measures" homotopical obstructions for an equivariant map to have an equivariant extension without zeros on the set composed of several orbit types. To be more specific, assume G is a compact Lie group, V is a finite-dimensional G-representation, $\Omega \subset \mathbb{R}^n \oplus V$ is open

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bounded and invariant (G is supposed to act trivially on $\mathbb{R}^n$), $f:\Omega \to V$ is an equivariant map and $f(x) \neq 0$ as $x \in \partial\Omega$. The equivariant degree $\deg_G(f,\Omega)$ is defined as an element of the group of equivariant homotopy classes of maps from $S(\mathbb{R}^{m+n} \oplus V)$ to $S(\mathbb{R}^m \oplus V)$ for m big enough (here $S(\,\cdot\,)$ stands for the unit sphere). Usually one distinguishes two parts of $\deg_G(f,\Omega)$: *primary and secondary* (they require different computation techniques). The primary part is related to topological obstructions for non-zero extension of f around the orbits from $f^{-1}(0)$ of type (H) with $\dim W(H) = n$ and requires Brouwer degree methods for its computation (see, for instance, [8], [21], [25], [26]; here $W(H)$ stands for the Weyl group of H). The orbits of type (H) with $\dim W(H) < n$ give rise to the secondary part of $\deg_G(f,\Omega)$ that requires either secondary obstruction geometric techniques (see [13]–[17]) or bordism theory (see [1]).

In particular, to compute the primary part of the $SO(3) \times S^1$-degree in the one-parameter case we use the so called Ulrich type formulae reducing the computations to the case of S^1-degree (see [20], [21], [27] and [30] for details). To compute the secondary part we use the main result from [1] where the general structure of a secondary obstructions equivariant homotopy group is given (the acting group is finite). We also refer to [13]–[17] where the case of abelian group actions (both for primary and secondary parts) is studied in detail.

In fact, we exclude the "non-orientable" part of the primary degree (defined (mod 2)) as we do not have effective tools for its computations. We call the remaining "orientable" part of the primary degree together with the modulo $\mathbb{Z}_2$ secondary part the *reduced* $SO(3) \times S^1$*-equivariant degree*.

An extremely important feature of the "orientable" primary $SO(3) \times S^1$-equivariant degree is the so called *multiplicativity property*. This property permits to evaluate the corresponding primary degrees of "basic maps" (related to individual isotypical components) and next to multiply them in the sense similar to the multiplication in a Burnside ring (see, for instance, [22], [29]).

1.2. Applications to Hopf and steady-state bifurcations. Let V be an orthogonal $SO(3)$-representation (possibly reducible) and $f:\mathbb{R} \oplus V \to V$ an $SO(3)$-equivariant C^1-map such that $f(\alpha,0) = 0$ for all $\alpha \in \mathbb{R}$. We consider the following one-parameter autonomous system of ODE's

$$
\begin{cases} x' = f(\alpha,x) & \text{for } x \in V, \\ x(0) = x(p) & \text{for a certain (unknown) } p > 0. \end{cases} \tag{1.1}
$$

We assume that $(\alpha_0,0)$ is an isolated center of the linearized system (1.1) at zero such that 0 is not a characteristic value of the linearized equation, i.e. the steady state solution $(\alpha_0,0)$ is isolated in $\mathbb{R} \oplus V$. If, in addition, there is a change of the stability of the stationary solutions $(\alpha,0)$ as α crosses α_0, a bifurcation of small amplitude periodic solutions takes place (the so called Hopf bifurcation). By the standard argument utilizing the time-shift symmetry, problem (1.1) (which in fact depends on two parameters α and p) can be reduced to an equation having one free parameter and respecting $SO(3) \times S^1$-symmetries (see, for instance, [11], [13]). The multiplicativity property allows us to compute the bifurcation invariants expressed by the primary degree and to use them to classify and analyze possible symmetries of the bifurcating solutions.

We also present a simple example of a steady-state bifurcation with $SO(3)$-symmetries, to which we apply our results related to the *secondary part* of the reduced degree to classify the symmetries of bifurcating stationary solutions.

We should also mention other methods for studying equivariant bifurcation problems, like the method of Fiedler (cf. [7]) based on the reduction to the fixed-point subspaces, or the method of Golubitsky et al. (cf. [9]–[12]) using the normal forms and equivariant classification of singularities (see [28]).

1.3. Overview. After the Introduction the paper is organized as follows. In Section 2 we fix terminology and present some known facts from the equivariant degree theory. In addition, combining the main results from [1] and Theorem 8.5 from [16], we compute secondary obstruction groups for abelian group actions in the case of one free parameter (under reasonable simply-connectness conditions).

In Section 3 we collect the necessary information on irreducible $SO(3) \times S^1$-representations (denoted by $\mathcal{V}_l^{c,m}$). In Section 4 we compute secondary obstruction groups related to $SO(3) \times S^1$-equivariant degree. Section 5 contains computations of the reduced $SO(3) \times S^1$-equivariant degree for certain "basic" equivariant maps $\mathbb{C} \oplus \mathcal{V}_l^{c,m} \to \mathbb{R} \oplus \mathcal{V}_l^{c,m}$ (for both, primary and secondary parts) that naturally appear in Hopf and steady-state bifurcation problems. Section 6 is devoted to the above mentioned multiplicativity property. In particular, we present several multiplication tables for the subgroups of $SO(3) \times S^1$ which are used in computations of the primary equivariant degree. Finally, in Section 7 we study problem (1.1) by combining the obtained "basic" map results for primary degree from Section 5 with the multiplication results from Section 6. In addition, we consider an example of two parameters $SO(3)$-symmetric steady-state bifurcation that is studied using our secondary degree results from Section 5.

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2. Preliminaries on equivariant degree theory

2.1. Equivariant jargon. We will recall the equivariant jargon frequently used throughout this paper.

Let G be a compact Lie group. Two closed subgroups H and K of G are *conjugate* (denoted by $H \sim K$) if there exists $g \in G$ such that $K = gHg^{-1}$. The relation $\sim$ is an equivalence relation. The equivalence class of H is called *conjugacy class* of H in G and will be denoted by (H). We denote by $O(G)$ the set of all conjugacy classes of closed subgroups of G. The set $O(G)$ is partially ordered: $(H) \geq (K)$ if and only if K is conjugate to a subgroup of H. For a closed subgroup H of G, we use $N(H)$ to denote the *normalizer* of H in G, and $W(H)$ to denote the *Weyl group* $N(H)/H$ of H in G. The group G is called *bi-orientable* if it has an orientation which is invariant under all left and right translations.

Let G act on a finite-dimensional manifold M and $x \in M$. We denote by $G_x := \{g \in G : gx = x\}$ the *isotropy group* of x. The conjugacy class (G_x) will be called the *orbit type* of x. For an invariant subset $X \subset M$, a closed subgroup H of G and an orbit type $\alpha \in O(G)$ we put

$$\begin{aligned}
X^H &:= \{x \in X : hx = x \text{ for all } h \in H\},\\
X_H &:= \{x \in X : G_x = H\},\\
X^{[H]} &:= X^H \setminus X_H,\\
X^\alpha &:= \{x \in X : (G_x) \geq \alpha\},\\
X_\alpha &:= \{x \in X : (G_x) = \alpha\}.
\end{aligned}$$

It is well-known (see, for instance, [18], [29]) that for every orbit type α in M, the set M_α is an invariant submanifold of M. We will denote by TM_α the tangent bundle to M_α and by $T_xM_{(G_x)}$ the tangent space to $M_{(G_x)}$ at $x \in M$.

2.2. Equivariant degree: construction. For a Euclidean space U denote by $S(U)$ the unit sphere in U and by $B(U)$ the closed unit ball in U.

Denote by V an orthogonal representation of a compact Lie group G. We will be using a slightly modified definition of the equivariant degree, originally introduced by Ize et al. (cf. [14]), in the case of equivariant maps $f\colon \mathbb{R}^n \oplus V \to V$ for some $n \geq 0$ (G is supposed to act trivially on $\mathbb{R}^n$).

It is well known (see [14]) that for $N > 1$ the set

$$\Pi_N := [B(\mathbb{R}^{N+n} \oplus V), S(\mathbb{R}^{N+n} \oplus V); \mathbb{R}^N \oplus V, (\mathbb{R}^N \oplus V) \setminus \{0\}]^G$$

of equivariant homotopy classes, has a natural structure of an abelian group. Moreover, the standard m-th suspension homomorphism $\xi_m\colon \Pi_N \to \Pi_{N+m}$, by an equivariant version of the Freudenthal Theorem, is an isomorphism for $N \geq n+3$ (cf. [14]). We define $\Pi^G := \Pi_N$ for $N \geq n+3$. Assume that $\Omega \subset \mathbb{R}^n \oplus V$ is a bounded invariant open set and $f\colon \mathbb{R}^n \oplus V \to V$ an equivariant map such that $f(x) \neq 0$ for $x \in \partial\Omega$ (such a map will be called Ω-*admissible*). It is clear that there exists an invariant neighbourhood $\mathcal{N}$ of $\partial\Omega$ such that $f(x) \neq 0$ for all $x \in \mathcal{N}$. Let $\Omega_{\mathcal{N}} := \Omega \cup \mathcal{N}$ and $R > 0$ be such that $\overline{\Omega_{\mathcal{N}}}$ is contained in the interior of the ball $B_R(0) := \{x : \|x\| \leq R\}$. Choose an invariant Urysohn function

$\eta : B_R(0) \to \mathbb{R}$ such that

$$\eta(x) = \begin{cases} 0 & \text{if } x \in \Omega, \\ 1 & \text{if } x \notin \Omega_{\mathcal{N}}, \end{cases}$$

and define $F\colon([-1,1]\times B_R(0), \partial([-1,1]\times B_R(0))) \to (\mathbb{R}\oplus V, (\mathbb{R}\oplus V)\setminus\{0\})$ by

$$F(t,x) = (t+2\eta(x), f(x)), \quad (t,x) \in [-1,1]\times B_R(0).$$

Since $[-1,1]\times B_R(0)$ is equivariantly homeomorfic to $B(\mathbb{R}^{n+1}\oplus V)$, F defines an equivariant homotopy class in Π_1. We put

$$\deg_G(f,\Omega) := \xi_{n+3}[F] \in \Pi^G,$$

and we call it the *equivariant* (or *G-equivariant*) *degree* of f in Ω. This degree has all standard properties like *existence, additivity, homotopy* and *suspension* properties (see [13]–[17]).

The following notion of *normality* is useful in computations of the equivariant degree. We say that an equivariant map $F\colon\mathbb{R}^{N+n}\oplus V \to \mathbb{R}^N\oplus V$ is *normal* on an open set $\Omega\subset\mathbb{R}^{N+n}\oplus V$ if

$$\forall_{x\in F^{-1}(0)\cap\Omega}\ \exists_{\delta>0}\ \forall_{v\perp T_x\Omega_{(G_x)}}\ \|v\|<\delta \Rightarrow F(x+v)=v,$$

(see [8], [24], [25] for more details). In addition, F is called *regular normal* in Ω if it is normal, of class C^1 in Ω, and for every orbit type (H) in Ω the restricted map $F_{|\Omega_H}\colon\Omega_H\to\mathbb{R}^N\oplus V^H$ has zero as a regular value.

For a bounded open invariant set Ω, the set of regular normal maps on Ω is dense in the space of continuous G-maps (cf. [24]), thus every equivariant map $F\colon\mathbb{R}^{N+n}\oplus V\to\mathbb{R}^N\oplus V$ can be approximated by a regular normal map on Ω. As a consequence, we obtain that

$$\Pi^G = \bigoplus_{\dim W(H)\le n} \Pi(H), \tag{2.1}$$

where $\Pi(H)$ is a subgroup of elements having a normal representative with only zeros of the orbit type (H) (see [1]). Consequently we have

$$\deg_G(f,\Omega) = \sum_{(H)} n_{(H)}, \tag{2.2}$$

where $n_{(H)}$ stands for the component of the degree $\deg_G(f,\Omega)$, belonging to $\Pi(H)$.

It is an immediate consequence that if in formula (2.2) $n_{(H)}\neq 0$ for some (H) (we use the additive notation for the group $\Pi(H)$), then there exists $x\in\Omega$ such that $f(x)=0$ and $(G_x)\ge(H)$.

2.3. Equivariant degree: primary part. We denote by $\Phi_n^+(G)$ (or simply $\Phi_0(G)$ if $n = 0$) the set of all orbit types (H) such that the Lie group $W(H)$ is *bi-orientable* and $\dim W(H) = n$, and for a subset $X \subset V$ we put $\Phi_n^+(G, X)$ to denote the set of all orbit types (H) in X such that $(H) \in \Phi_n^+(G)$. It is well-known (see [8], [16], [17], [25], [26]) that $\Pi(H)$ is isomorphic to $\mathbb{Z}$ for $(H) \in \Phi_n^+(G, V)$. Choosing an invariant orientation on $W(H)$ is equivalent to choosing a generator in $\Pi(H)$ (see [8] for more details). Thus for each $(H) \in \Phi_n^+(G, V)$ the element $n_{(H)}$ from formula (2.2) can be written as $n_H \cdot (H)$ with $n_H \in \mathbb{Z}$.

Definition 2.1. If $f\colon \mathbb{R}^n \oplus V \to V$ is an Ω-admissible map then

$$G\text{-Deg}\,(f, \Omega) := \sum_{(H)\in\Phi_n^+(G,V)} n_H \cdot (H) \in \Pi^G \tag{2.3}$$

is called *primary degree* of f in Ω.

The primary degree was introduced (using a different construction) by Gęba et al. (cf. [8]).

Observe that in the general case of an arbitrary $n > 0$ the computation of the primary degree is a complicated task. In what follows we will be studying the case where $n = 1$, i.e. the case of one-parameter bifurcation. We will also be focused on the case where $G = \Gamma \times S^1$ for $\Gamma = SO(3)$ but we should mention that some of our results are also valid for an arbitrary compact Lie group G.

In the case $n = 1$, there is an effective way to reduce the computations of G-Deg (f, Ω) to computations of the S^1-degree using the so called *Ulrich type formula*. To be more specific, let V be a finite-dimensional S^1-representation, $\Omega \subset \mathbb{R} \oplus V$ a bounded open S^1-invariant subset, $f\colon \mathbb{R} \oplus V \to V$ an Ω-admissible S^1-equivariant map. Then $S^1\text{-Deg}\,(f, \Omega) = \sum_k n_{\mathbb{Z}_k} \cdot (\mathbb{Z}_k)$. Set $n_{\mathbb{Z}_1} = I_{S^1}(\pi_V - f)_{\mathbb{Z}_1}$, where I_{S^1} denotes the S^1-*fixed-point index* associated with the S^1-degree and π_V is the projection on V.

Assume that G is a compact Lie group, V is a finite-dimensional G-representation, $\Omega \subset \mathbb{R} \oplus V$ a bounded open G-invariant subset and $f\colon \mathbb{R} \oplus V \to V$ an Ω-admissible G-equivariant map. Let $(H) \in \Phi_1^+(G, V)$. Then S^1 can be canonically identified with the connected component of $1 \in W(H)$. Notice that the restriction $f^H\colon \mathbb{R} \oplus V^H \to V^H$ (respectively, $f^{[H]}\colon \mathbb{R} \times V^{[H]} \to V^{[H]}$) is $W(H)$-equivariant and Ω^H-admissible (respectively, $\Omega^{[H]}$-admissible). In particular, $I_{S^1}(F^H)_{\mathbb{Z}_1}$ (respectively, $I_{S^1}(F^{[H]})_{\mathbb{Z}_1}$) is correctly defined, where $F^H = \pi_V - f^H\colon \mathbb{R} \oplus V^H \to V^H$ (respectively, $F^{[H]} = \pi_V - f^{[H]}\colon \mathbb{R} \times V^{[H]} \to V^{[H]}$).

Theorem (see [20]). *Under the above assumptions each n_H in (2.3) can be computed according to the formula:*

$$n_H = [I_{S^1}(F^H)_{\mathbb{Z}_1} - I_{S^1}(F^{[H]})_{\mathbb{Z}_1}] \Big/ \left|\frac{W(H)}{S^1}\right|,$$

where $|W(H)/S^1|$ denotes the number of connected components in $W(H)$.

2.4. Equivariant degree: secondary part. Return to the decomposition (2.1). Under the assumption $n = 1$, the summands $\Pi(H)$, with $\dim W(H) < n$, i.e. in the case when $W(H)$ is finite, can be effectively studied (see [16] for an abelian G and [1] in the general case). More precisely (see [1]), we have the following exact sequence of groups:

$$(2.4)\quad 0 \to \mathbb{Z}_2 \to \Pi(H) \to H_1(\mathcal{R}_H/W(H)) = \frac{\pi_1(\mathcal{R}_H/W(H))}{[\pi_1(\mathcal{R}_H/W(H)), \pi_1(\mathcal{R}_H/W(H))]} \to 0,$$

where $\mathcal{R}_H = (\mathbb{R}^{N+1} \oplus V)_H$, $N \geq 4$, $H_1(*)$ denotes the functor of the first singular homology, and $[\mathcal{A}, \mathcal{A}]$ stands for the commutator group of $\mathcal{A}$. In the case where the set $\mathcal{R}_H$ is simply connected we obtain (see [1]) that $\pi_1(\mathcal{R}_H/W(H)) = W(H)$, therefore

$$H_1(\mathcal{R}_H/W(H)) = W(H)/[W(H), W(H)]$$

and consequently the exact sequence (2.4) can be written as

$$0 \to \mathbb{Z}_2 \to \Pi(H) \to W(H)/[W(H), W(H)] \to 0,$$

so, if the Lie Group G is abelian, we have the following exact sequence

$$(2.5)\qquad 0 \to \mathbb{Z}_2 \to \Pi(H) \to G/H \to 0.$$

The following result obtained by Ize et al. (see [16]) in the case of an abelian group G, describes the summands $\Pi(H)$ with $W(H) = G/H$ finite.

Theorem 2.3 (see [16, Theorem 8.5]). *If G is abelian, $(\mathbb{R}^{N+1} \oplus V)_H$ is simply connected and $G/H = \mathbb{Z}_{p_1} \oplus \ldots \oplus \mathbb{Z}_{p_m}$, then*

$$\Pi(H) = \mathbb{Z}_{q_0} \oplus \ldots \oplus \mathbb{Z}_{q_m},$$

where $q_0 = \gcd(p_0, \ldots, p_m)$ with $p_0 = 2$, $q_m = \operatorname{lcm}(p_0, \ldots, p_m)$, and for $0 < j < m$, $q_j = h_j/h_{j-1}$, where h_j is the largest common factor of all possible products of $j+1$ numbers from $\{p_0, \ldots, p_m\}$.

Corollary 2.4. *Under the assumptions of Theorem* 2.3*, the exact sequence* (2.5) *splits and we have*

$$\Pi(H) = \mathbb{Z}_2 \oplus G/H.$$

Proof. Consider the Sylov decomposition of the group G/H

$$(2.6)\qquad G/H = \mathbb{Z}_{2^{k_1}} \oplus \ldots \oplus \mathbb{Z}_{2^{k_r}} \oplus \mathbb{Z}_{\rho_1^{l_1}} \oplus \ldots \oplus \mathbb{Z}_{\rho_s^{l_s}},$$

where $k_1 \leq \ldots \leq k_r$ and ρ_j are primes greater than 2 (some of them can be the same). Denote $\Psi = \mathbb{Z}_{\rho_1^{l_1}} \oplus \ldots \oplus \mathbb{Z}_{\rho_s^{l_s}}$. It follows from (2.5) that $\Psi \subset \Pi(H)$ and $\Pi(H)/\Psi$ contains no element of odd order. We will show, by applying

Theorem 2.3 to the decomposition (2.6), that $\Pi(H)/\Psi$ contains the subgroup $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^{k_1}} \oplus \ldots \oplus \mathbb{Z}_{2^{k_r}}$, from which the required splitting follows. Put

$$p_0 = 2,\ p_1 = 2^{k_1},\ \ldots,\ p_r = 2^{k_r},\ p_{r+1} = \rho_1^{l_1},\ \ldots,\ p_m = \rho_s^{l_s}.$$

By the definition of the numbers h_j (see Theorem 2.3), we have

$$h_0 = 1,\ h_1 = n_1,\ \ldots\ h_{s-1} = n_{s-1},$$
$$h_s = 2 \cdot n_s,\ h_{s+1} = 2^{1+k_1} \cdot n_{s+1},\ \ldots\ h_m = h_{s+r} = 2^{1+k_1+\ldots+k_r} \cdot n_{s+r},$$

where n_j are odd integers. Therefore,

$$q_s = 2 \cdot m_s,\ q_{s+1} = 2^{k_1} \cdot m_{s+1},\ \ldots,\ q_{s+r} = q_m = 2^{k_r} \cdot m_{s+r},$$

where m_j are odd integers. Thus $\Pi(H)/\Psi$ contains $\mathbb{Z}_2 \oplus \mathbb{Z}_{2^{k_1}} \oplus \ldots \oplus \mathbb{Z}_{2^{k_r}}$ and the result follows. $\square$

2.5. Equivariant degree: reduction. Using the same notation as in the previous subsection (compare with (2.4)), we define $\Pi_*(H) := H_1(\mathcal{R}_H/W(H))$. In other words, since $\mathbb{Z}_2$ is a subgroup of $\Pi(H)$, we can simply write

$$\Pi_*(H) = \Pi(H)/\mathbb{Z}_2.$$

Definition 2.5. Let $G = \Gamma \times S^1$ where Γ is a compact Lie group. Set

$$\Pi_*^G := \bigoplus_{(H)\in\Phi_1^+(G,V)} \Pi(H) \oplus \bigoplus_{\dim W(K)=0} \Pi_*(K \times S^1),$$

where K is a subgroup of Γ. Let $\rho\colon \Pi^G \to \Pi_*^G$ be the natural projection. We define $\deg_G^*(f,\Omega) = \rho(\deg_G(f,\Omega))$ and call it the *reduced equivariant degree* of f in Ω.

For an Ω-admissible regular normal map $f\colon \mathbb{R}\oplus V \to V$ and a secondary orbit type (H) in Ω (i.e. $\dim W(H) = 0$), the component $n_H \in \Pi_*(H)$ of the reduced degree $\deg_G^*(f,\Omega)$ can be computed as follows: Let $M := f^{-1}(0) \cap (\mathbb{R} \oplus V)_H$ denote the oriented submanifold consisting of all zeros of f in $\mathcal{R}_H := (\mathbb{R}\oplus V)_H$. Then $M/W(H)$ has also an orientation $o \in H_1(M/W(H))$ (see [4, p. 254]), and $n_H = i_*(o)$, where $i_*\colon H_1(M/W(H)) \to H_1(\mathcal{R}_H/W(H)) =: \Pi_*(H)$ is induced by the inclusion $i\colon M/W(H) \hookrightarrow \mathcal{R}_H/W(H)$.

2.6. Weakly-normal mappings.

Definition 2.6. We will say that a compact Lie group G satisfies the *property* (S_k), where k stands either for 0 or 1, if the following condition is satisfied

(S_k) $\quad (G) > (K) > (H) \Rightarrow (K) \in \Phi_k^+(G) \quad$ for all $(H) \in \Phi_k^+(G)$ and all (K)

Example 2.7. (i) Any finite group G satisfies the property (S_0).

(ii) The only subgroups H of $G = SO(3) \times S^1$ with one-dimensional Weyl group $W(H)$, which are not twisted subgroups (see the next section for more details on twisted subgroups), are $\mathbb{Z}_n \times S^1$, $n \geq 2$. Since $N(\mathbb{Z}_n \times S^1) = O(2) \times S^1$, it follows that $W(\mathbb{Z}_n \times S^1) = O(2)$ is not bi-orientable, thus $\Phi_1^+(G)$ consists only of twisted subgroups $K^{\varphi,m}$, with $(K) \in \Phi_0(SO(3))$. Consequently $SO(3) \times S^1$ satisfies the property (S_1).

(iii) Let Γ be any finite extension of $SO(3)$. Then Γ satisfies the property (S_0). In particular, the groups $SO(3)$, $O(3)$ and $SU(2)$ satisfy the property (S_0).

Remark 2.8. Notice that if Γ satisfies the property (S_0) and every subgroup H of $G = \Gamma \times S^1$ with $(H) \in \Phi_1^+(G)$ is of type $K^{\varphi,m}$, where K is a subgroup of Γ with finite Weyl group $W(K)$, then the group G also satisfies the property (S_1). If $\Gamma = SO(3)$, then we have exactly such a case.

Definition 2.9. Assume that G is a compact Lie group satisfying the property (S_k) for either $k = 0$ or $k = 1$. Let V be an orthogonal representation of G, $\Omega \subset \mathbb{R}^k \oplus V$ an invariant open bounded set and $f\colon \mathbb{R}^k \oplus V \to V$ an equivariant map. We say that f is *weakly normal* in Ω if for every orbit type (H) in Ω such that $(H) \in \Phi_k^+(G,V)$ we have that $f^H\colon \mathbb{R}^k \oplus V^H \to V^H$ is a $W(H)$-equivariant normal map in Ω^H, where f^H is the restriction of f to $\mathbb{R}^k \oplus V^H$.

Remark 2.10. It is clear that every normal map is also weakly normal. However, not every weakly normal map is normal. Intuitively, a weakly normal map is just a map for which the condition of normality is required (in some sense) only for the orbit types from $\Phi_k^+(G,V)$.

Proposition 2.11. *Let V be an orthogonal representation of a compact Lie group G satisfying the propery* (S_k)*, where either $k = 0$ or $k = 1$, $\Omega \subset \mathbb{R}^k \oplus V$ an invariant open bounded set, and $f\colon \mathbb{R}^k \oplus V \to V$ a weakly normal map in Ω. Then there exists a normal in Ω map $\widetilde{f}\colon \mathbb{R}^k \oplus V \to V$ such that for all orbit types (H) in Ω satisfying $(H) \in \Phi_k^+(G)$ one has $f^H \equiv \widetilde{f}^H$.*

Proof. We apply the standard induction over the orbit types in Ω. We extend the partial order of the orbit types in Ω to a linear order in such a way that bi-orientable orbit types are greater than any non-bi-orientable orbit type. For any orbit type (H), we apply the standard induction step as follows. All the zeros $Z_{(H)}$ of f in $\Omega_{(H)}$ form a compact invariant set. We can consider a sufficiently small normal neighbourhood U of the submanifold $\Omega_{(H)}$ near the set $Z_{(H)}$ and consider an invariant Urysohn function φ such that it is zero on a smaller neighbourhood of $Z_{(H)}$ and is one on a neighbourhood of the boundary of U. Then we can define the "corrected" equivariant map $\bar f$ by the formula

$$\overline{f}(x) = \begin{cases} \varphi(z,v)f(z,v) + (1-\varphi(z,v))(f(z,0)+v) & \text{for } x=(z,v)\in U, \\ f(x) & \text{for } x \notin U, \end{cases}$$

where (z,v) are the coordinates in the normal neighbourhood with v representing the normal vector. Notice that in the case of an orbit type $(H)\in\Phi_k^+(G,\Omega)$, due to the assumption that f is weakly normal in Ω, it follows that this "correction" can result in appearance of new zeros only of the orbit types that do not belong to $\Phi_k^+(G,V)$. □

The rest of the paper is devoted to the computations and applications of the reduced equivariant degree in the case $G = SO(3)\times S^1$.

3. Irreducible $SO(3)\times S^1$-representations

3.1. Subgroups of $SO(3)\times S^1$. The subgroups of $SO(3)$, classified up to their conjugacy classes are: $O(2)$, $SO(2)$, D_n, $n\ge 2$ (D_2 is also denoted by V_4 and is called *Klein group*), $\mathbb{Z}_n$, $n\ge 1$, and the exceptional groups A_4 (or $\mathbb{T}$ – the *tetrahedral group*), S_4 (or $\mathbb{O}$ – the *octahedral group*) and A_5 (or $\mathbb{I}$ – the *icosahedral group*). The elements of $\Phi_0(SO(3))$ are $(SO(3))$, $(O(2))$, $SO(2)$, (A_4), (A_5), (S_4), (D_n), $n\ge 3$, and (V_4). Let $G = SO(3)\times S^1$ and V be an orthogonal representation of G.

$H = K^\varphi$	K	$\varphi(K)$	Ker φ	$N(H)$	$W(H)$	Comments
$O(2)$	$O(2)$	$\mathbb{Z}_1$	$O(2)$	$O(2)\times S^1$	S^1	
$SO(2)$	$SO(2)$	$\mathbb{Z}_1$	$SO(2)$	$O(2)\times S^1$	$\mathbb{Z}_2\times S^1$	
D_n	D_n	$\mathbb{Z}_1$	D_n	$D_{2n}\times S^1$	$\mathbb{Z}_2\times S^1$	$n\ge 3$
V_4	V_4	$\mathbb{Z}_1$	V_4	$S_4\times S^1$	$S_3\times S^1$	
S_4	S_4	$\mathbb{Z}_1$	S_4	$S_4\times S^1$	S^1	
A_5	A_5	$\mathbb{Z}_1$	A_5	$A_5\times S^1$	S^1	
A_4	A_4	$\mathbb{Z}_1$	A_4	$S_4\times S^1$	$\mathbb{Z}_2\times S^1$	
$O(2)^-$	$O(2)$	$\mathbb{Z}_2$	$SO(2)$	$O(2)\times S^1$	S^1	
A_4^t	A_4	$\mathbb{Z}_3$	V_4	$A_4\times S^1$	S^1	
S_4^-	S_4	$\mathbb{Z}_2$	A_4	$S_4\times S^1$	S^1	
$SO(2)^k$	$SO(2)$	S^1	$\mathbb{Z}_k$	$SO(2)\times S^1$	S^1	$\varphi(\gamma)=\gamma^k$, $k=1,2,\dots$
D_n^z	D_n	$\mathbb{Z}_2$	$\mathbb{Z}_n$	$D_{2n}\times S^1$	$\mathbb{Z}_2\times S^1$	$n=4,5,6,\dots$
D_{2k}^d	D_{2k}	$\mathbb{Z}_2$	D_k	$D_{2k}\times S^1$	S^1	$k=2,3,\dots$
V_4^-	V_4	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$D_4\times S^1$	$\mathbb{Z}_2\times S^1$	

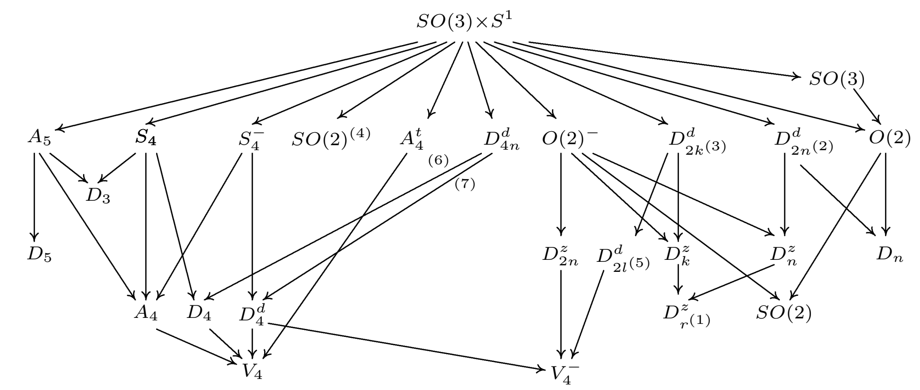
Table 3.1. Twisted subgroups of $SO(3)\times S^1$ ($D_2 = V_4$)

The subgroups of $SO(3) \times S^1$ can be classified as follows: the subgroups of type $H \times S^1$ and the twisted (m-folded) subgroups

$$K^{\varphi,m} := \{(\gamma, z) \in K \times S^1 : \ \varphi(\gamma) = z^m\},$$

where $\varphi: K \to S^1$ is a homomorphism and $m > 0$ is an integer. In Table 3.1 we have the twisted (1-folded) subgroups H in $SO(3) \times S^1$ having one-dimensional Weyl group $W(H) = N(H)/H$.

Twisted subgroups H of $SO(3) \times S^1$ with one-dimensional bi-orientable Weyl group $W(H)$ constitute the following lattice. An arrow indicates that a subgroup (at the higher level) contains a copy of a conjugate of the subgroup to which the arrow is pointing to.



(1) $r=\gcd(n,k)$, (2) $n>2$, (3) $k>2$, (4) $s\geq 1$, (5) k/l, l are odd, (6) n even, (7) n odd.

Figure 3.2. Twisted subgroup lattice for $SO(3) \times S^1$

On the other hand, the subgroups H of $SO(3) \times S^1$ having finite Weyl group are of the form $H = K \times S^1$, where K is $SO(3)$, $O(2)$, $SO(2)$, A_4, A_5, S_4, D_n, $n \geq 3$, or V_4.

3.2. Fixed-point spaces of irreducible $SO(3)\times S^1$-representations. Let V be an orthogonal representation of $SO(3)\times S^1$. For a subgroup $H \subset SO(3)\times S^1$ we denote by $\mathrm{Fix}(H)$ the subspace V^H and by $d(H)$ its dimension. In the case $H = K \times S^1$, V^H is a subspace of V^{S^1}, which is an orthogonal representation of $SO(3)$. In order to analyze irreducible representations of $SO(3)\times S^1$ we need first some additional results about irreducible $SO(3)$-representations $\mathcal{V}_j$ of dimension $2j+1$. The following result can be found in [10] or [21].

Theorem 3.1. *For the representation space $V = \mathcal{V}_j$ we have the following dimensions of the fixed point subspaces V^K:*

(i) $d(\mathbb{Z}_m) = 2[j/m] + 1$ *for* $m \geq 1$,

(ii) $d(D_m) = \begin{cases} [j/m] & \textit{if } j \textit{ is odd}, \\ [j/m]+1 & \textit{if } j \textit{ is even}, \end{cases}$

(iii) $d(SO(2)) = 1$,

(iv) $d(O(2)) = \begin{cases} 0 & \textit{if } j \textit{ is odd}, \\ 1 & \textit{if } j \textit{ is even}, \end{cases}$

(v) $d(A_4) = 2[j/3] + [j/2] - j + 1$,

(vi) $d(S_4) = [j/4] + [j/3] + [j/2] - j + 1$,

(vii) $d(A_5) = [j/5] + [j/3] + [j/2] - j + 1$.

A representation $\mathcal{V}_j$ considered as an $SO(3) \times S^1$-representation with the trivial S^1-action is an irreducible $SO(3)\times S^1$-representation. On the other hand, the complexification $\mathcal{V}_j^c = \mathcal{V}_j \oplus \mathcal{V}_j$ of $\mathcal{V}_j$, with the action of S^1 on $\mathcal{V}_j^c$ defined by the formula $\gamma v = \gamma^m \cdot v$, $\gamma \in S^1$, $v \in \mathcal{V}_j^c$, $m \in \mathbb{N}$ (where "$\cdot$" denotes the usual complex multiplication) is also an irreducible $SO(3)\times S^1$-representation denoted by $\mathcal{V}_j^{c,m}$. All the irreducible representations of $SO(3) \times S^1$ can be constructed in this way. In the case $m = 1$ we will simply write $\mathcal{V}_j^{c,1} = \mathcal{V}_j^c$.

We need also more information on the orbit types of the irreducible representations $\mathcal{V}_j^c$ of $SO(3) \times S^1$. The Tables 3.3 and 3.4 were obtained according to the results from [3]. These tables can be used to establish the isotropy lattices for all representations $\mathcal{V}_j^c$. In particular, we compute the isotropy lattices for $\mathcal{V}_j^c$, $j = 2$, 3, 4 and 5, which are presented following the tables (the symbol $[d]$, standing on the right margin, denotes the dimension of the corresponding fixed-point subspace).

Subgroup H	j	$\dim \mathrm{Fix}(H)$	$N(H)$	$W(H)$
$O(2)$	all j	2	$O(2) \times S^1$	S^1
D_m, $m > 2$	$m \le j$	$2([\frac{j}{m}] + 1)$	$D_{2m} \times S^1$	$\mathbb{Z}_2 \times S^1$
V_4	all j	$2(\frac{j}{2} + 1)$	$S_4 \times S^1$	$S_3 \times S^1$
S_4	$j \ne 2$	$2([\frac{j}{4}] + [\frac{j}{3}] + [\frac{j}{2}] - j + 1)$	$S_4 \times S^1$	S^1
A_5	$j = 6, 10, 12, 16, 18,$	$2([\frac{j}{5}] + [\frac{j}{3}] + [\frac{j}{2}] - j + 1)$	$A_5 \times S^1$	S^1
	$j \ge 20$			
A_4	$j = 6$, $j \ge 10$	$2(2[\frac{j}{3}] + [\frac{j}{2}] - j + 1)$	$S_4 \times S^1$	$\mathbb{Z}_2 \times S^1$
A_4^t	all j	$j - 2[\frac{j}{3}]$	$A_4 \times S^1$	S^1
S_4^-	$j = 6$, $j \ge 10$	$2([\frac{j}{3}] - [\frac{j}{4}])$	$S_4 \times S^1$	S^1
V_4^-	all j	j	$D_4 \times S^1$	$\mathbb{Z}_2 \times S^1$
D_n^z, $n \ge 3$	$n \le \frac{j}{2}$	$2[\frac{j}{n}]$	$D_{2n} \times S^1$	$\mathbb{Z}_2 \times S^1$
D_{2k}^d, $k \ge 2$	$k \le j$	$2([\frac{j}{k}] - [\frac{j}{2k}])$	$D_{2k} \times S^1$	S^1
$SO(2)^s$, $s \ge 1$	$s \le j$	2	$SO(2) \times S^1$	S^1

Table 3.3. Twisted isotropy subgroups of $SO(3) \times S^1$ for $\mathcal{V}_j^c$, j even

Subgroup H	j	dim Fix(H)	$N(H)$	$W(H)$
$O(2)$	never	2	$O(2) \times S^1$	S^1
$SO(2)$	never	2	$O(2) \times S^1$	$\mathbb{Z}_2 \times S^1$
D_4	$j \geq 2$	$2[\frac{j}{4}]$	$D_8 \times S^1$	$\mathbb{Z}_2 \times S^1$
D_m, $m > 2$, $m \neq 4$	$j > 2m$	$2[\frac{j}{m}]$	$D_{2m} \times S^1$	$\mathbb{Z}_2 \times S^1$
S_4	$j = 9$; $j \geq 13$	$2([\frac{j}{4}] + [\frac{j}{3}] + [\frac{j}{2}] - j + 1)$	$S_4 \times S^1$	S^1
A_5	$j = 15, 21, 25, 27$; $j \geq 31$	$2([\frac{j}{5}] + [\frac{j}{3}] + [\frac{j}{2}] - j + 1)$	$A_5 \times S^1$	S^1
A_4	$j = 9$, $j \geq 13$	$2(2[\frac{j}{3}] + [\frac{j}{2}] - j + 1)$	$S_4 \times S^1$	$\mathbb{Z}_2 \times S^1$
V_4	$j \geq 5$	$2[\frac{j}{2}]$	$S_4 \times S^1$	$S_3 \times S^1$
A_4^t	$j = 5, 7, 9, 13, 15, 19, 21, 23,$ $j \geq 27$	$j - 2[\frac{j}{3}] - 1$	$A_4 \times S^1$	S^1
S_4^-	$j = 3$, $j \geq 7$	$2([\frac{j}{3}] - [\frac{j}{4}])$	$S_4 \times S^1$	S^1
V_4^-	$j \geq 3$	$2([\frac{j}{2}] + 1)$	$D_4 \times S^1$	$\mathbb{Z}_2 \times S^1$
D_n^z, $n \geq 3$	$n \leq j$	$2([\frac{j}{n}] + 1)$	$D_{2n} \times S^1$	$\mathbb{Z}_2 \times S^1$
D_{2k}^d, $k \geq 3$	$k < j < 2k$	$2([\frac{j}{k}] - [\frac{j}{2k}])$	$D_{2k} \times S^1$	S^1
$O(2)^-$	all j	2	$O(2) \times S^1$	S^1
$SO(2)^k$, $k \geq 1$	$k \leq j$	2	$SO(2) \times S^1$	S^1
D_4^d	$j \geq 5$	$2([\frac{j}{2}] - [\frac{j}{4}])$	$D_4 \times S^1$	S^1

Table 3.4. Twisted isotropy subgroups of $SO(3) \times S^1$ for $\mathcal{V}_j^c$, j odd

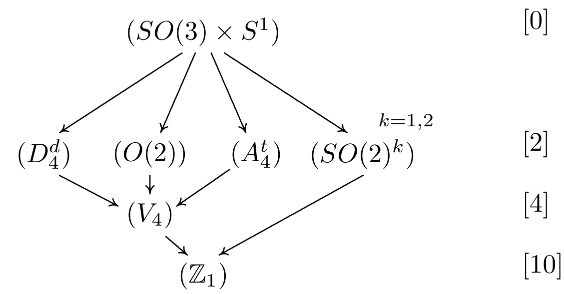


Figure 3.5. Twisted isotropy subgroups for $\mathcal{V}_2^c$

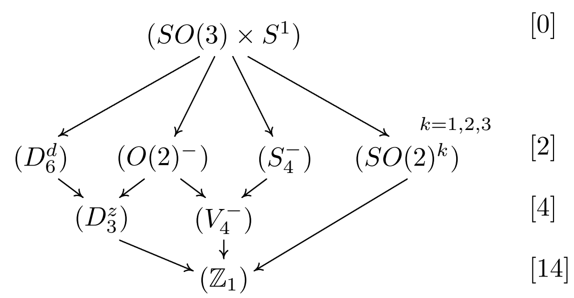


Figure 3.6. Twisted isotropy subgroups for $\mathcal{V}_3^c$

We have also the lattice of the isotropy subgroups for $\mathcal{V}_4^c$.

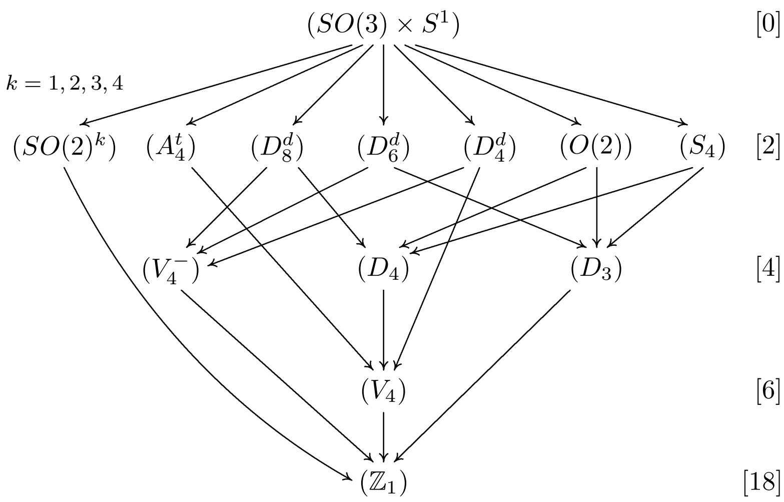


Figure 3.7. Twisted isotropy subgroups for $\mathcal{V}_4^c$

And finally we have the lattice for the representation V_5^c.

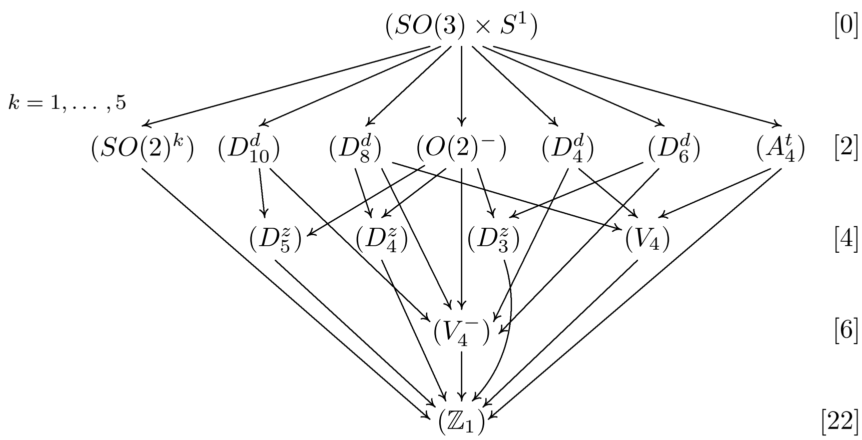


Figure 3.8. Twisted isotropy subgroups for $\mathcal{V}_5^c$

3.3. The numbers $n(L,H)$. Given subgroups L and H of a group G, we put

$$N(L,H)=\{g\in G : gLg^{-1}\subset H\}$$

and define the number

$$n(L,H)=\left|\frac{N(L,H)}{N(H)}\right|.$$

It is easy to notice that the number $n(L,H)$ coincides with the number of different subgroups H' such that H and H' are conjugate and $L \subset H'$. If, in addition, L and H are two isotropy groups in a representation V of G, then $n(L,H)$ can also be identified with the number of subspaces $V^{H'}$ contained in V^L.

The following table was borrowed from [21].

No.	L	H	$n(L,H)$	Condition
1.	$SO(2)$	$O(2)$	1	
2.	D_m	D_n	1	$n/m \in \mathbb{Z},\ m \geq 3$
3.	V_4	D_{2n}	3	$n \geq 2$
4.	D_m	$O(2)$	1	$m \geq 3$
5.	V_4	$O(2)$	3	
6.	D_3	S_4	2	
7.	D_4	S_4	2	
8.	D_3	A_5	2	
9.	D_5	A_5	2	
10.	V_4	A_4	1	
11.	V_4	S_4	4	
12.	V_4	A_5	2	
13.	A_4	S_4	1	
14.	A_4	A_5	2	

Table 3.9. The numbers $n(L,H)$ for subgroups of $SO(3)$

Using the definition of $N(L,H)$ one can prove the following

Proposition 3.2. *For subgroups $L \subset H \subset SO(3)$ we can write $N(L,H)$ as a disjoint union $N(L,H) = N(H) \cup N(H)g_2 \cup \ldots \cup N(H)g_k$, where $k = n(L,H)$ and $g_2, \ldots, g_k \in N(L,H)$.*

As a consequence of Proposition 3.2 we obtain

Proposition 3.3. *Let $L \subset H \subset SO(3)$ be such that (L), $(H) \in \Phi_0(SO(3))$. Then*

(a) $N(L^{\theta,m}, H^{\theta,m}) \subset N(L,H) \times S^1$.

(b) *If $N(H^{\theta,m}) = N(H) \times S^1$, then there exist $j_2, \ldots, j_r \in \{2, \ldots, k\}$ ($j_s \neq j_t$ for $s \neq t$) such that*

$$N(L^{\theta,m}, H^{\theta,m}) = (N(H) \cup N(H)g_{j_2} \cup \ldots \cup N(H)g_{j_r}) \times S^1,$$

in particular $n(L^{\theta,m}, H^{\theta,m}) = r \leq k = n(L,H)$.

Remark 3.4. Notice that in contrast to Proposition 3.3(b), in the case of $L^{\theta,m} = V_4$ and $H^{\theta,m} = A_4^t$ we have $n(V_4, A_4^t) = 2 \not\leq 1 = n(V_4, A_4)$.

By Proposition 3.3(b), if $n(L,H) = 1$ and $N(H^{\theta,m}) = N(H) \times S^1$, then $n(L^{\theta,m}, H^{\theta,m}) = 1$. All the remaining $n(L^{\theta,m}, H^{\theta,m})$ are represented in the Table 3.10.

No.	L	H	$n(L,H)$	Condition
1.	V_4^-	$O(2)^-$	1	
2.	V_4	S_4^-	1	
3.	V_4^-	S_4^-	1	
4.	V_4	D_{2n}^d	3	$n \in 2\mathbb{N}$
5.	V_4^-	D_{2n}^d	1	$n \in 2\mathbb{N}$
6.	V_4^-	D_{2n}^d	2	$n \in 2\mathbb{N}+1$
7.	V_4^-	D_n^z	1	$n \in \mathbb{N}$
8.	D_3^z	S_4^-	2	
9.	D_4^d	S_4^-	1	
10.	V_4	A_4^t	2	
11.	D_n	D_{2m}^d	1	$m/n \in \mathbb{N}$, $n \geq 3$
12.	D_{2n}^d	D_{2m}^d	1	$m/n \in 2\mathbb{N}+1$, $n \geq 2$
13.	D_n^z	D_{2m}^d	1	$m/n \in \mathbb{N}$

Table 3.10. The numbers $n(L,H)$

The numbers $n(L,H)$, for (non-twisted) subgroups L, H of the type $K \times S^1$, where $(K) \in \Phi_0(SO(3))$ are, in fact, listed in Table 3.9.

4. Reduced secondary groups $\Pi_*(K \times S^1)$

4.1. Direct consequences of formula (2.4). For a subgroup $H = K \times S^1$, $K \subset SO(3)$ in order to compute the group $\Pi_*(K \times S^1)$, we consider the space $\mathcal{R}_H = V^H \setminus V^{[H]} = V^K \setminus V^{[K]}$. The set $V^{[K]}$ is composed of all the subspaces V^L, such that $K \subset L$, $K \neq L$. Since $V^{SO(3)} \subset V^L$ for every L, the set $\mathcal{R}_H$ is $W(K)$-equivariantly homotopically equivalent to $\widetilde{V}^K \setminus \widetilde{V}^{[K]}$, where $\widetilde{V}$ is the orthogonal complement to $V^{SO(3)}$. The Weyl group $W(K)$ acts freely on $\mathcal{R}_H$, therefore $\pi: \mathcal{R}_H \to \mathcal{R}_H/W(H)$ is a covering map. In the case $\mathcal{R}_H$ is simply-connected, we have that $\pi_1(\mathcal{R}_H/W(H)) = W(H)$, and consequently (see formula (2.4)), we obtain

$$\Pi_*(K \times S^1) = W(H)/[W(H), W(H)].$$

It is easy to indicate the cases when this condition is satisfied, see Table 4.1

$H = K \times S^1$	$W(K)$	Conditions	$\Pi_*(H)$
$O(2) \times S^1$	$\mathbb{Z}_1$	$d(O(2)) - d(SO(3)) > 2$	0
$S_4 \times S^1$	$\mathbb{Z}_1$	$d(S_4) - d(SO(3)) > 2$	0
$A_5 \times S^1$	$\mathbb{Z}_1$	$d(A_5) - d(SO(3)) > 2$	0
$SO(2) \times S^1$	$\mathbb{Z}_2$	$d(SO(2)) - d(O(2)) > 2$	$\mathbb{Z}_2$
$D_n \times S^1$ $n \geq 6$	$\mathbb{Z}_2$	$d(D_n) - d(D_m) > 2$, $m/n \in \mathbb{Z}$, (D_m) and (D_n) – the only orbit types in $V^{(D_n)}$	$\mathbb{Z}_2$

Table 4.1. Groups $\Pi_*(H)$

Let $H = K \times S^1$ and assume that V^K contains only one orbit type (L) such that $K \neq L$ and $n(K, L) = 1$. This is the case for maximal orbit types $(O(2) \times S^1)$, $(S_4 \times S^1)$, $(A_5 \times S^1)$, or for $(SO(2) \times S^1)$. If $d(K) - d(L) = 2$ then $\mathcal{R}_H \sim V^K \setminus V^{[K]} = V^K \setminus V^L$ is $W(K)$-equivariantly homotopically equivalent to S^1 on which $W(K)$ acts freely. Since $\mathcal{R}_H/W(H) \sim S^1$, it follows that $\pi_1(\mathcal{R}_H/W(H)) = \mathbb{Z}$. Consequently, we have the following table:

$H = K \times S^1$	$W(K)$	Conditions	$\Pi_*(H)$
$O(2) \times S^1$	$\mathbb{Z}_1$	$d(O(2)) - d(SO(3)) = 2$	$\mathbb{Z}$
$S_4 \times S^1$	$\mathbb{Z}_1$	$d(S_4) - d(SO(3)) = 2$	$\mathbb{Z}$
$A_5 \times S^1$	$\mathbb{Z}_1$	$d(A_5) - d(SO(3)) = 2$	$\mathbb{Z}$
$SO(2) \times S^1$	$\mathbb{Z}_2$	$d(SO(2)) - d(O(2)) = 2$	$\mathbb{Z}$
$D_n \times S^1$ $n \geq 6$	$\mathbb{Z}_2$	$d(D_n) - d(D_m) = 2$, $m/n \in \mathbb{Z}$, (D_m) and (D_n) – the only orbit types in $V^{(D_n)}$	$\mathbb{Z}$

Table 4.2. Groups $\Pi_*(H)$

4.2. Secondary groups $\Pi_*(K \times S^1)$ for representations $\mathcal{V}_j \oplus \mathcal{V}_j$. In the case of concrete representations of $SO(3) \times S^1$, the computations of the reduced secondary groups $\Pi_*(K \times S^1)$ need more work. Usually, we should not expect that the space $\mathcal{R}_H$, where $H = K \times S^1$, is simply connected. In what follows, we will compute the groups $\Pi_*(H) = H_1(\mathcal{R}_H/W(H))$ for the representations $\mathcal{V}_j^c$, $j = 1, \dots, 5$ with no action of S^1 (what is required by the form of the isotropy subgroups H related to secondary obstructions). We put $\mathcal{V}_j^t = \mathcal{V}_j \oplus \mathcal{V}_j$, where $\mathcal{V}_j$ is the j-th irreducible representation of $SO(3)$ of dimension $2j+1$ and point out that triviality of the S^1-action on $\mathcal{V}_j^t$ is indicated by the letter t in the symbol $\mathcal{V}_j^t$.

For $j=1$, $j=2$ and $j=3$ we have the following lattices of isotropy groups

$$
\begin{tikzcd}
(SO(3)\times S^1) \arrow[d] & (SO(3)\times S^1) \arrow[d] & & (SO(3)\times S^1) \arrow[dl] \arrow[d] \arrow[dr] & & {[0]} \\
(SO(2)\times S^1) & (O(2)\times S^1) \arrow[d] & (A_4\times S^1) & (SO(2)\times S^1) & (D_3\times S^1) & {[2]} \\
 & (V_4\times S^1) & & & & {[4]} \\
j=1 & j=2 & & j=3 & &
\end{tikzcd}
$$

Figure 4.3

For the representation $\mathcal{V}_1^t$ we have (according to the Tables 4.1 and 4.2)

$$\Pi_*(SO(2)\times S^1)=\mathbb{Z}.$$

For the representation $\mathcal{V}_2^t$, by the same reason, we have

$$\Pi_*(O(2)\times S^1)=\mathbb{Z}.$$

Computations of the group $\Pi_*(V_4\times S^1)$ require more work. The fixed-point subspace for the subgroup $V_4\times S^1$ is four-dimensional, where there are three two-dimensional subspaces L_1, L_2 and L_3 that are fixed by the corresponding conjugate copies of the subgroup $O(2)\times S^1$. Observe that the origin is the only common point for these subspaces. The Weyl group $W(V_4\times S^1)\simeq S_3\simeq D_3$ consists of six element $\{1,\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2\}$ with $\alpha_j^2=\beta_k^3=1$. Notice that the $V_4\times S^1$-fixed-point space is a complexification of an irreducible two-dimensional representation of S_3, consequently it can be identified with the space $\mathbb{C}^2$ where S_3 acts as follows:

$$
\begin{aligned}
&\beta_1:(z_1,z_2)\mapsto(e^{2\pi i/3}z_1,e^{-2\pi i/3}z_2),\quad \beta_2=\beta_1^2,\\
&\alpha_1:(z_1,z_2)\mapsto(z_2,z_1),\\
&\alpha_2:(z_1,z_2)\mapsto(e^{2\pi i/3}z_2,e^{-2\pi i/3}z_1),\\
&\alpha_3:(z_1,z_2)\mapsto(e^{-2\pi i/3}z_2,e^{2\pi i/3}z_1).
\end{aligned}
$$

Notice, that $L_1=\{(z,z):z\in\mathbb{C}\}$, $L_2=\{(z,e^{-2\pi i/3}z):z\in\mathbb{C}\}$ and $L_3=\{(z,e^{2\pi i/3}z):z\in\mathbb{C}\}$. The set

$$\mathcal{R}:=\mathbb{C}^2\setminus(L_1\cup L_2\cup L_3)$$

is exactly the set $\mathcal{R}_H$ for $H=V_4\times S^1$ (see (2.4)). In order to compute $H_1(\mathcal{R}/S_3)$, first we equivariantly deform $\mathcal{R}$ to a two-dimensional subset $\mathcal{C}$ $S^3\subset\mathbb{C}^2$, and next, we construct a fundamental cell for $\mathcal{C}$ for which we describe the boundary identifications. As the first step of the equivariant deformation, we contract $\mathcal{R}$ to the set $\mathcal{X} := \mathcal{R} \cap S^3$, which consists of the sphere S^3 with removed three linked circles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$. We choose a point on the circle $\mathcal{C}_1$ and consider the stereographic projection from this point. Then the image of the set $\mathcal{X}$ can be illustrated by Figure 4.4.

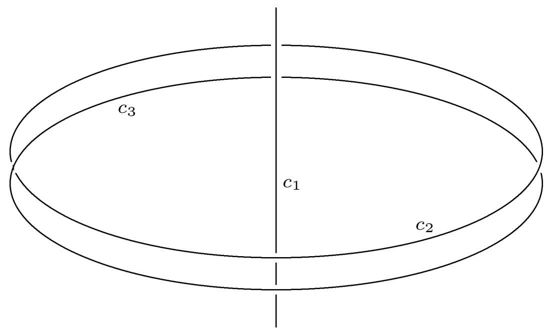


Figure 4.4

By enlarging equivariantly the circles $\mathcal{C}_j$, $j = 1, 2, 3$, we deform the set $\mathcal{X}$ to a solid torus, from which are removed two “enlarged” circles (solid tori), see Figure 4.5.

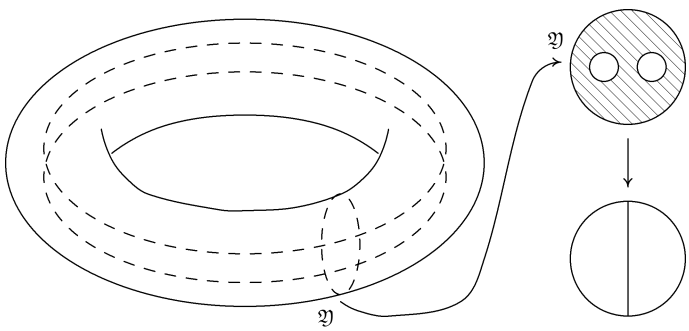


Figure 4.5

It is easy to see that the final result of these deformations will be the set $\mathcal{C}$, which can be described as the union $\mathcal{C} = S_1 \cup S_2 \cup J_1 \cup J_2 \cup J_3$, where $S_1 := S^1 \times \{0\}$, $S_2 := \{0\} \times S^1$ and, for k=1,2,3,

$$J_k := \left\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1,\ z_1, z_2 \neq 0,\ \arg \frac{z_1}{z_2} = \frac{\pi}{3}(2k+1) \right\}.$$

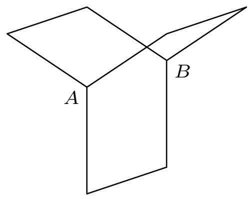


Figure 4.6

The set $\mathcal{C}$ is homeomorphic to a "divided torus" (see Figure 4.5) $\mathbb{Y} \times S^1$, for which the set $\mathbb{Y}$ can be illustrated as in Figure 4.6
where $A \in S_1$ and $B \in S_2$. Observe that:

(i) β_1 and β_2 permute cyclicly the sets J_k and keep S_1 and S_2 invariant,
(ii) α_k, $k = 1, 2, 3$, permutes S_1 and S_2 and reflects the set J_k at the central line $|z_1| = |z_2|$ with a rotation by π.

As a consequence, we obtain the quotient space $\mathcal{C}/S_3$ as the closure of

$$\mathcal{F} := \{(z_1, z_2) \in J_1 : |z_1| > |z_2|\},$$

with appropriate identifications of the boundary points as shown in the Figure 4.7.

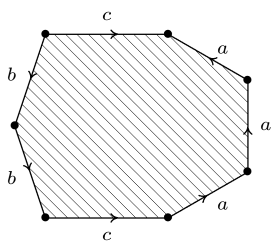


Figure 4.7

Therefore $\Pi_*(V_4 \times S^1) = H_1(\mathcal{C}/S_3) \simeq \mathbb{Z}$.

Consider now $\mathcal{V}_3^t$

$$\Pi_*(A_4 \times S^1) = \mathbb{Z}, \quad \Pi_*(SO(2) \times S^1) = \mathbb{Z}, \quad \Pi_*(D_3 \times S^1) = \mathbb{Z}.$$

Let us consider the representation $\mathcal{V}_4^t$. We have the following lattice of isotropy subgroups in $\mathcal{V}_4^t$ (Figure 4.8)

It is clear that $\Pi_*(O(2) \times S^1) = \mathbb{Z}$ and $\Pi_*(S_4 \times S^1) = \mathbb{Z}$.

Now we will compute the groups $\Pi_*(D_4 \times S^1)$ and $\Pi_*(D_3 \times S^1)$. For the subgroup D_4 we have that the space V^{D_4} (where for simplicity we write V instead $\mathcal{V}_4^t$) contains 1 copy of $V^{O(2)}$ (since $n(D_4, O(2)) = 1$) and two copies of V^{S_4} (since $n(D_4, S_4) = 2$). Since S_4 denotes the octahedral group of symmetries

$$
\begin{tikzcd}[column sep=small]
 & (SO(3)\times S^1) \arrow[dl] \arrow[dr] & & {[0]} \\
(O(2)\times S^1) \arrow[d] \arrow[drr] & & (S_4\times S^1) \arrow[d] \arrow[dll] & {[2]} \\
(D_3\times S^1) & & (D_4\times S^1) \arrow[d] & {[4]} \\
 & & (V_4\times S^1) & {[6]}
\end{tikzcd}
$$

$$j = 4$$

Figure 4.8

of a cube Q, it is clear that for a fixed face A (of the cube Q), for which the group D_4 is its symmetry group, there exists another cube Q', such that D_4 is also a symmetry group for the corresponding face A' (see the Figure 4.9 below).

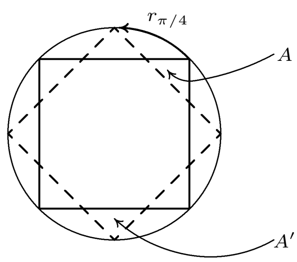


Figure 4.9

The rotation $r_{\pi/4}$ maps the face A onto the face A', so the two octahedral symmetry groups S_4 and S_4' of Q and Q' are conjugate by $r_{\pi/4}$. Thus $r_{\pi/4}$ gives rise to the generator of $W(H) = \mathbb{Z}_2$, which maps the subspace V^{S_4} onto $V^{S_4'}$. We put $C = S(V^{S_4})$, $C' = S(V^{S_4'})$ and $C_0 = S(V^{O(2)})$. The space V_H is $\mathbb{Z}_2$-homotopically equivalent to $S(V)_H = S^3 \setminus (C \cup C' \cup C_0)$. Notice that $\mathbb{Z}_2$ acts freely on V_H, maps circle C onto C', and C_0 is contained in the fixed-point set of the $W(H)$-action. We can choose a point P in C_0 and consider a stereographic projection from the point P to identify the set V_H with a subset of $\mathbb{R}^3$. It is clear that the space $S(V)_H/W(H)$ can be identified with the space $\mathbb{R}^3$ from which a line (corresponding to C_0) and a closed simple curve turning twice around the line (corresponding to C and C' identified by the $\mathbb{Z}_2$-action) were removed (see the Figure 4.10).

Hence, the space $V_H/W(H)$ is homotopically equivalent to a figure obtained by revolving the figure ∞ about an axis in $\mathbb{R}^3$ with a twist of $180°$ (similar to

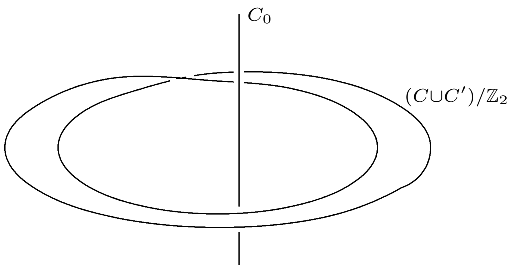


Figure 4.10

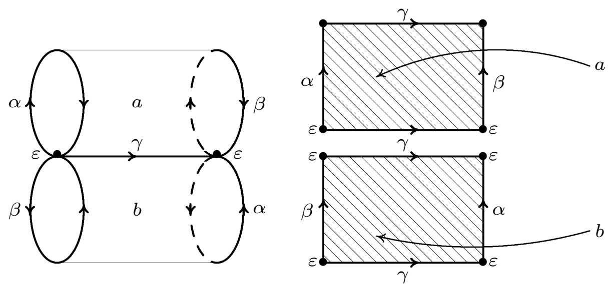


Figure 4.11

a Möbius band). This figure is a CW-complex with the cell decomposition (see Figure 4.11).

Hence, the homology groups of $V_H/W(H)$ can be computed from the chain complex

$$C_2 \to C_1 \to C_0$$

where $C_2 = \mathbb{Z} \oplus \mathbb{Z}$ is generated by a and b, $C_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ by α, β and γ, and $C_0 = \mathbb{Z}$ by ε, the boundary homomorphism $\partial_2 \colon C_2 \to C_1$ is given by $\partial_2(m,k) = (m+k, -m-k, 0)$, and $\partial_1 \equiv 0$. It is easy to verify that $H_2(V_H/W(H)) = \mathbb{Z}$, $H_1(V_H/W(H)) = \mathbb{Z} \oplus \mathbb{Z}$ and $H_0(V_H/W(H)) = \mathbb{Z}$. Therefore

$$\Pi_*(D_4 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad \Pi(D_4 \times S^1) = \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}.$$

In a very similar way, one can also prove that $\Pi_*(D_3 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$.

Consider the orbit type $(H) = (V_4 \times S^1)$. Since $n(V_4, D_4) = 3$ there are exactly three conjugate copies of D_4, which we will denote by D_4, D_4' and D_4'', containing the group V_4. These three subgroups are exactly symmetry groups of three parallel pairs of faces of the cube Q. The dimension of each of the subspaces V^{D_4}, $V^{D_4'}$ and $V^{D_4''}$ is equal to 4 and it is also clear that these subspaces intersect across the subspace V^{S_4}, which is 2-dimensional. By contracting the subspace V^{S_4} to a point, we obtain that $\mathcal{R}_H$ is equivariantly homotopically equivalent to $S^3 \setminus (C_1 \cup C_2 \cup C_3)$, where C_1, C_2 and C_3 denote three linked circles in S^3, on which the group S_3 acts freely. Therefore, this is exactly the same situation as in the case of the isotropy group $V_4 \times S^1$ for the representation $\mathcal{V}_2^t$, so by the same arguments we obtain

$$\Pi_*(V_4 \times S^1) = \mathbb{Z}.$$

Finally, we consider the representation $V := \mathcal{V}_5^t$ for which we have the following lattice of isotropy groups:

$$\begin{tikzcd}
 & & (SO(3)\times S^1) \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] & & {[0]} \\
(SO(2)\times S^1) & (D_5\times S^1) & (D_4\times S^1) \arrow[d] & (D_3\times S^1) & {[2]} \\
 & & (V_4\times S^1) & & {[4]}
\end{tikzcd}$$

$$j = 5$$

Figure 4.12

By following the same idea as above, we show that

$$\begin{gathered}
\Pi_*(SO(2) \times S^1) = \mathbb{Z}, \quad \Pi_*(D_5 \times S^1) = \mathbb{Z}, \quad \Pi_*(D_4 \times S^1) = \mathbb{Z}, \\
\Pi_*(D_3 \times S^1) = \mathbb{Z}, \quad \Pi_*(V_4 \times S^1) = \mathbb{Z}.
\end{gathered}$$

5. Equivariant degree of basic maps

5.1. Primary degree. Consider the representation $V := \mathcal{V}_l^{c,m}$ of $G := SO(3) \times S^1$. We can define the maps $f_{l,m} : \mathbb{C} \oplus V \to \mathbb{R} \oplus V$, $l = 1, 2, \ldots$ by

$$f_{l,m}(z, v) = (|z|(\|v\| - 1) + \|v\| + 1, z \cdot v), \tag{5.1}$$

which are $SO(3) \times S^1$-equivariant. In the case $m = 1$, instead of $f_{l,1}$ we will write f_l.

The maps $f_{l,m}$ are important for the computation of the bifurcation invariants related to $SO(3)$-symmetric Hopf bifurcation. More precisely, let $\Omega_{l,m} = \{(z, v) \in \mathbb{C} \oplus V;\ \|v\| < 1,\ 1/2 < |z| < 2\}$, then the reduced degree $\deg_G^*(f_{l,m}, \Omega_l)$ represents the bifurcation invariant related to a *crossing number* associated with the isotypical component corresponding to the irreducible representation $\mathcal{V}_{l,m}^c$ (see Section 7 for more details).

In order to compute G-Deg$(f_l, \Omega_{l,1})$ for $l = 2, \dots, 5$ we will use the Ulrich Type Formula for G-Degree (cf. Theorem 2.2) to derive recurrence formulae (with respect to the orbit types). More precisely, suppose that $f\colon \mathbb{C} \oplus V \to \mathbb{R} \oplus V$ is an $SO(3) \times S^1$-equivariant map given by formula (5.1) and $\Omega = \{(z, v) \in \mathbb{C} \oplus V : \|v\| < 1,\ 1/2 < |z| < 2\}$. Then the primary degree $SO(3) \times S^1$-Deg$(f, \Omega) = \sum_{(H)} n_H(H)$ can be computed as follows. First, if (H) is a maximal orbit type in V (i.e. if (K) is another orbit type in V such that $(K) > (H)$ then $(K) = (G)$), so

$$n_{(H)} = (\dim V^H - \dim V^G)/2.$$

For a sub-maximal orbit type (H) in V such that there exist other orbit types $(K_1), \dots, (K_m)$, different from (H) with $(K_k) > (H)$, $k = 1, \dots, m$, we compute n_H by the formula

$$n_H = \left[(\dim V^H - \dim V^G)/2 - \sum_{k=1}^{m} n_{K_k} \cdot n(H, K_k) \cdot |W(K_k)/S^1| \right] \Big/ |W(H)/S^1|,$$

where the numbers n_{K_k}, $k = 1, \dots, m$, which are related to larger orbit types, are assumed to be already evaluated. This algorithm can be applied to any arbitrary representation V.

For $l = 2$, since (D_4^d), $(O(2))$, (A_4^t), $(SO(2)^1)$ and $(SO(2)^2)$ are maximal orbit types, it follows from the Ulrich type formula that for each of these orbit types (H) the corresponding coefficient of the degree is exactly equal to the $(\mathbb{Z}_1)$-coefficients of the S^1-degree of the map f_2 restricted to the fixed point space V^H, where $V = \mathcal{V}_2^c$. According to the isotropy lattice, the dimension of V^H is equal to 2, thus all these coefficients are equal to 1. In order to compute the coefficient of (V_4) we notice that for $H = V_4$, the dimension of V^H is equal to 4, therefore the $(\mathbb{Z}_1)$-coefficient of the S^1-degree of the restriction of f_2 to the space V^H is equal 2. Consequently, the coefficient n_H of (V_4) can be computed by the following formula:

$$\begin{aligned} n_H &= (2 - n(V_4, D_4^d) \cdot 1 - n(V_4, O(2)) \cdot 1 - n(V_4, A_4^t) \cdot 1)/|W(V_4)/S^1| \\ &= (2 - 3 - 3 - 2)/6 = -1, \end{aligned}$$

where $|W(V_4)/S^1|$ denotes the number of elements in the homogenous space $W(V_4)/S^1$, which is equal to 6. In the above computations we also used the fact that $n(V_4, O(2)) = 3$. In the case K and L are two subgroups of $SO(3)$, the numbers $n(K, L)$ with respect to the action of $SO(3) \times S^1$ are exactly the same as for the group $SO(3)$ (see [3] or [21]).

Therefore we obtain

$$G\text{-Deg}\,(f_2, \Omega_{2,1}) = (D_4^d) + (O(2)) + (A_4^t) + (SO(2)^1) + (SO(2)^2) - (V_4).$$

In a similar way, we obtain

$$G\text{-Deg}\,(f_3,\Omega_{3,1}) = (O(2)^-) + (D_3) + (S_4^-) - (V_4^-) + \sum_{k=1}^{3}(SO(2)^k).$$

The case $l=4$ requires a little more of work, but again all the coefficients can be easily evaluated using the Ulrich type formula. Namely,

$$\begin{aligned} G\text{-Deg}\,(f_4,\Omega_{4,1}) &= (A_4^t) + (D_8^d) + (D_6^d) + (D_4^d) + (O(2)) + (S_4) - (V_4^-)\\ &\quad + \sum_{k=1}^{4}(SO(2)^k) - (D_4) - (D_3) - (V_4). \end{aligned}$$

For example, in order to compute the coefficient n_{V_4} corresponding to the orbit type (V_4), we consider the restriction of the map f_4 to the fixed-point subspace $V^{V_4} := \mathrm{Fix}(V_4)$ of dimension 6. Since the only orbit types that are contained in V^{V_4} are (V_4), (A_4^t), (D_8^d), (D_4^d), $(O(2))$, (S_4), and (D_4), for a regular normal approximation $\widetilde{f}_4$ of f_4, the contribution to S^1-Deg(f_4,V^{V_4}) made by the orbit types different from (V_4) is exactly

$$\begin{aligned} &n_{A_4^t}n(V_4,A_4^t)|W(A_4^t)/S^1| + n_{D_8^d}n(V_4,D_8^d)|W(D_8^d)/S^1|\\ &\quad + n_{D_4^d}n(V_4,D_4^d)|W(D_4^d)/S^1| + n_{O(2)}n(V_4,O(2))|W(O(2))/S^1|\\ &\qquad + n_{S_4}n(V_4,S_4)|W(S_4)/S^1| + n_{D_4}n(V_4,D_4)|W(D_4)/S^1|, \end{aligned}$$

thus

$$\begin{aligned} n_{V_4} &= [S^1\text{-Deg}(f_4,V^{V_4}) - (n_{A_4^t}n(V_4,A_4^t)\cdot 1 + n_{D_8^d}n(V_4,D_8^d)\cdot 1\\ &\quad + n_{D_4^d}n(V_4,D_4^d)\cdot 1 + n_{O(2)}n(V_4,O(2))\cdot 1 + n_{S_4}n(V_4,S_4)\cdot 1\\ &\quad + n_{D_4}n(V_4,D_4)\cdot 2)]/|W(V_4)/S^1|\\ &= \big(3-(2+3+3+3+4-6)\big)/6 = -1. \end{aligned}$$

Finally, for $l=5$, we have

$$\begin{aligned} G\text{-Deg}\,(f_5,\Omega_{5,1}) &= (A_4^t) + (D_4^d) + (D_6^d) + (D_8^d) + (D_{10}^d)\\ &\quad - (V_4) + (O(2)^-) - 2(V_4^-) + \sum_{k=1}^{5}(SO(2)^k). \end{aligned}$$

Similar computations could be also carried out for subsequent irreducible representations $\mathcal{V}_l^c$, $l=6,7,\dots$

5.2. Reduced secondary degree. Finally, we conclude this section with few remarks on the computations of the secondary obstructions for the reduced degree $\deg_G^*(f,\Omega)$, where $G = SO(3)\times S^1$. Suppose that V is an orthogonal representation of $\Gamma = SO(3)$. For the sake of simplicity we consider only representations of the form $\mathbb{R}^{k+1}\oplus V\oplus V$, $k\ge 0$, where we assume that Γ acts trivially on $\mathbb{R}^{k+1}$. Since $V\oplus V$ can be given a complex structure, i.e. $V\oplus V = V\otimes_{\mathbb{R}}\mathbb{C}$, there is a natural semi-free action of $S^1 \subset \mathbb{C}$ on $V \oplus V$, and consequently $\mathbb{R}^{k+1} \oplus V \oplus V$ is an orthogonal (real) representation of $G = SO(3) \times S^1$. Put $W := \mathbb{R}^k \oplus V \oplus V$. Let $\Omega \subset \mathbb{R} \oplus W$ be a G-invariant open bounded subset and $f\colon \mathbb{R} \oplus W \to W$ an Ω-admissible G-equivariant regular normal map. Since S^1 acts freely on $V \oplus V \setminus \{0\}$, the only isotropy groups in $V \oplus V \setminus \{0\}$ are twisted groups H^θ, where $H \subset SO(3)$ and $\theta\colon H \to S^1$ is a homomorphism. Consider an orbit type $(H^\theta) \in \Phi_1^+(G, V \oplus V)$. Since f is normal, then for every subgroup $L^\theta \subset H^\theta$, where $L^\theta \in \Phi_1^+(G, V \oplus V)$, the map $f^{L^\theta}\colon \mathbb{R} \oplus W^{L^\theta} \to W^{L^\theta}$ is $W(L^\theta)$-normal in Ω^{L^θ}. In particular, if $K = \operatorname{Ker} \theta$, then for $L = K \times \{1\}$ the map f^K is $W(K)$-normal in $\mathbb{R} \oplus W^K$. This implies that the map f is a weakly normal (see Section 2.6) Γ-equivariant map (it is easy to find an example where f won't be Γ-normal). Using the same idea as in Proposition 2.11, we have the following result:

Proposition 5.1. *Assume that $W := \mathbb{R}^k \oplus V \oplus V$, where V is an orthogonal representation of $\Gamma = SO(3)$, $V \oplus V$ has the natural complex structure described above (so it is also a representation of $G = SO(3) \times S^1$), and $\Omega \subset \mathbb{R} \oplus W$ a G-invariant open bounded subset. If $f\colon \mathbb{R} \oplus W \to W$ is a G-equivariant regular normal map in Ω, then there exists a regular normal Γ-equivariant map $\widehat{f}\colon \mathbb{R} \oplus W \to W$ such that for every $(K) \in \Phi_1^+(\Gamma, \Omega)$ we have $\widehat{f}^K \equiv f^K$.*

In the setting described in Proposition 5.1, we will compute $\deg_\Gamma^*(f, \Omega)$ by finding the relations between G-$\mathrm{Deg}\,(f, \Omega)$, which can be evaluated using the developed algorithms, and $\deg_\Gamma^*(f, \Omega)$. These relations can be used to express $\deg_\Gamma^*(f, \Omega)$ in terms of G-$\mathrm{Deg}\,(f, \Omega)$.

Since f is a regular normal G-equivariant map in Ω, for an orbit type $H^\theta \in \Phi_1^+(G, \Omega)$ the $W(H^\theta)$-equivariant map $f^{H^\theta}\colon (\mathbb{R} \oplus W)_{H^\theta} \to W^{H^\theta}$ has the zero set M_{H^θ} in Ω_{H^θ} composed of isolated $W(H^\theta)$-orbits, i.e.

$$M_{H^\theta} := W(H^\theta)x_1 \cup \ldots \cup W(H^\theta)x_N.$$

Let $K = \operatorname{Ker} \theta$ and assume that $K \in \Phi_0(\Gamma)$. In the fixed point space $\mathbb{R} \oplus W^{K \times \{1\}}$ there are exactly $n = n(K \times \{1\}, H^\theta)$ copies of the subspaces $V^{H_j^{\theta_j}}$, where $H_j^{\theta_j}$ is conjugate to H^θ. It is also clear that the union of these subspaces $V^{H_j^{\theta_j}}$ is invariant with respect to the action of $W(K)$. Denote by M the set of zeros of $f^{K \times \{1\}}$ of the orbit type (H^θ), i.e.

$$M = f^{-1}(0) \cap (\mathbb{R} \oplus W)_{(H^\theta)} = M_{H_1^{\theta_1}} \cup \ldots \cup M_{H_n^{\theta_n}}.$$

According to the properties of the secondary degree, the inclusion

$$i\colon M/W(K) \hookrightarrow (\mathbb{R} \oplus W)/W(K)$$

determines the element

$$m_K = i_*(o) \in H_1((\mathbb{R} \oplus W)_K/W(K)),$$

where $o \in H_1(M/W(K))$ is the orientation of $M/W(K)$. Consequently, we obtain a correspondence $n_{H^\theta} \mapsto m_K$ which clearly defines (on generators) the homomorphism

$$\nabla\colon \bigoplus_{(H^\theta)\in\Phi_1^+(G,\Omega)} \Pi(H^\theta) \to \bigoplus_{K\in\Phi_0(\Gamma)} \Pi(K).$$

Consequently, we obtain the following result

Proposition 5.2. *Assume that* $W := \mathbb{R}^k \oplus V \oplus V$ *and* $\Omega \subset \mathbb{R} \oplus W$ *are as in Proposition* 5.1. *If* $f\colon \mathbb{R} \oplus W \to W$ *is a* G*-equivariant* Ω*-admissible map, then*

$$\nabla[G\text{-Deg}\,(f,\Omega)] = \deg^*_\Gamma(f,\Omega).$$

In some cases, it is possible to evaluate the exact values for the homomorphism ∇. We have the following

Examples 5.3. Consider again the complex irreducible representation $\mathcal{V}_j^c$ of the group $SO(3)$, where $j \in \mathbb{N}$. Assume that S^1 acts trivially on V_j^c and denote this representation (considered as an $SO(3) \times S^1$-representation) by V. Using formula (5.1), we can define the $SO(3) \times S^1$-equivariant maps

$$f_j\colon \mathbb{C} \oplus V \to \mathbb{R} \oplus V, \quad j = 1, 2, \dots$$

Put $\Omega_j = \{(z,v) \in \mathbb{C} \oplus V : \|v\| < 1,\ 1/2 < |z| < 2\}$. We evaluate the reduced degree $\deg^*_G(f_j,\Omega_j)$, in the case $j = 2$ and 3, using Proposition 5.2. First, we find the primary degrees G-Deg(f_j,Ω_j) (see Section 5.1), i.e. we have

$$\begin{aligned} G\text{-Deg}\,(f_2,\Omega_2) &= (D_4^d) + (O(2)) + (A_4^t) + (SO(2)^1) + (SO(2)^2) - (V_4),\\ G\text{-Deg}\,(f_3,\Omega_3) &= (O(2)^-) + (D_6^d) + (S_4^-) - (V_4^-) + \sum_{k=1}^{3} (SO(2)^k). \end{aligned}$$

Next, we apply the homomorphism ∇. Since for $j = 2$ and $j = 3$ all the groups $\Pi_*(K) \simeq \mathbb{Z}$, and the isotropy subgroups are maximal with two-dimensional fixed-point spaces (except for the subgroup V_4), we obtain that $\nabla((O(2)) = ((O(2))$, for $j = 2$, and $\nabla(O(2)^-) = SO(2)$, $\nabla(D_6^d) = (D_3)$, $\nabla(S_4^-) = (A_4)$, for $j = 3$, and $\nabla(SO(2)^k) = 0$. In the case of the orbit types (H^θ), where H^θ is one of the groups D_4^d, A_4^t or V_4, we have that $K = V_4 = \operatorname{Ker}\theta$. Recall that the space $\mathcal{R}_K$ is equivariantly homotopically equivalent to a "divided torus" $\mathcal{C} = \mathbb{Y} \times S^1$ (see Section 4.2), on which the group S^1 acts naturally on the second component. Let us consider a single G-orbit M in $\mathbb{Y} \times S^1$ contributing to the component (H^θ) of the primary degree. It is clear that this orbit consists of two circles passing through the points A and B, so $M/S_3 \simeq S^1$ is mapped onto a cell a

in $\mathcal{C}/S_3$, which is a generator of $\Pi_*(V_4) \simeq \mathbb{Z}$. Consequently, $\nabla(H^\theta) = (V_4)$, and we obtain:

$$\begin{aligned}
\deg^*_\Gamma(f_2, \Omega_2) &= \nabla[G\text{-Deg}\,(f_2, \Omega_2)] = (V_4) + (O(2)),\\
\deg^*_\Gamma(f_3, \Omega_3) &= \nabla[G\text{-Deg}\,(f_3, \Omega_3)] = (A_4) + (SO(2)) + (D_3).
\end{aligned}$$

In general, computation of the secondary parts of the reduced degree can be very difficult, however, we believe it is a feasible task. If we assume that we are dealing with the G-action, where $G = SO(3) \times S^1$, such that S^1 acts trivially, then of course we have

$$\begin{aligned}
\deg^*_G(f_2, \Omega_2) &= (V_4 \times S^1) + (O(2) \times S^1),\\
\deg^*_G(f_3, \Omega_3) &= (A_4 \times S^1) + (SO(2) \times S^1) + (D_3 \times S^1).
\end{aligned}$$

6. Multiplicativity property for $SO(3) \times S^1$-equivariant deegree

This section is devoted to a special case of the multiplicativity property (cf. [20]) which will be needed later for the computations of the equivariant degree related to the Hopf bifurcation with $SO(3)$-symmetries.

Lemma 6.1. *Assume that Γ satisfies the property* (S_0)*. Then for every three subgroups K, H, L of Γ such that $W(K)$, $W(H)$ and $W(L)$ are finite, $L \subset H$, and for every homomorphism $\varphi\colon H \to S^1$, the set*

$$\left(\frac{\Gamma}{K} \times \frac{\Gamma \times S^1}{H^{\varphi,m}}\right)^{L^{\varphi,m}} \Big/ W(L^{\varphi,m})$$

is finite.

Proof. Let M be a subgroup of Γ such that $L \subset M$. We put

$$N(L, M) = \{\gamma \in \Gamma : \gamma^{-1} L \gamma \subset M\}$$

and we denote by $n(L, M)$ the number of elements in $N(L, M)/N(M)$. If L and M have finite Weyl groups then the number $n(L, M)$ is finite (see [18]). We have

$$(\Gamma/K \times \Gamma \times S^1/H^{\varphi,m})^{L^{\varphi,m}} \subset N(L, K)/K \times N(L, H) \times S^1/H^{\varphi,m},$$

therefore we have the following estimate for the number of elements in $(\Gamma/K \times \Gamma \times S^1/H^{\varphi,m})^{L^{\varphi,m}}/W(L^{\varphi,m})$:

$$\left|\left(\frac{\Gamma}{K} \times \frac{\Gamma \times S^1}{H^{\varphi,m}}\right)^{L^{\varphi,m}} \Big/ W(L^{\varphi,m})\right| \le \frac{n(L, K) \cdot |W(K)| \cdot n(L, H) \cdot |W(H)|}{|W(L)|},$$

where the symbol $|X|$ stands for the number of elements in the set X. $\square$

Example 6.2. Notice that the statement of Lemma 6.1 is not true in general for non-bi-orientable subgroups of type $\mathbb{Z}_n \times S^1$. Indeed, suppose that n is a multiple of k and that n, $k \geq 3$. Then we have

$$\begin{aligned}\left(\frac{SO(3)}{D_n} \times \frac{SO(3) \times S^1}{\mathbb{Z}_n \times S^1}\right)^{\mathbb{Z}_k \times S^1} &= \frac{N(\mathbb{Z}_k, D_n)}{D_n} \times \frac{N(\mathbb{Z}_k, \mathbb{Z}_n)}{\mathbb{Z}_n} \\ &\supset \frac{N(\mathbb{Z}_k)}{D_n} \times \frac{N(\mathbb{Z}_k)}{\mathbb{Z}_n} = \frac{O(2)}{D_n} \times \frac{O(2)}{\mathbb{Z}_n} = S^1 \times O(2).\end{aligned}$$

Therefore the set

$$\left(\frac{SO(3)}{D_n} \times \frac{SO(3) \times S^1}{\mathbb{Z}_n \times S^1}\right)^{\mathbb{Z}_k \times S^1} \Big/ W(\mathbb{Z}_k \times S^1) \simeq S^1$$

is not finite.

Let Γ be a compact Lie group and $G = \Gamma \times S^1$. Recall that the $\mathbb{Z}$-module $A(\Gamma) = \mathbb{Z}[\Phi_0(\Gamma)]$ admits the natural multiplication structure (see [29] or [2]) and the corresponding ring is called the Burnside ring. We also define the $\mathbb{Z}$-module $A_1^+(G)$ generated by $(H^{\varphi,m}) \in \Phi_1^+(G)$, i.e. $A_1^+(G) = \mathbb{Z}[\Phi_1^+(G)]$. Then we have the following

Theorem 6.3. *Let Γ be a compact Lie group satisfying* (S_0) *and $G = \Gamma \times S^1$. Assume that G has the property that every subgroup $H \subset G$ with $(H) \in \Phi_1^+(G)$ is of the form $K^{\varphi,m}$ with $(K) \in \Phi_0(\Gamma)$. Then there is a "multiplication" function $\cdot : A(\Gamma) \times A_1^+(G) \to A_1^+(G)$ defined on the generators (K), $(K) \in \Phi_0(\Gamma)$ and $(H^{\varphi,m}) \in \Phi_1^+(G)$ as follows*

$$(K) \cdot (H^{\varphi,m}) = \sum_{(L)} n_L (L^{\varphi,m}),$$

where the summation is taken over all subgroups L such that $W(L)$ is finite and $L = \gamma^{-1} K \gamma \cap H$, for some $\gamma \in \Gamma$, and

$$n_L = \left| \left(\frac{G}{K \times S^1} \times \frac{G}{H^{\varphi,m}}\right)_{(L^{\varphi,m})} \Big/ G \right|.$$

The $\mathbb{Z}$-module $A_1^+(G)$ equipped with the multiplication "$\cdot$" becomes an $A(\Gamma)$-module.

Proof. It is enough to notice that the number n_L is a well defined finite number. Indeed, we have

$$n_L \leq \left| \left(\frac{\Gamma}{K} \times \frac{\Gamma \times S^1}{H^{\varphi,m}}\right)^{L^{\varphi,m}} \Big/ W(L^{\varphi,m}) \right|$$

and therefore the homomorphism "$\cdot$" is well defined.

Lemma 6.4. *Assume that a compact Lie group* Γ *satisfies the property* (S_0) *and has the property that every subgroup* $H \subset G := \Gamma \times S^1$ *with* $(H) \in \Phi_1^+(G)$ *is of the form* $K^{\varphi,m}$ *with* $(K) \in \Phi_0(\Gamma)$. *Let* V (*resp.* W) *be an orthogonal representation of* G (*resp. of* Γ), $\Omega_1 \subset \mathbb{R} \oplus V$ (*resp.* $\Omega_2 \subset W$) *an invariant open bounded set and* $f_1: \mathbb{R} \oplus V \to V$ (*resp.* $f_2: W \to W$) *an* Ω_1*-admissible* (*resp.* Ω_2*-admissible*) *equivariant map. If* f_1 *is weakly normal in* Ω_1 *and* f_2 *is normal in* Ω_2 *then the map* $f = f_1 \times f_2: \mathbb{R} \oplus V \oplus W \to V \oplus W$ *is weakly normal in* $\Omega_1 \times \Omega_2$.

Proof. Let $(H) \in \Phi_1^+(G)$ or $(H) \in \Phi(G)$ be an orbit type in $\Omega_1 \times \Omega_2$. Then by assumption, $H = K^{\varphi,m}$ or $H = K \times S^1$ for some $(K) \in \Phi_0(\Gamma)$. We have that $f^H = f_1^H \times f_2^K: \mathbb{R} \oplus V^H \oplus W^K \to V^H \oplus W^K$. Suppose that (L) is an orbit type in $\Omega_1^H \times \Omega_2^K$. The set $V^{(L)}$ is a finite union of $V^{L'}$ such that $L' \in (L)$ and since f_1 is weakly normal in Ω_1 and f_2 is normal in Ω_2, it follows that $f_1 \times f_2$ is weakly normal in $\Omega_1 \times \Omega_2$. $\square$

Theorem 6.5 (Multiplicativity property). *Let* Γ *be a compact Lie group,* V (*resp.* W) *be an orthogonal representation of* $G := \Gamma \times S^1$ (*resp. of* Γ), $\Omega_1 \subset \mathbb{R} \oplus V$ (*resp.* $\Omega_2 \subset W$) *an invariant open bounded set and* $f_1: \mathbb{R} \oplus V \to V$ (*resp.* $f_2: W \to W$) *an* Ω_1*-admissible* (*resp.* Ω_2*-admissible*) *equivariant map. Then*

$$G\text{-Deg}\,(f_1 \times f_2, \Omega_1 \times \Omega_2) = \Gamma\text{-Deg}\,(f_2, \Omega_2) \cdot G\text{-Deg}\,(f_1, \Omega_1),$$

where the multiplication is taken in the $A(\Gamma)$*-module* $A_1^+(G)$.

Proof. Applying the additivity and homotopy properties for the reduced degree we can assume that both f_1 and f_2 are regular normal, $f_i^{-1}(0) \cap \Omega_i = G(x_i)$, $i = 1, 2$, where $G_{x_1} = K^{\varphi,m}$ (or resp. $G_{x_1} = K \times S^1$) and $G_{x_2} = H \times S^1$, with $W(K)$ and $W(H)$ finite. We can also assume that Ω_1 and Ω_2 are tubes around the orbits $G(x_1)$ and $G(x_2)$ respectively. Thus $G(x_1) \times \Gamma(x_2) = (f_1 \times f_2)^{-1}(0) \cap (\Omega_1 \times \Omega_2)$ and $G(x_1) \times \Gamma(x_2)$ is G-diffeomorphic to $(\Gamma \times S^1)/H \times (\Gamma \times S^1)/K^{\varphi,m}$. Since by Lemma 6.4, $f = f_1 \times f_2$ is weakly normal, thus, by Proposition 2.11, there exists a regular normal map $\widehat{f}: \mathbb{R} \oplus V \oplus W \to V \oplus W$ such that $f^H \equiv \widehat{f}^H$ for all orbit types in $\Omega_1 \times \Omega_2$. Consequently, the conclusion follows from the fact that the orbits in $\widehat{f}^{-1}(0) \cap (\Omega_1 \times \Omega_2)$ of type $(L^{\varphi,m})$ are "counted" by the coefficient n_L in the multiplication formula, i.e.

$$(H) \cdot (K^{\varphi,m}) = \sum_{(L)} n_L (L^{\varphi,m}),$$

thus

$$G\text{-Deg}\,(f_1 \times f_2, \Omega_1 \times \Omega_2) = (H) \cdot (K^{\varphi,m}) = \Gamma\text{-Deg}\,(f_2, \Omega_2) \cdot G\text{-Deg}\,(f_1, \Omega_1). \quad \square$$

	$(O(2)^-)$	(S_4^-)	(A_4^t)	(D_5^z)
$(O(2))$	$(O(2)^-)+(V_4^-)$	$(D_4^d)+(D_3^z)+(V_4^-)$	(V_4)	(D_5^z)
$(SO(2))$	$(SO(2))$	0	0	0
(S_4)	$(D_4^z)+(D_3^z)+(V_4^-)$	$(S_4^-)+(D_4^d)+(D_3^z)+(V_4^-)$	$(A_4^t)+(V_4)$	0
(A_5)	$(D_5^z)+(D_3^z)+(V_4)$	$(A_4)+2(D_3^z)+(V_4^-)$	$2(A_4^t)$	$2(D_5^z)$
(A_4)	(V_4^-)	$(A_4)+(V_4^-)$	$2(A_4^t)$	0
(D_5)	(D_5^z)	0	0	$2(D_5^z)$
(D_4)	$(D_4^z)+2(V_4^-)$	$2(D_4^d)+2(V_4^-)$	$2(V_4)$	0
(D_3)	(D_3^z)	$2(D_3^z)$	0	0
(V_4)	$3(V_4^-)$	$(V_4)+3(V_4^-)$	$2(V_4)$	0
(D_n),	$(D_n^z)+2\left\|\frac{n}{2}\right\|(V_4^-)$	$2\left\|\frac{n}{4}\right\|(D_4^d)+2\left\|\frac{n}{3}\right\|(D_3^z)$	$2\left\|\frac{n}{2}\right\|(V_4)$	$2\left\|\frac{n}{5}\right\|(D_5^z)$
$n \geq 3$		$+\left(1-\left\|\frac{n}{4}\right\|\right)\left\|\frac{n}{2}\right\|(V_4)$		
		$+\left(3-\left\|\frac{n}{4}\right\|\right)\left\|\frac{n}{2}\right\|(V_4^-)$		

Table 6.1

	(D_4^z)	(D_4^d)	(D_3^z)	(V_4^-)
$(O(2))$	$(D_4^z)+2(V_4^-)$	$(D_4^d)+(V_4)+(V_4^-)$	(D_3^z)	$3(V_4^-)$
$(SO(2))$	0	0	0	0
(S_4)	$2(D_4^z)+2(V_4^-)$	$2(D_4^d)+(V_4)+(V_4^-)$	$2(D_3^z)$	$4(V_4^-)$
(A_5)	$2(V_4^-)$	$(V_4)+(V_4^-)$	$2(D_3^z)$	$2(V_4^-)$
(A_4)	$2(V_4^-)$	$(V_4)+(V_4^-)$	0	$2(V_4^-)$
(D_5)	0	0	0	0
(D_4)	$2(D_4^z)+4(V_4^-)$	$2(D_4^d)+2(V_4)+2(V_4^-)$	0	$6(V_4^-)$
(D_3)	0	0	$2(D_3^z)$	0
(V_4)	$6(V_4^-)$	$3(V_4)+3(V_4^-)$	0	$6(V_4^-)$
(D_n)	$2\left\|\frac{n}{4}\right\|(D_4^z)$	$2\left\|\frac{n}{4}\right\|(D_4^d)+\left(3-\left\|\frac{n}{4}\right\|\right)\left\|\frac{n}{2}\right\|(V_4)$	$2\left\|\frac{n}{3}\right\|(D_3^z)$	$6\left\|\frac{n}{2}\right\|(V_4^-)$
$n \geq 3$	$+2\left(3-\left\|\frac{n}{4}\right\|\right)\left\|\frac{n}{2}\right\|(V_4^-)$	$+\left(3-\left\|\frac{n}{4}\right\|\right)\left\|\frac{n}{2}\right\|(V_4)$		
		$+\left(3-\left\|\frac{n}{4}\right\|\right)\left\|\frac{n}{2}\right\|(V_4^-)$		

Table 6.2

Finally, we present the multiplication tables (Tables 6.1–6.3) for the primary orbit types of $SO(3) \times S^1$-action, where

$$\begin{vmatrix} n \\ k \end{vmatrix} = \begin{cases} 1 & \text{if divides } n, \\ 0 & \text{otherwise.} \end{cases}$$

	(D_k^z)	(D_{2k}^d)
$(O(2))$	$(D_k^z) + 2\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4^-)$	$(D_{2k}^d) + \begin{vmatrix} k \\ 2 \end{vmatrix}(V_4) + \left(2 - \begin{vmatrix} k \\ 2 \end{vmatrix}\right)(V_4^-)$
$(SO(2))$	0	0
(S_4)	$2\begin{vmatrix} k \\ 4 \end{vmatrix}(D_4^z) + 2\begin{vmatrix} k \\ 3 \end{vmatrix}(D_3^z) + 2\left(2 - \begin{vmatrix} k \\ 4 \end{vmatrix}\right)\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4^-)$	$\begin{vmatrix} k \\ 4 \end{vmatrix}(D_4) + \begin{vmatrix} k \\ 4 \end{vmatrix}(D_4^z) + 2\left(\begin{vmatrix} k \\ 2 \end{vmatrix} - \begin{vmatrix} k \\ 4 \end{vmatrix}\right)(D_4^d)$
		$+\begin{vmatrix} k \\ 3 \end{vmatrix}(D_3) + \begin{vmatrix} k \\ 3 \end{vmatrix}(D_3^z) + \begin{vmatrix} k \\ 2 \end{vmatrix}(V_4) + \left(4 - 3\begin{vmatrix} k \\ 2 \end{vmatrix}\right)(V_4^-)$
(A_5)	$2\begin{vmatrix} k \\ 5 \end{vmatrix}(D_5^z) + 2\begin{vmatrix} k \\ 3 \end{vmatrix}(D_3^z) + 2\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4^-)$	$\begin{vmatrix} k \\ 5 \end{vmatrix}(D_5) + \begin{vmatrix} k \\ 5 \end{vmatrix}(D_5^z) + \begin{vmatrix} k \\ 3 \end{vmatrix}(D_3) + \begin{vmatrix} k \\ 3 \end{vmatrix}(D_3^z)$
		$+\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4) + \left(2 - \begin{vmatrix} k \\ 2 \end{vmatrix}\right)(V_4^-)$
(A_4)	$2\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4^-)$	$\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4) + \left(2 - \begin{vmatrix} k \\ 2 \end{vmatrix}\right)(V_4^-)$
(D_5)	$2\begin{vmatrix} k \\ 5 \end{vmatrix}(D_5^z)$	$\begin{vmatrix} k \\ 5 \end{vmatrix}(D_5) + \begin{vmatrix} k \\ 5 \end{vmatrix}(D_5^z)$
(D_4)	$2\begin{vmatrix} k \\ 4 \end{vmatrix}(D_4^z) + 2\left(3 - \begin{vmatrix} k \\ 4 \end{vmatrix}\right)\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4^-)$	$\begin{vmatrix} k \\ 4 \end{vmatrix}(D_4) + 2\left(\begin{vmatrix} k \\ 2 \end{vmatrix} - \begin{vmatrix} k \\ 4 \end{vmatrix}\right)(D_4^d) + \begin{vmatrix} k \\ 4 \end{vmatrix}(D_4^z)$
		$+2\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4) + 2\left(3 - 2\begin{vmatrix} k \\ 2 \end{vmatrix}\right)(V_4^-)$
(D_3)	$2\begin{vmatrix} k \\ 3 \end{vmatrix}(D_3^z)$	$\begin{vmatrix} k \\ 3 \end{vmatrix}(D_3) + \begin{vmatrix} k \\ 3 \end{vmatrix}(D_3^z)$
(V_4)	$6\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4^-)$	$3\begin{vmatrix} k \\ 2 \end{vmatrix}(V_4) + 3\left(2 - \begin{vmatrix} k \\ 2 \end{vmatrix}\right)(V_4^-)$
(D_n)	$2[l](D_l^z) + 2(3 - [l])\begin{vmatrix} l \\ 2 \end{vmatrix}(V_4^-)$	$\begin{vmatrix} k \\ l \end{vmatrix}[l](D_l) + \begin{vmatrix} k \\ l \end{vmatrix}[l](D_l^z) + 2\begin{vmatrix} l \\ 2 \end{vmatrix}\left(1 - \begin{vmatrix} k \\ l \end{vmatrix}\right)[l](D_l^d)$
$n \geq 3$	where $l := \gcd(n, k)$,	$+\begin{vmatrix} l \\ 2 \end{vmatrix}\begin{vmatrix} k \\ 2 \end{vmatrix}\left(3 - [l]\left(\begin{vmatrix} k \\ l \end{vmatrix} + \begin{vmatrix} l \\ 4 \end{vmatrix}(1 - \begin{vmatrix} k \\ l \end{vmatrix})\right)\right)(V_4)$
	$[l] := \begin{cases} 0 & l < 3 \\ 1 & l \geq 3 \end{cases}$	$+\begin{vmatrix} l \\ 2 \end{vmatrix}\left(6 - 3\begin{vmatrix} k \\ 2 \end{vmatrix} - [l]\left(2 - \begin{vmatrix} k \\ l \end{vmatrix} - \begin{vmatrix} l \\ 4 \end{vmatrix}(1 - \begin{vmatrix} k \\ l \end{vmatrix})\right)\right)(V_4^-)$
		where $l := \gcd(n, 2k)$

Table 6.3

In addition,

$$\begin{aligned} (SO(2)) \cdot (SO(2)^k) &= (SO(2)^k), && (O(2)) \cdot (SO(2)^k) = 2(SO(2)^k), \\ (H) \cdot (SO(2)^k) &= 0, && \text{if } (H) \neq (SO(2)), (O(2)), \\ (SO(3)) \cdot (H) &= (H), && \text{if } (H) \in \Phi_1^+(SO(3) \times S^1). \end{aligned}$$

7. Bifurcations with $SO(3)$-symmetries

7.1. Hopf bifurcation with $SO(3)$-symmetries. Let V be an orthogonal $SO(3)$-representation and $f\colon \mathbb{R}\oplus V \to V$ an $SO(3)$-equivariant C^1-map such that $f(\alpha,0)=0$ for all $\alpha,\in\mathbb{R}$. We consider the following bifurcation problem

$$\begin{cases} \dot{x} = f(\alpha,x) & \text{for } x\in V,\\ x(0)=x(p) & \text{for a certain (unknown) } p>0. \end{cases} \tag{7.1}$$

Our goal is to classify possible symmetries of small-amplitude periodic solutions to (7.1) bifurcating from the trivial one. The method used here is based on the standard reduction of (7.1) to the study of basic maps (cf. [6], [20], [21], [23], [24]), and for this reason we include here only a brief presentation of its key steps. Roughly speaking, problem (7.1) can be reformulated as an equation $F_\theta(\lambda,v)=0$ in such a way that the primary $SO(3)\times S^1$-degree of the map F_θ turns out to be responsible for the classification of symmetries of non-trivial periodic solutions.

Step 1. Operator formulation in a functional space. The equation (7.1) is equivalent to

$$\begin{cases} \dot{w} = \dfrac{p}{2\pi} f(\alpha,w),\\ w(0)=w(2\pi). \end{cases} \tag{7.2}$$

We put $W=C^1(S^1,V)$ with $S^1=\mathbb{R}/2\pi\mathbb{Z} := \{e^{it} : t\in[0,2\pi]\}$. Define

$$\begin{aligned} L\colon C^1(S^1,V) &\to C(S^1,V), & Lw &= \dot{w},\\ N_f\colon \mathbb{R}\oplus C(S^1,V) &\to C(S^1,V), & N_f(\alpha,w)(t) &= f(\alpha,w(t)),\\ j\colon C^1(S^1,V) &\to C(S^1,V), & j(w) &= w. \end{aligned}$$

Then (7.2) is equivalent to

$$Lw = \frac{p}{2\pi} N_f(\alpha, j(w))$$

and consequently to

$$w-(L-P)^{-1}\left[\frac{p}{2\pi}N_f(\alpha,j(w))-Pw\right]=0, \tag{7.3}$$

where $Pw=(1/(2\pi))\int_0^{2\pi} w(t)\,dt$. It is useful to replace the parameter $p/(2\pi)$ by $1/\beta$ and consider the following two-parameter equation:

$$F(\alpha,\beta,w) := w-(L-P)^{-1}\left[\frac{1}{\beta}N_f(\alpha,j(w))-Pw\right]=0. \tag{7.4}$$

It is clear that F is a completely continuous vector field.

Step 2. Characteristic equation. Let $V^c = V \otimes_{\mathbb{R}} \mathbb{C}$ denote the standard complexification of V. We put

$$\Delta_\alpha(\lambda) := \lambda \mathrm{Id} - D_x f(\alpha, 0) \colon V^c \to V^c, \quad \lambda \in \mathbb{C}.$$

Clearly, $\Delta_\alpha(\lambda)$ is an $SO(3)$-equivariant linear operator. Recall that

$$\det{}_{\mathbb{C}} \Delta_\alpha(\lambda) = 0, \quad \lambda \in \mathbb{C} \tag{7.5}$$

is the *characteristic equation* associated with the stationary solution $(\alpha, 0)$. A stationary point for (7.1) is called a center if (7.5) has a purely imaginary root. A center $(\alpha_0, 0)$ is said to be isolated, if there are no other centers in its neighbourhood in $\mathbb{R} \oplus V$. We assume that $(\alpha_0, 0)$ is an isolated center for equation (7.1) with a characteristic root $\lambda = i\beta_0$, $\beta_0 > 0$. We also assume that $D_x f(\alpha_0, 0)$ is an isomorphism, i.e. there is no steady-state bifurcation taking place at $(\alpha_0, 0)$.

Step 3. Isotypical decomposition. The space W is an isometric Banach representation of the group $G = SO(3) \times S^1$ and it has the following S^1-isotypical decomposition

$$W = \overline{V \oplus V_1^c \oplus \ldots \oplus V_m^c \oplus \ldots}$$

where $V_m^c = V \otimes_{\mathbb{R}} \mathbb{C}_m$ and $\mathbb{C}_m$ denotes the m-th irreducible representation of S^1. One can identify the elements of V_m^c, $m \geq 1$, with the functions $t \mapsto e^{imt} w$, $w \in V^c$, i.e. $\dim V_m^c = 2 \dim V$ for $m \geq 1$. The above decomposition leads to the $SO(3) \times S^1$-isotypical decompositions

$$\begin{aligned} V &:= U_0 \oplus \ldots \oplus U_k, \\ V_m^c &= U_0^{c,m} \oplus \ldots \oplus U_k^{c,m}, \quad \text{for } m \geq 1, \end{aligned}$$

with $U_0 = V^G$, where U_j, $j > 0$, corresponds to a certain irreducible representation $\mathcal{V}_{r_j}$ of $\Gamma = SO(3)$ and $U_j^{c,m}$ corresponds to $\mathcal{V}_{r_j}^{c,m}$ (cf. Section 3.2). Observe that we use the index j to indicate $SO(3)$-isotypical components, and m to indicate S^1-isotypical components.

Step 4. Linearization procedure. The necessary bifurcation condition implies that if $(\alpha_0, \beta_0, 0)$ is a bifurcation point, then $D_w F(\alpha_0, \beta_0, \cdot\,)$ can not be an isomorphism and this will be the case when $(\alpha_0, 0)$ is a center with the characteristic root $i\beta_0$. We put

$$A(\alpha, \beta) := DF_w(\alpha, \beta, 0) \colon W \to W \quad \text{and} \quad A_{j,m}(\alpha, \beta) = A(\alpha, \beta)_{|U_j^{c,m}}.$$

Notice that (cf. [19]–[23])

$$A_{j,m}(\alpha, \beta) = \frac{1}{im\beta} \Delta_\alpha(im\beta)_{|U_j^{c,m}} \quad \text{and} \quad A_{j,0}(\alpha, \beta) := \frac{1}{\beta} D_x f(\alpha, 0)_{|U_j}.$$

Since for $m \geq 1$ the isotypical component U_j is a union

$$\underbrace{\mathcal{V}_{r_j}^{c,m} \oplus \ldots \oplus \mathcal{V}_{r_j}^{c,m}}_{l \text{ times}},$$

and $\mathcal{V}_{r_j}$ is $\mathbb{R}$-absolutely irreducible (thus $\mathcal{V}_{r_j}^{c,m}$ are also absolutely $\mathbb{C}$-irreducible), the operators $A_{j,m}$ can be identified with an $l \times l$-complex matrix, which we will denote by the same symbol. Similarly, the operator $A_{j,0}$ will be identified with an $l \times l$-real matrix that we also denote by the same symbol.

Since $(\alpha_0, \beta_0, 0)$ is an isolated center, in a small neighbourhood of (α_0, β_0), we have that $A_{j,m}(\alpha, \beta)$ is an isomorphism if $(\alpha, \beta) \neq (\alpha_0, \beta_0)$.

Step 5. Finite-dimensional reduction. By the compactness argument ([22]), we can assume that W is finite-dimensional and $F\colon \mathbb{C} \oplus W \to W$ is a G-equivariant map of class C^1 such that:

(i) $F(z, 0) = 0$ for all $z = \alpha + i\beta \in \mathbb{C}$,
(ii) $(z_0, 0)$ with $z_0 = \alpha_0 + i\beta_0$, is an isolated singular point of F,
(iii) $D_w F(z, 0)_{|V_m^c} = (1/im\beta)\Delta_\alpha(i\beta)$ and $D_w F(z, 0)_{|V} = (1/\beta) D_x f(\alpha, 0)$, for $z = \alpha + i\beta$ in a neighbourhood of z_0.

Step 6. Equivariant degree associated with a bifurcation point. Following the well-known idea of J. Ize, we can use an invariant complementing function θ, defined in a special neighbourhood

$$\Omega := \{(z, v) \in \mathbb{C} \oplus W : |z - z_0| < \varepsilon,\ \|v\| < r\}$$

(where $\varepsilon > 0$ and $r > 0$ are chosen to be sufficiently small) by

$$\theta(z, v) := |z - z_0|(\|v\| - r) + \|v\|, \quad (z, v) \in \overline{\Omega}.$$

Put $F_\theta(z, v) = (\theta(z, v), F(z, v)) \in \mathbb{R} \oplus W$. Clearly, F_θ is G-equivariant and, moreover, we can always choose $\varepsilon > 0$ and $r > 0$ such that $F_\theta(z, v) \neq (0, 0)$ for $(z, v) \in \partial\Omega$. Therefore, F_θ is Ω-admissible and G-$\mathrm{Deg}\,(F_\theta, \Omega)$ is well-defined. This degree (see [21,22]) characterizes possible symmetries of the bifurcating periodic solutions to (7.1) in the following sense: For G-$\mathrm{Deg}\,(F_\theta, \Omega) = \sum n_H(H)$, if $n_H \neq 0$ then there exists a branch of periodic solutions $(\alpha, \beta, x(t))$ with symmetries at least H. Moreover, in the "generic" situation, there will exist at least n_H branches with symmetries exactly H (cf. [22]).

Step 7. Bifurcation invariants. Let $\Sigma = \{z \in \mathbb{C} : |z - z_0| = \delta\}$, where $\delta > 0$ is a sufficiently small number and let D^2 denote the interior of the circle Σ. We define

$$\nu_j = \begin{cases} 1 & \text{if } \det_{\mathbb{R}} A_{j,0}(\alpha_0, \beta_0) > 0, \\ -1 & \text{if } \det_{\mathbb{R}} A_{j,0}(\alpha_0, \beta_0) < 0, \end{cases}$$

and

$$\xi_{j,m} = \deg(\det_{\mathbb{C}}(A_{j,m}), D^2).$$

The integer $\xi_{j,m}$ is the (j,m)-th *crossing number* associated with $z_0 = \alpha_0 + i\beta_0$, i.e.

$$\xi_{j,m} = \deg(\det_{\mathbb{C}}\Delta_{\alpha_-}(\,\cdot\,)_{|U_j^c}, O_m) - \deg(\det_{\mathbb{C}}\Delta_{\alpha_+}(\,\cdot\,)_{|U_j^c}, O_m),$$

where U_j^c is the complexification of U_j, $O_m := \{u+i\beta : |\beta - m\beta_0| < \delta,\ 0 < -u < \delta\}$, $\delta > 0$ is a sufficiently small number and $0 < \alpha_0 - \alpha_- < \delta$, $0 < \alpha_+ - \alpha_0 < \delta$ (see [18]–[22] for more details). Recall that in the all above formulae we use the matrix representation of the equivariant operators. In simple words, the crossing number $\xi_{j,m}$ is equal to the number of characteristic roots (counting according to their multiplicities) of

$$\det_{\mathbb{C}}\Delta_\alpha(\lambda)_{|U_j^c} = 0$$

that move through the value $im\beta_0$ from the left complex half-plane to the right half-plane as α crosses α_0, divided by $\dim \mathcal{V}_{r_j}$.

We put

$$\mu(z_0) = \sum_{j,m} \xi_{j,m} G\text{-Deg}\,(f_{j,m}, \Omega_{j,m}),$$

where $\Omega_{j,m} = \{(z,v) \in \mathbb{C} \oplus \mathcal{V}_{r_j}^{c,m} : \|v\| < 1,\ 1/2 < |z| < 2\}$ and $f_{j,m} : \mathbb{C} \oplus \mathcal{V}_{r_j}^{c,m} \to \mathbb{R} \oplus \mathcal{V}_{r_j}^{c,m}$ is given by formula (5.1). We also put

$$\nu(z_0) = \prod_j SO(3)\text{-Deg}(\nu_j \mathrm{Id}; D_j),$$

where D_j is the unit ball in $\mathcal{V}_{r_j}$.

The above bifurcation invariants turn out to contain full information needed to evaluate the primary equivariant degree G-Deg$\,(F_\theta, \Omega)$ (see Step 6). Namely, by applying the same ideas as in [21], [23], [24], one can establish the following result

Theorem. *Under the above assumptions, the primary $SO(3) \times S^1$-equivariant degree associated with the bifurcation point* $(\alpha_0, \beta_0, 0)$ *for the equation* (7.1) *is given by the following formula:*

$$\begin{aligned} G\text{-Deg}\,(F_\theta, \Omega) &= \nu(z_0)\mu(z_0) \\ &= \prod_j SO(3)\text{-Deg}(\nu_j Id, D_j)\Big[\sum_{j,m} \xi_{j,m} G\text{-Deg}\,(f_{j,m}, \Omega_{j,m})\Big] \end{aligned}$$

Example. We consider the following orthogonal $SO(3)$-representation

$$V = U_1 \oplus U_2,$$

where $U_1 = \mathcal{V}_3 \oplus \mathcal{V}_3$ and $U_2 = \mathcal{V}_4 \oplus \mathcal{V}_4 \oplus \mathcal{V}_4$. For $x \in V$ we denote $x = (x_1, x_2, x_3, x_4, x_5)$ the components of the vector x with x_1, $x_2 \in \mathcal{V}_3$ and x_3, x_4, $x_5 \in \mathcal{V}_4$. As an example we consider the following system of equations

$$\begin{cases} \dot{x}_1 &= \alpha x_1 - x_2 + \varphi_1(\alpha, x), \\ \dot{x}_2 &= x_1 + \alpha x_2 + \varphi_2(\alpha, x), \\ \dot{x}_3 &= -\alpha x_3 - x_4 + \varphi_3(\alpha, x), \\ \dot{x}_4 &= x_3 - \alpha x_4 + \varphi_4(\alpha, x), \\ \dot{x}_5 &= -x_5 + \varphi_5(\alpha, x), \end{cases}$$

where $\varphi_j(\alpha, x) = o(\|x\|)$ as $x \to 0$ are C^1-differentiable $SO(3)$-equivariant functions for $j = 1, \dots, 5$. It is easy to notice by studying the characteristic equation for this system that there is a Hopf bifurcation for $\alpha_0 = 0$ with the frequency $\beta_0 = 1$. By inspection, we obtain $\nu_1 = 1$, $\nu_2 = -1$ and $\mu_1 = -1$, $\mu_2 = 1$. Consequently, we have for $z_0 = (0, 1)$

$$\begin{aligned} \nu(z_0, 0) &= SO(3)\text{-Deg}\,(-\mathrm{Id}\,, D_4) \\ &= (SO(3)) - 2(S_4) - 2(O(2)) + 3(D_4) + 3(D_3) - (V_4), \end{aligned}$$

and

$$\begin{aligned} \mu(z_0, 0) = &- G\text{-Deg}\,(f_{3,1}, \Omega_{3,1}) + G\text{-Deg}\,(f_4, \Omega_{4,1}) \\ = &- [(O(2)^-) + (S_4^-) - (V_4^-) + (D_6^d) \\ &+ (SO(2)^1) + (SO(2)^2) + (SO(2)^3)] \\ &+ \Big[(A_4^t) + (D_8^d) + (D_6^d) + (D_4^d) + (O(2)) + (S_4) - (V_4^-) \\ &+ \sum_{k=1}^{4} (SO(2)^k) - (D_4) - (D_3) - (V_4)\Big] \\ = &\,(A_4^t) + (D_8^d) + 2(D_6^d) + (D_4^d) + (O(2)) - (O(2)^-) \\ &+ (S_4) - (S_4^-) + (SO(2)^4) - (D_4) - (D_3) - (V_4). \end{aligned}$$

Therefore, applying our previous computations and multiplication tables, we obtain

$$\begin{aligned} G\text{-Deg}\,(F_\theta, \Omega) = &\,\nu(z_0, 0) \cdot \mu(z_0, 0) \\ = &\,((SO(3)) - 2(S_4) - 2(O(2)) + 3(D_4) + 3(D_3) - (V_4)) \\ &\cdot ((A_4^t) + (D_8^d) + 2(D_6^d) + (D_4^d) + (O(2)) - (O(2)^-) \\ &+ (S_4) - (S_4^-) + (SO(2)^4) - (D_4) - (D_3) - (V_4)) \\ = &- (A_4^t) - (D_8^d) - 2(D_6^d) - (D_4^d) \\ &- (O(2)) - 3(O(2)^-) - (S_4) + (S_4^-) - 3(SO(2)^4) \\ &+ 3(D_4) + 3(D_4) - 25(V_4) - 13(V_4^-) + 5(D_3^z). \end{aligned}$$

7.2. Steady-state bifurcation with $SO(3)$-symmetries. In the following example we present an application of the reduced equivariant degree to $SO(3)$-symmetric steady-state bifurcation problem with two parameters.

Example. For $\Gamma = SO(3)$, we consider the following orthogonal representation $V := \mathcal{V}_2 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 \oplus \mathcal{V}_3$. We denote by $x = (x_1, x_2, x_3, x_4)$ the components of the vector $x \in V$. We consider the following two-parameter system of equations

$$\begin{cases} \dot{x}_1 &= \sqrt{\alpha^2+\beta^2}\,(\cos(\alpha)\,x_1 - \sin(\alpha)\,x_2) + \varphi_1(\alpha,\beta,x), \\ \dot{x}_2 &= \sqrt{\alpha^2+\beta^2}\,(\sin(\alpha)\,x_1 + \cos(\alpha)\,x_2) + \varphi_2(\alpha,\beta,x), \\ \dot{x}_3 &= \sqrt{\alpha^2+\beta^2}\,(\cos(\alpha)\,x_3 - \sin(\alpha)\,x_4) + \varphi_3(\alpha,\beta,x), \\ \dot{x}_4 &= \sqrt{\alpha^2+\beta^2}\,(\sin(\alpha)\,x_3 + \cos(\alpha)\,x_4) + \varphi_4(\alpha,\beta,x), \end{cases}$$

where $\varphi_j(\alpha,\beta,x) = o(\|x\|)$ as $x \to 0$ are continuous $SO(3)$-equivariant functions for $j = 1,2,3,4$. We are looking for stationary points bifurcating from the trivial solutions $\{(\alpha,\beta,0) : \alpha,\beta \in \mathbb{R}\}$. Consider the map $f_\theta\colon \mathbb{R}^2 \oplus V \to \mathbb{R} \oplus V$, defined by

$$f_\theta(\alpha,\beta,x) = \begin{bmatrix} \theta(\alpha,\beta,x) \\ \sqrt{\alpha^2+\beta^2}\,(\cos(\alpha)\,x_1 - \sin(\alpha)\,x_2) + \varphi_1(\alpha,\beta,x) \\ \sqrt{\alpha^2+\beta^2}\,(\sin(\alpha)\,x_1 + \cos(\alpha)\,x_2) + \varphi_2(\alpha,\beta,x)) \\ \sqrt{\alpha^2+\beta^2}\,(\cos(\alpha)\,x_3 - \sin(\alpha)\,x_4) + \varphi_3(\alpha,\beta,x) \\ \sqrt{\alpha^2+\beta^2}\,(\sin(\alpha)\,x_3 + \cos(\alpha)\,x_4) + \varphi_4(\alpha,\beta,x) \end{bmatrix},$$

where θ is an invariant complementing function defined in a special neighbourhood U of the point $(0,0) \in \mathbb{R}^2 \oplus V$. By applying the standard deformations (cf. [20]), we can show that

$$\deg^*_\Gamma(f_\theta, U) = \deg^*_\Gamma(f_2, \Omega_2) + \deg^*_\Gamma(f_3, \Omega_3),$$

where f_j and Ω_j, $j = 2,3$, are given by formula (5.1), and $z = \beta + i\alpha \in \mathbb{C} = \mathbb{R}^2$. Consequently, we have

$$\deg^*_\Gamma(f_\theta, U) = (O(2)) + (V_4) + (A_4) + (SO(2)) + (D_3).$$

We can now predict the occurrence of the bifurcations of stationary solutions with $(O(2))$, (A_4), $(SO(2))$ and (D_3) symmetries and possibly with (V_4)-symmetries.

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