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AN EXISTENCE RESULT FOR A CLASS OF QUASILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS WITH JUMPING NONLINEARITIES

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Abstract. We establish an existence result for a class of quasilinear elliptic boundary value problems with jumping nonlinearities using variational arguments. First we calculate certain homotopy groups of sublevel sets of the asymptotic part of the variational functional. Then we use these groups to show that the full functional admits a linking geometry and hence a min-max critical point.

1. Introduction

Consider the quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} + f(x,u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$ is the p-Laplacian, $1 < p < \infty$, $u^\pm = \max\{\pm u, 0\}$, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying a growth condition

$$|f(x,t)| \leq V(x)^{p-q}\,|t|^{q-1} + W(x)^{p-1} \tag{1.2}$$

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with $1 \leq q < p$ and $V, W \in L^p(\Omega)$. When $a = b \notin \sigma(-\Delta_p)$ and $q = 1$, the solvability of (1.1) was recently established by Drábek and Robinson ([10]). This result was generalized to include certain off-diagonal points (a, b), $a \neq b$ and $1 \leq q < p$ in Perera ([22]). In particular, (1.1) was solved there when (a, b) is in a neighbourhood of a point of the form (a_0, a_0) with $a_0 \notin \sigma(-\Delta_p)$. In the present paper we prove an existence result for (1.1) that extends and complements the theorems of [10] and [22].

The set Σ_p of those points $(a, b) \in \mathbb{R}^2$ for which the asymptotic problem

$$\begin{cases} -\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

has a nontrivial solution is called the Fučík spectrum of the p-Laplacian on Ω (see Fučík [12] and Dancer [6]). This notion generalizes the usual spectrum, which corresponds to $a = b$. In the ODE case $n = 1$, Fučík ([12]) and Drábek ([9]) have shown that Σ_p consists of a sequence of decreasing curves emanating from the points (λ_l, λ_l), where $\{\lambda_l\}$ are the eigenvalues of $-\Delta_p$, with one or two curves coming from each point. It has also been shown that in the semilinear PDE case $p = 2$, $n \geq 2$, Σ_2 consists locally of curves emanating from (λ_l, λ_l) (see e.g. [3], [5], [6], [11]–[13], [15]–[17], [20], [21], [26]).

In the quasilinear PDE case $p \neq 2$, $n \geq 2$, it is known that $\sigma(-\Delta_p)$ has an unbounded sequence of "variational" eigenvalues $\{\lambda_l\}$ satisfying a standard min-max characterization, but it is not known whether this is a complete list of eigenvalues. It was shown in Lindqvist ([19]) that the first eigenvalue λ_1 is positive, simple, and admits a positive eigenfunction φ_1, so Σ_p clearly contains the two lines $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$. A first nontrivial curve in Σ_p through (λ_2, λ_2) asymptotic to $\lambda_1 \times \mathbb{R}$ and $\mathbb{R} \times \lambda_1$ at infinity was recently constructed and variationally characterized by a mountain-pass procedure in Cuesta, de Figueiredo, and Gossez ([4]). More recently, an unbounded sequence of continuous and strictly decreasing curves in Σ_p was constructed and variationally characterized by a min-max procedure in Perera ([22]).

Let us also mention that some Morse theoretical aspects of the Fučík spectrum have been studied in Dancer and Perera ([8]), Dancer ([7]), Perera and Schechter ([23]–[25]), and Li, Perera, and Su ([18]).

Our main result here is the following.

Theorem 1.1. *Assume that (a, b) can be joined to a point of the form (a_0, a_0) with $a_0 \notin \sigma(-\Delta_p)$ by a path that does not intersect Σ_p. Then* (1.1) *has a solution. In particular, the problem*

$$\begin{cases} -\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

has a solution for each $f \in L^{p/(p-1)}(\Omega)$.

For $p = 2$, this result is known and is obtained via the Leray–Schauder degree, which cannot be applied when $p \neq 2$. As is well-known, solutions of (1.1) are the critical points of the C^1-functional

$$\Phi(u) = \int_\Omega |\nabla u|^p - a(u^+)^p - b(u^-)^p - pF(x,u), \quad u \in X = W_0^{1,p}(\Omega) \tag{1.5}$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. We will obtain a critical point of Φ using homotopy theory and linking type arguments.

We use the customary notation

$$\Phi^\alpha = \{u \in X : \Phi(u) \leq \alpha\} \tag{1.6}$$

for sublevel sets. For a pair of topological spaces (X, A) with basepoint $x_0 \in A$, $\pi_*(X, x_0)$ and $\pi_*(X, A, x_0)$ denote the absolute and relative homotopy groups, respectively.

2. Preliminaries on the Fučík spectrum

Consider the C^1-functional

$$J_s(u) = \int_\Omega |\nabla u|^p - s(u^+)^p \tag{2.1}$$

defined on the C^1-Finsler manifold

$$S = \{u \in X : \|u\|_{L^p(\Omega)} = 1\} \tag{2.2}$$

where $X = W_0^{1,p}(\Omega)$. It is easily seen from the Lagrange multiplier rule that the points in Σ_p on the line parallel to the diagonal $a = b$ and passing through $(s, 0)$ are exactly of the form $(s + c, c)$ with c a critical value of J_s. Moreover, J_s satisfies the Palais-Smale compactness condition (PS) (see [4]).

In particular, the eigenvalues of $-\Delta_p$ correspond to the critical values of the even functional $J = J_0$ and we can define an unbounded sequence of variational eigenvalues as follows. For $l \in \mathbb{N}$, let S^{l-1} be the unit sphere in $\mathbb{R}^l$ with basepoint $e_1 = (1, 0, \ldots, 0)$, denote by $\mathcal{S}_{l-1}$ the set of odd continuous maps $\phi\colon (S^{l-1}, e_1) \to (S, \varphi_1)$ (i.e. $\phi\colon S^{l-1} \to S$ and $\phi(e_1) = \varphi_1$), and set

$$\lambda_l := \inf_{\phi \in \mathcal{S}_{l-1}} \max_{u \in \phi(S^{l-1})} J(u). \tag{2.3}$$

Lemma 2.1. *λ_l is an eigenvalue of $-\Delta_p$ and $\lambda_l \to \infty$.*

Proof. Clearly, $\lambda_1 = \inf J(S)$ is the first eigenvalue. Denoting by $\{\mu_l\}$ the usual Lusternik–Schnirelmann eigenvalues, we have $\lambda_l \geq \mu_l$ since the genus of $\phi(S^{l-1})$ is l for each $\phi \in \mathcal{S}_{l-1}$. In particular, for $l \geq 2$, $\lambda_l \geq \mu_2 > \mu_1 = \lambda_1 = J(\varphi_1)$, so if λ_l is a regular value of J, then there is an $\varepsilon > 0$ and an odd homeomorphism $\eta\colon S \to S$ such that $\eta(\varphi_1) = \varphi_1$ and $\eta(J^{\lambda_l+\varepsilon}) \subset J^{\lambda_l-\varepsilon}$ by a lemma of Bonnet ([1]). But then taking $\phi \in \mathcal{S}_{l-1}$ with $\max J(\phi(S^{l-1})) < \lambda_l + \varepsilon$ and setting $\widetilde{\phi} = \eta \circ \phi$, we get a map in $\mathcal{S}_{l-1}$ for which $\max J(\widetilde{\phi}(S^{l-1})) < \lambda_l - \varepsilon$, contradicting the definition of λ_l. Finally, $\lambda_l \to \infty$ since $\mu_l \to \infty$. □

In the above proof we used the deformation lemma of Bonnet instead of the standard first deformation lemma because S is not of class $C^{1,1}$ when $p < 2$.

LEMMA 2.2. *If $\lambda_l < a_0 < \lambda_{l+1}$, then*

$$\pi_{l-1}(J^{a_0}, \varphi_1) \neq 0. \tag{2.4}$$

PROOF. Since $a_0 > \lambda_l$, there is a map $\phi \in \mathcal{S}_{l-1}$ such that $\phi(S^{l-1}) \subset J^{a_0}$. We claim that the homotopy class of ϕ is nontrivial in $\pi_{l-1}(J^{a_0}, \varphi_1)$. Suppose that ϕ is homotopic to the constant map sending S^{l-1} to φ_1. This gives a continuous map $\widetilde{\phi}^+$ of the upper hemisphere S^l_+ in $\mathbb{R}^{l+1}$ with boundary S^{l-1} into J^{a_0} such that $\widetilde{\phi}^+|_{S^{l-1}} = \phi$, which can then be extended to a map $\widetilde{\phi} \in \mathcal{S}_l$ such that $\widetilde{\phi}(S^l) \subset J^{a_0}$. This is a contradiction since $a_0 < \lambda_{l+1}$. □

LEMMA 2.3. *If $\alpha > \lambda_1 - s$ and J_s has no critical values in $[\alpha, \beta]$, then*

$$\pi_*(J_s^\beta, J_s^\alpha, \varphi_1) = 0, \tag{2.5}$$

and hence

$$\pi_*(J_s^\alpha, \varphi_1) \cong \pi_*(J_s^\beta, \varphi_1). \tag{2.6}$$

When $p \geq 2$, this lemma is an immediate consequence of the well-known second deformation lemma. The case $p < 2$ is more delicate again because S is only C^1 then, so we will use a deformation lemma of Ghoussoub ([14]).

PROOF OF LEMMA 2.3. Suppose that the homotopy class of the map

$$\phi\colon (D^q, S^{q-1}, e_1) \to (J_s^\beta, J_s^\alpha, \varphi_1),$$

where D^q is the unit disk in $\mathbb{R}^q$ with boundary S^{q-1}, is nontrivial in $\pi_q(J_s^\beta, J_s^\alpha, \varphi_1)$. Set

$$c := \inf_{\phi' \in [\phi]} \max_{u \in \phi'(D^q)} J_s(u). \tag{2.7}$$

Clearly, $c \in [\alpha, \beta]$. We will show that c is a critical value of J_s.

We follow the proof of Theorem 3.2 in Ghoussoub ([14]). Consider the subspace $\mathcal{L}$ of $C([0,1] \times S; S)$ consisting of all continuous deformations η such that

(i) $\eta(t,u) = u$ for all $(t,u) \in (\{0\} \times S) \cup ([0,1] \times \{\varphi_1\})$,
(ii) $\sup\{\rho(\eta(t,u), u) : (t,u) \in [0,1] \times S\} < \infty$ where ρ is the Finsler metric on S,
(iii) $J_s(\eta(t,u)) \leq J_s(u)$ for all $(t,u) \in [0,1] \times S$.

$\mathcal{L}$ equipped with the metric

$$\delta(\eta,\eta') = \sup\{\rho(\eta(t,u),\eta'(t,u)) : (t,u) \in [0,1] \times S\} \tag{2.8}$$

is a complete metric space. For any $\eta \in \mathcal{L}$, (i) and (iii) imply that $\eta(t,\cdot\,) : (J_s^\beta, J_s^\alpha, \varphi_1) \to (J_s^\beta, J_s^\alpha, \varphi_1)$ for each $t \in [0,1]$, so $\eta(1,\phi'(\,\cdot\,)) \in [\phi]$ for any $\phi' \in [\phi]$. Fix $\varepsilon > 0$, take $\phi' \in [\phi]$ such that

$$c \le \max J_s(\phi'(D^q)) < c + \varepsilon^2, \tag{2.9}$$

and define a continuous function $I\colon \mathcal{L} \to [c, c+\varepsilon^2)$ by

$$I(\eta) = \max J_s(\eta(1,\phi'(D^q))). \tag{2.10}$$

Let $\overline{\eta}$ be the identity in $\mathcal{L}$ (i.e. $\overline{\eta}(t,u) = u$ for all $(t,u) \in [0,1] \times S$), and note that

$$I(\overline{\eta}) < c + \varepsilon^2 \le \inf_{\mathcal{L}} I + \varepsilon^2. \tag{2.11}$$

Applying the Ekeland's principle, we get an $\eta_0 \in \mathcal{L}$ such that

$$I(\eta_0) \le I(\overline{\eta}), \tag{2.12}$$

$$\delta(\eta_0,\overline{\eta}) \le \varepsilon, \tag{2.13}$$

$$I(\eta) \ge I(\eta_0) - \varepsilon\delta(\eta,\eta_0) \quad \text{for all } \eta \in \mathcal{L}. \tag{2.14}$$

Let $C = \{u \in \eta_0(1,\phi'(D^q)) : J_s(u) = I(\eta_0)\}$. Since J_s satisfies (PS) and $c \le J_s(u) < c+\varepsilon^2$ for all $u \in C$, it is enough to show that there is a $u_\varepsilon \in C$ such that $\|J_s'(u_\varepsilon)\| \le 4\varepsilon$.

Suppose now that $\|J_s'(u)\| > 4\varepsilon$ for all $u \in C$. Applying Lemma 3.7 of [14], we get $t_0 \in (0,1]$, $\alpha \in C([0,t_0) \times S; S)$, and $g \in C(S;(0,\infty))$ such that

(i) $\alpha(t,\varphi_1) = \varphi_1$,
(ii) $\rho(\alpha(t,u),u) \le 3t/2$ for all $(t,u) \in [0,t_0) \times S$,
(iii) $J_s(\alpha(t,u)) - J_s(u) \le -2\varepsilon g(u)t$ for all $(t,u) \in [0,t_0) \times S$,
(iv) $g(u) = 1$ for all $u \in C$.

For $0 < \lambda < t_0$, let $\eta_\lambda(t,u) = \alpha(t\lambda,\eta_0(t,u))$. Clearly, $\eta_\lambda \in \mathcal{L}$. Since $\delta(\eta_\lambda,\eta_0) \le 3t\lambda/2 < 3\lambda/2$ by (ii), (2.14) gives $I(\eta_\lambda) > I(\eta_0) - 3\varepsilon\lambda/2$. Since $\phi'(D^q)$ is compact, there is a $u_\lambda \in \phi'(D^q)$ such that $J_s(\eta_\lambda(1,u_\lambda)) = I(\eta_\lambda)$, so we have

$$J_s(\eta_\lambda(1,u_\lambda)) - J_s(\eta_0(1,u)) > -3\varepsilon\lambda/2 \quad \text{for all } u \in \phi'(D^q). \tag{2.15}$$

In particular, if u_0 is any cluster point of (u_λ) as $\lambda \to 0$, then $\eta_0(1,u_0) \in C$, and hence

$$g(\eta_0(1,u_0)) = 1 \tag{2.16}$$

by (iv). On the other hand,

$$
\begin{aligned}
(2.17)\qquad J_s(\eta_\lambda(1,u_\lambda)) - J_s(\eta_0(1,u_\lambda)) &= J_s(\alpha(\lambda,\eta_0(1,u_\lambda)) - J_s(\eta_0(1,u_\lambda)) \\
&\le -2\varepsilon\lambda g(\eta_0(1,u_\lambda))
\end{aligned}
$$

by (iii), and combining this with (2.15) gives

$$
(2.18)\qquad g(\eta_0(1,u_\lambda)) < 3/4,
$$

which contradicts (2.16).

Finally, (2.6) follows from (2.5) and the long exact sequence of the pair (J_s^β, J_s^α). □

Next we note that on S the asymptotic part $\int_\Omega |\nabla u|^p - a(u^+)^p - b(u^-)^p$ of the functional Φ in (1.5) can be written as $J_{a-b}(u) - b$ and that $\varphi_1 \in J_{a-b}^b$ if $a > \lambda_1$. The main result of this section is the following.

LEMMA 2.4. *If $a_0 > \lambda_1$ and (a,b) can be joined to (a_0,b_0) by a path that does not intersect Σ_p, then $a > \lambda_1$ and*

$$
(2.19)\qquad \pi_*(J_{a-b}^b, \varphi_1) \cong \pi_*(J_{a_0-b_0}^{b_0}, \varphi_1).
$$

PROOF. Since $\lambda_1 \times \mathbb{R} \subset \Sigma_p$, $a > \lambda_1$. We will show that if $(a_0,b_0) \notin \Sigma_p$ and (a,b) is sufficiently close to (a_0,b_0), then (2.19) holds.

Choose $\varepsilon > 0$ so small that $(a,b) \notin \Sigma_p$ whenever

$$
(2.20)\qquad |a-a_0| + |b-b_0| < 5\varepsilon,
$$

which is possible since Σ_p is closed, and suppose that $|a-a_0|+|b-b_0| < \varepsilon$. Then

$$
(2.21)\qquad |J_{a-b}(u) - J_{a_0-b_0}(u)| = |(a-a_0)-(b-b_0)| \int_\Omega (u^+)^p < \varepsilon,
$$

for all $u \in S$, so we have the inclusions

$$
(2.22)\qquad J_{a_0-b_0}^{b-\varepsilon} \hookrightarrow J_{a-b}^b \overset{i}{\hookrightarrow} J_{a_0-b_0}^{b+\varepsilon} \hookrightarrow J_{a-b}^{b+2\varepsilon},
$$

which induce the commutative diagram

$$
(2.23)\qquad
\begin{tikzcd}
& \pi_*(J_{a-b}^b, \varphi_1) \arrow[dd, "i_*"] \arrow[dr, "i''_*"] & \\
\pi_*(J_{a_0-b_0}^{b-\varepsilon}, \varphi_1) \arrow[ur] \arrow[dr, "i''_*"] & & \pi_*(J_{a-b}^{b+2\varepsilon}, \varphi_1) \\
& \pi_*(J_{a_0-b_0}^{b+\varepsilon}, \varphi_1) \arrow[ur] &
\end{tikzcd}
$$

Since the points $(a_0-b_0+b-\varepsilon, b-\varepsilon)$ and $(a_0-b_0+b+\varepsilon, b+\varepsilon)$ (resp. (a,b) and $(a+2\varepsilon, b+2\varepsilon)$) satisfy (2.20), $J_{a_0-b_0}$ (resp. J_{a-b}) has no critical values in

$[b-\varepsilon, b+\varepsilon]$ (resp. $[b, b+2\varepsilon]$), so i'_* (resp. i''_*) is an isomorphism by Lemma 2.3. Thus i_* is an isomorphism. Finally,

$$\pi_*(J^{b+\varepsilon}_{a_0-b_0}, \varphi_1) \cong \pi_*(J^{b_0}_{a_0-b_0}, \varphi_1) \tag{2.24}$$

as $J_{a_0-b_0}$ has no critical values between b_0 and $b+\varepsilon$. $\square$

3. Proof of Theorem 1.1

Let Φ be the functional given by (1.5).

Lemma 3.1. *If $(a,b) \notin \Sigma_p$, then Φ satisfies* (PS), *i.e. any sequence (u_j) in X such that $\Phi(u_j)$ is bounded and $\|\Phi'(u_j)\| \to 0$ has a convergent subsequence.*

Proof. It suffices to show that (u_j) is bounded by a standard argument, so suppose that $\rho_j = \|u_j\| \to \infty$. Setting $v_j = u_j/\rho_j$ and passing to a subsequence, $v_j \to v$ weakly in X, strongly in $L^p(\Omega)$, and a.e. in Ω. Thus

$$\begin{aligned}\int_\Omega |\nabla v_j|^{p-2}\nabla v_j \cdot \nabla(v_j - v) &= \frac{(\Phi'(u_j), v_j - v)}{p\rho_j^{p-1}} \\ &\quad + \int_\Omega \left[a\,(v_j^+)^{p-1} - b(v_j^-)^{p-1} + \frac{f(x,u_j)}{\rho_j^{p-1}}\right](v_j - v) \to 0,\end{aligned} \tag{3.1}$$

and we deduce that $v_j \to v$ strongly in X (see, e.g., Browder [2]). In particular, $\|v\| = 1$, so $v \neq 0$. Now passing to the limit in

$$\frac{(\Phi'(u_j)w)}{p\rho_j^{p-1}} = \int_\Omega |\nabla v_j|^{p-2}\nabla v_j \cdot \nabla w - \left[a(v_j^+)^{p-1} - b(v_j^-)^{p-1} + \frac{f(x,u_j)}{\rho_j^{p-1}}\right] w \tag{3.2}$$

gives

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla w - (a(v^+)^{p-1} - b(v^-)^{p-1})w = 0 \quad \text{for all } w \in X, \tag{3.3}$$

i.e. v is a nontrivial solution of (1.1). Impossible since $(a,b) \notin \Sigma_p$. $\square$

We are now ready to prove Theorem 1.1. If $a_0 < \lambda_1$, then $a, b < \lambda_1$ since $(\lambda_1 \times \mathbb{R}) \cup (\mathbb{R} \times \lambda_1) \subset \Sigma_p$, so it is easily seen from (1.2) and the Sobolev imbedding that Φ is bounded below and hence admits a minimizer. So suppose that $\lambda_l < a_0 < \lambda_{l+1}$. Then it follows from Lemmas 2.2 and 2.4 that

$$\pi_{l-1}(J^b_{a-b}, \varphi_1) \neq 0. \tag{3.4}$$

Choose $\varepsilon > 0$ so small that the line segment joining $(a-\varepsilon, b-\varepsilon)$ and $(a+\varepsilon, b+\varepsilon)$ does not intersect Σ_p. Then J_{a-b} has no critical values in $[b-\varepsilon, b+\varepsilon]$ and hence Lemma 2.3 and (3.4) imply that

$$\pi_{l-1}(J^{b-\varepsilon}_{a-b}, \varphi_1) \neq 0 \tag{3.5}$$

and that the inclusion $J^{b-\varepsilon}_{a-b} \hookrightarrow J^{b+\varepsilon}_{a-b}$ induces an isomorphism

$$\pi_{l-1}(J^{b-\varepsilon}_{a-b}, \varphi_1) \to \pi_{l-1}(J^{b+\varepsilon}_{a-b}, \varphi_1). \tag{3.6}$$

So there is a continuous map $\phi\colon (S^{l-1}, e_1) \to (J^{b-\varepsilon}_{a-b}, \varphi_1)$ whose homotopy class is nontrivial in $\pi_{l-1}(J^{b+\varepsilon}_{a-b}, \varphi_1)$, i.e. for every continuous map $\widetilde{\phi}\colon D^l \to S$ such that $\widetilde{\phi}|_{S^{l-1}} = \phi$,

$$\widetilde{\phi}(D^l) \cap (S \setminus J^{b+\varepsilon}_{a-b}) \neq \emptyset. \tag{3.7}$$

Let

$$F = \{su : s \geq 0,\ u \in S \setminus J^{b+\varepsilon}_{a-b}\} \tag{3.8}$$

and note that, for $su \in F$,

$$\Phi(su) = s^p(J_{a-b}(u) - b) - p \int_\Omega F(x, su) \geq \varepsilon s^p - C(s^q + s) \tag{3.9}$$

by (1.2). Similarly, for $u \in \phi(S^{l-1})$ and $R > 0$,

$$\Phi(Ru) \leq -\varepsilon R^p + C(R^q + R). \tag{3.10}$$

Fix R so large that

$$\sup \Phi(B) < \inf \Phi(F) \tag{3.11}$$

where $B = R\phi(S^{l-1})$, and let

$$\mathcal{F} = \{A \subset X : \text{there is a continuous surjection } \widetilde{\phi} : D^l \to A \text{ such that } \widetilde{\phi}|_{S^{l-1}} = R\phi\}. \tag{3.12}$$

Then $\mathcal{F}$ is a homotopy-stable family of compact subsets of X with closed boundary B, i.e.

(i) every set $A \in \mathcal{F}$ contains B,
(i) for any set $A \in \mathcal{F}$ and any $\eta \in C([0,1] \times X; X)$ satisfying $\eta(t, u) = u$ for all $(t, u) \in (\{0\} \times X) \cup ([0,1] \times B)$ we have that $\eta(\{1\} \times A) \in \mathcal{F}$.

Moreover, we claim that the set F is dual to the class $\mathcal{F}$, i.e.

$$F \cap B = \emptyset, \quad F \cap A \neq \emptyset \quad \text{for all } A \in \mathcal{F}. \tag{3.13}$$

It is clear from (3.11) that $F \cap B = \emptyset$. Let $A = \widetilde{\phi}(D^l) \in \mathcal{F}$. If $0 \in A$, then we are done. Otherwise, denoting by π the radial projection onto S, $(\pi \circ \widetilde{\phi})|_{S^{l-1}} = \phi$ and hence $\pi(A) \cap F \neq \emptyset$ by (3.7), which implies $A \cap F \neq \emptyset$.

Now set

$$c := \inf_{A \in \mathcal{F}} \max_{u \in A} \Phi(u). \tag{3.14}$$

Since $\sup \Phi(B) < c$ by (3.11) and (3.13), it follows from Theorem 3.2 of Ghoussoub ([14]) and Lemma 3.1 that c is a critical value of Φ.

REFERENCES

[1] A. BONNET, *A deformation lemma on a C^1-manifold*, Manuscripta Math. **81** (1993), 339–359.

[2] F. BROWDER, *Nonlinear eigenvalue problems and group invariance*, Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. of Chicago, Chicago, Ill., 1968), Springer, New York, 1970, pp. 1–58.

[3] N. CÁC, *On nontrivial solutions of a Dirichlet problem whose jumping nonlinearity crosses a multiple eigenvalue*, J. Differential Equations **80** (1989), 379–404.

[4] M. CUESTA, D. DE FIGUEIREDO AND J.-P. GOSSEZ, *The beginning of the Fučík spectrum for the p-Laplacian*, J. Differential Equations **159** (1999), 212–238.

[5] M. CUESTA AND J.-P. GOSSEZ, *A variational approach to nonresonance with respect to the Fučik spectrum*, Nonlinear Anal. **19** (1992), 487–500.

[6] E. N. DANCER, *On the Dirichlet problem for weakly non-linear elliptic partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **76** (1976/77), 283–300; Corrigendum: Proc. Roy. Soc. Edinburgh Sect. A **89** (1981), 15.

[7] ———, *Remarks on jumping nonlinearities*, Topics in Nonlinear Analysis, Birkhäuser, Basel, 1999, pp. 101–116.

[8] E. N. DANCER AND K. PERERA, *Some remarks on the Fučík spectrum of the p-Laplacian and critical groups*, J. Math. Anal. Appl. **254** (2001), 164–177.

[9] P. DRÁBEK, *Solvability and bifurcations of nonlinear equations* (1992), Longman Scientific & Technical, Harlow.

[10] P. DRÁBEK AND S. ROBINSON, *Resonance problems for the p-Laplacian.*, J. Funct. Anal. **169** (1999), 189–200.

[11] D. DE FIGUEIREDO AND J.-P. GOSSEZ, *On the first curve of the Fučik spectrum of an elliptic operator*, Differential Integral Equations **7** (1994), 1285–1302.

[12] S. FUČÍK, *Boundary value problems with jumping nonlinearities*, Časopis Pěst. Mat. **101** (1976), 69–87.

[13] T. GALLOUËT AND O. KAVIAN, *Résultats d'existence et de non-existence pour certains problèmes demi-linéaires à l'infini*, Ann. Fac. Sci. Toulouse Math. (5) **3** (1981, 1982), 201–246.

[14] N. GHOUSSOUB, *Duality and Perturbation Methods in Critical Point Theory*, Cambridge University Press, Cambridge, 1993.

[15] A. LAZER, *Introduction to multiplicity theory for boundary value problems with asymmetric nonlinearities*, Partial differential equations (Rio de Janeiro, 1986), Springer, Berlin, 1988, pp. 137–165.

[16] A. LAZER AND P. MCKENNA, *Critical point theory and boundary value problems with nonlinearities crossing multiple eigenvalues*, Comm. Partial Differential Equations **10** (1985), 107–150.

[17] ———, *Critical point theory and boundary value problems with nonlinearities crossing multiple eigenvalues II*, Comm. Partial Differential Equations **11** (1986), 1653–1676.

[18] S. J. LI, K. PERERA, AND J. B. SU, *On the role played by the Fučík spectrum in the determination of critical groups in elliptic problems where the asymptotic limits may not exist*, Nonlinear Anal. **49** (2002), 603–611.

[19] P. LINDQVIST, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc. **109** (1990), 157–164; Addendum: Proc. Amer. Math. Soc., **116** (1992), 583–584.

[20] C. MAGALHÃES, *Semilinear elliptic problem with crossing of multiple eigenvalues*, Comm. Partial Differential Equations **15** (1990), 1265–1292.

[21] C. MARGULIES AND W. MARGULIES, *An example of the Fučik spectrum*, Nonlinear Anal. **29** (1997), 1373–1378.

[22] K. Perera, *On the Fučík spectrum of the p-Laplacian*, NoDEA Nonlinear Differential Equations Appl. (to appear).

[23] K. Perera and M. Schechter, *Type* (II) *regions between curves of the Fučík spectrum and critical groups*, Topol. Methods Nonlinear Anal. **12** (1998), 227–243.

[24] ______, *A generalization of the Amann–Zehnder theorem to nonresonance problems with jumping nonlinearities*, NoDEA Nonlinear Differential Equations Appl. **7** (2000), 361–367.

[25] ______, *The Fučík spectrum and critical groups*, Proc. Amer. Math. Soc. **129** (2001), 2301–2308.

[26] M. Schechter, *The Fučík spectrum*, Indiana Univ. Math. J. **43** (1994), 1139–1157.

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