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ASYMPTOTICAL MULTIPLICITY AND SOME REVERSED VARIATIONAL INEQUALITIES

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Dedicated to Professor Andrzej Granas

Abstract. We are concerned with multiplicity results for solutions of some reversed variational inequalities, in which the inequality is opposite with respect to the classical inequalities introduced by Lions and Stampacchia. The inequalities we study arise from a family (P_ω) of elliptic problems of the fourth order when ω tends to ∞. We use two basic tools: the ∇-theorems and a theorem about the multiplicity of "asymptotically critical" points. In the last section some open problems are listed.

1. Introducing the problem

Some sequences of elliptic problems lead in a natural way to study some variational inequalities whose sign is opposite to the one of the usual inequalities of Lions–Stampacchia's type (see [8]). For this reason we call them "reversed" variational inequalities and, at least for the moment, we have found several difficulties in finding their deep sense. In short, we have many questions and few answers.

In order to present a possible genesis of the reversed variational inequalities, let us consider, for example, the bounce problem: if an open subset Ω of $\mathbb{R}^N$ represents the "billiard" and V is the potential energy of a conservative force field in $\mathbb{R}^N$, we can think to obtain the bounce trajectories $\gamma\colon[0,1]\to\overline{\Omega}$ between

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two points A and B of Ω, as limits of the sequences $(\gamma_n)_n$ of solutions of the problems

$$\ddot{\gamma}_n + \nabla V(\gamma_n) + \nabla U_n(\gamma_n) = 0, \quad \gamma_n(0) = A, \quad \gamma_n(1) = B, \tag{1.1}$$

where $(U_n)_n$ is a sequence of functions on $\mathbb{R}^N$ such that $U_n = 0$ in $\overline{\Omega}$ and $U_n \uparrow \infty$ outside $\overline{\Omega}$. If the functions U_n satisfy suitable conditions, denoted by $\nu(x)$ the inward unit normal to Ω in a point x of $\partial\Omega$, one finds that the limit trajectories between A and B satisfy the following reversed variational inequality:

$$\begin{cases} \displaystyle\int_0^1 \dot{\gamma} \cdot \dot{\delta}\, dt - \int_0^1 \nabla V(\gamma) \cdot \delta\, dt \leq 0 \\ \text{for all } \delta\colon [0,1] \to \mathbb{R}^N \text{ such that } \delta(0) = \delta(1) = 0 \\ \text{and } \delta(t) \cdot \nu(\gamma(t)) \geq 0 \text{ for all } t \in \{t \in [0,1] \mid \gamma(t) \in \partial\Omega\}. \end{cases} \tag{1.2}$$

Moreover, the condition of conservation of the energy

$$\frac{1}{2}|\dot{\gamma}|^2 + V(\gamma) = \text{constant} \tag{1.3}$$

holds if, for example, $f_n(\gamma_n) \to f(\gamma)$, where f_n are the functionals defined on the set $\Gamma_{AB} = \{\gamma \in H^1([0,1];\mathbb{R}^N) \mid \gamma(0) = A,\ \gamma(1) = B\}$ by

$$f_n(\gamma) = \frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2\, dt - \int_0^1 V(\gamma(t))\, dt - \int_0^1 U_n(\gamma(t))\, dt \tag{1.4}$$

and

$$f(u) = \begin{cases} \displaystyle\frac{1}{2}\int_0^1 |\dot{\gamma}(t)|^2\, dt - \int_0^1 V(\gamma(t))\, dt & \text{if } \gamma([0,1]) \subset \overline{\Omega}, \\ -\infty & \text{otherwise.} \end{cases}$$

Note that (1.2) does *not* imply (1.3). For example, if $V \equiv 0$, there exist continuous families of polygonal curves which solve (1.2) and do not verify (1.3).

Maybe it is also for this reason that there are no general results (to our knowledge) for the bounce problem between two given points, except for the case in which Ω is convex (see [4]).

Anyway, in the case in which Ω is not convex, one must keep in mind the counter-example by Penrose, which shows that there might be couples of points in Ω which cannot be joined by bounce trajectories (see [11]).

A problem in several variables which is analogous to the previous one is the following one. Let Ω be an open and bounded subset of $\mathbb{R}^N$, $\phi\colon \Omega \to \mathbb{R}$ a given negative measurable function and $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory's function. Consider the family of problems

$$\begin{cases} \Delta u + g(x,u) = \omega((u-\phi)^-)^p & \text{in } \Omega\ (p > 0), \\ u \in H_0^1(\Omega), \end{cases} \tag{1.5}$$

where ω is a real parameter. For every $\omega \neq 0$, (1.5) is a deeply asymmetric problem with respect to the values $u \ll 0$ or $u \gg 0$.

If $\omega \to -\infty$ we get an usual variational inequality. If, instead, $\omega \to +\infty$, the solutions of (1.5) tend to solutions of the following reversed variational inequality:

$$\text{(1.6)} \qquad \begin{cases} u \in K_\phi, \\ \displaystyle\int_\Omega Du \cdot D(v-u)\,dx - \int_\Omega g(x,u)(v-u)\,dx \le 0 \quad \text{for all } v \in K_\phi, \end{cases}$$

where $K_\phi = \{u \in H_0^1(\Omega) \mid u \ge \phi\}$. Note that it is also possible to obtain a relation analogous to (1.3). We must immediately remark that also inequality (1.6) can have a continuous family of solutions which don't seem meaningfully related to problems (1.5). However, problem (1.6) is, to our knowledge, still open.

In this paper we study the problem

$$\text{(P)} \qquad \begin{cases} u \in K_\phi, \\ \displaystyle\int_\Omega \Delta u \Delta(v-u)\,dx - c\int_\Omega Du \cdot D(v-u)\,dx \\ \qquad\qquad \displaystyle -\alpha\int_\Omega u(v-u)\,dx \le 0 \qquad\qquad \text{for all } v \in K_\phi, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $\alpha, c \in \mathbb{R}$, $\phi\colon \Omega \to \mathbb{R}$ is a given negative measurable function and $K_\phi = \{u \in H_0^1(\Omega) \cap H^2(\Omega) \mid u \ge \phi\}$.

This problem can be considered as the limit case of the problems of the fourth order

$$\text{(P}_\omega\text{)} \qquad \begin{cases} \Delta^2 u + c\Delta u - \alpha u + \omega((u-\phi)^-)^p = 0 & \text{in } \Omega (p > 0), \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

as ω tends to ∞.

In the case of problems (P$_\omega$), we can say that the part of "plate" which is below ϕ is subjected to an always intenser force which pulls it down. For this reason the equilibrium positions are always higher. In some sense, in the case of problem (P), the "plate" u is hooked to the rigid wall ϕ.

Remark 1.1. Note that (contrary to what happens for (1.2)), if for example $N = 1$, $c = \alpha = 0$, then (P) has a unique nontrivial solution, as it is easy to show.

In [15] it was proved that problems (P$_\omega$) admit at least a nontrivial solution and, for some values of c and α, at least 3 nontrivial solutions. It is important to note that 2 of this points, say $u_{1\omega}$ and $u_{2\omega}$, can possibly be at the same level of the functionals $f_\omega\colon H_0^1(\Omega) \cap H^2(\Omega) \to \mathbb{R}$ (see Section 3), whose critical points are the solutions of (P$_\omega$). The existence of such 2 points is obtained by using one of the "∇-theorems" introduced in [12].

The possible coincidence of the values $f_\omega(u_{1\omega})$ and $f_\omega(u_{2\omega})$ makes it hard to forecast if the limits u_1 and u_2 of $u_{1\omega}$ and $u_{2\omega}$ respectively as $\omega \to \infty$ are still distinct (if $N \le 3$ it is proved that such limits do exist). Therefore in [15] it is shown that, if $N \leq 3$, then (P) has at least 2 nontrivial solutions, for some c and α's.

In this paper we prove that for the same c and α's, problem (P) has at least 3 nontrivial solutions if $N \leq 3$; two of these solutions can be at the same level for the functional f whose "upper" critical points solve (P) (see Section 4). One of these two solutions and the third one are actually limits of solutions of problems (P_ω) (see [15]), while the other solution is only limit of functions u_ω such that $\nabla f_\omega(u_\omega) \to 0$ and $f_\omega(u_\omega) \to f(u)$ if $\omega \to \infty$.

We also remark that such solutions of problem (P) satisfy only the inequality (P) and *not* the corresponding equation.

In order to obtain the results above, we essentially used two tools.

(I) In this approach an essential role was played by the theorem about the multiplicity of the asymptotically critical points for a sequence $(h_n)_n$ of functionals which tend, in some sense, to a functional h (see Section 2. In Appendix we also give a nonsmooth version of this theorem).

We had already introduced this theorem in [10], inspired by some techniques adopted by [9] and [7] and then by [1] and [2] to study the multiplicity of the critical points of one functional with Galerkin type methods. In Section 2 we give a version of this theorem which is suitable for problem (P). Roughly speaking, this theorem gives an estimate for defect of the number of critical points of the functional h, by the topological properties of the functionals h_n.

The problem that the critical points of h_n may converge, for example, to a unique critical point of h (in the case the corresponding critical values of h_n converge to the same critical value of h), is by–passed, provided the forecast number of critical points of h is obtained by limits of points u_n such that $\nabla h_n(u_n) \to 0$ and not necessarily $\nabla h_n(u_n) = 0$ for all n.

(II) The other important tool for this paper is the ∇-Theorem 2.7, by which we resume one of the ∇-theorems introduced in [12], giving a version which is suitable for the problem of the asymptotically critical points we have just recalled.

In the classical version, the basic idea of this ∇-theorems is to use some properties of the gradient of a functional h and some properties of its sublevels, in order to reduce the study of critical points of h to that of a functional G which is topologically richer than h. G is obtained, roughly speaking, by introducing a constraint for h which enriches the topological properties of the sublevels. The properties of the gradient of h let the critical points of G give rise to critical points of h.

The technique used here to introduce this sort of constraint for h ("blow up") without making too restrictive hypotheses, is quickly recalled in Section 5.

2. "Asymptotically critical" points and multiplicity

This Section has two aims. First of all we need to recall the theory of "asymptotically critical" points exposed in [10], adapting it to the requirements of this work. The second aim is to give an "asymptotic" version of one of the ∇-Theorems of [12] that we will use in the following.

We also need to recall the notion of subdifferential and superdifferential. For the following definition see, for example, [3], [5], [6], [10].

Definition 2.1. Let H be a Hilbert space and E a subset of H. Let us consider a function $h: E \to \mathbb{R} \cup \{-\infty\}$ (resp. $h: E \to \mathbb{R} \cup \{+\infty\}$). If $u \in E$, $h(u) \in \mathbb{R}$ and $\eta \in H$, we say that η is a superdifferential (resp. a subdifferential) of h in u if

$$\limsup_{\substack{v \to u \\ v \in E}} \frac{h(v) - h(u) - \langle \eta, v-u \rangle}{\|v-u\|} \leq 0$$

$$\left(\text{resp. } \liminf_{\substack{v \to u \\ v \in E}} \frac{h(v) - h(u) - \langle \eta, v-u \rangle}{\|v-u\|} \geq 0\right).$$

We will also say that $\eta \in \partial^+ h(u)$ (resp. $\eta \in \partial^- h(u)$).

We will say that u is an upper critical point (resp. lower critical point) for h if $0 \in \partial^+ h(u)$ (resp. $0 \in \partial^- h(u)$), that is

$$\limsup_{\substack{v \to u \\ v \in E}} \frac{h(v) - h(u)}{\|v-u\|} \leq 0 \quad \left(\text{resp. } \liminf_{\substack{v \to u \\ v \in E}} \frac{h(v) - h(u)}{\|v-u\|} \geq 0\right).$$

Now let M be the closure of an open regular subset of H and let $g: M \to \mathbb{R}$ be a functional of class C^1. If $u \in \partial M$ we will denote by $\nu(u)$ the outward unit normal to M in u.

Definition 2.2. We set

$$\operatorname{grad}_M g(u) = \begin{cases} \operatorname{grad} g(u) & \text{if } u \in \overset{\circ}{M}, \\ \operatorname{grad} g(u) + \langle \operatorname{grad} g(u), \nu(u) \rangle^- \nu(u) & \text{if } u \in \partial M. \end{cases}$$

In other words, if $u \in \partial M$, $\operatorname{grad}_M g(u)$ is the part of $\operatorname{grad} g(u)$ which "points out of M".

Let us now consider a sequence of functionals $(h_n)_n$ on M, for example of class C^1, and let $h: M \to \mathbb{R} \cup \{-\infty\}$ be a given functional.

Definition 2.3. If $u \in M$ and $h(u) \in \mathbb{R}$, we say that u is an asymptotically critical point for the couple $((h_n)_n, h)$ if there exist a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and $(u_k)_k$ in M such that

$$\operatorname{grad}_M h_{n_k}(u_k) \to 0, \quad u_k \to u \quad \text{and} \quad h_{n_k}(u_k) \to h(u).$$

We will also say that $h(u)$ is an asymptotically critical value for $((h_n)_n, h)$.

We explicitly remark that here we do not need to require that u is a "critical" point for h, that is $0 \in \partial^+ h(u)$, although this fact is verified in the case we are interested in (see Proposition 4.1).

The following property is very important: it expresses a sort of Palais–Smale condition for the couple $((h_n)_n, h)$ and, at the same time, it relates the functionals h_n to the functional h (in a very weak sense).

Definition 2.4 (∇-compactness). Let $\mathfrak{c}$ be a real number. We say that the couple $((h_n)_n, h)$ is ∇-compact at level $\mathfrak{c}$, or that the condition $\nabla(h_n, h; \mathfrak{c})$ holds, if for every strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and for every $(u_k)_k$ in M such that

$$\mathrm{grad}_M h_{n_k}(u_k) \to 0 \quad \text{and} \quad h_{n_k}(u_k) \to \mathfrak{c},$$

there exists a strictly increasing sequence $(k_j)_j$ in $\mathbb{N}$ and there exists u in M such that $u_{k_j} \to u$ and $h(u) = \mathfrak{c}$.

If a and b are real numbers with $a \leq b$ and $\nabla(h_n, h; \mathfrak{c})$ holds for all $\mathfrak{c}$ in $[a, b]$, we say that $\nabla(h_n, h; a, b)$ holds.

In this paper we will use the following version of Theorem 1.3 of [10], which can be proved in an analogous way, since the problem

$$\begin{cases} \mathcal{U}: [0, \varepsilon] \to M, \\ \mathcal{U}' = -\mathrm{grad}_M g(\mathcal{U}), \\ \mathcal{U}(0) = \mathcal{U}_0 \in M, \end{cases}$$

has a unique solution for a suitable $\varepsilon > 0$ and the solution depends continuously from $\mathcal{U}_0$ in the usual sense (see, for example [6]).

Theorem 2.5. *Let a and b be real numbers with $a \leq b$ and let $\nabla(h_n, h; a, b)$ holds. Then the number of asymptotically critical points for $((h_n)_n, h)$ with asymptotically critical value in $[a, b]$ is greater than or equal to*

$$\limsup_{n\to\infty} \mathrm{cat}_M(h_n^b, h_n^a).$$

Here $\mathrm{cat}_M(h_n^b, h_n^a)$ denotes the relative category in M of the set $h_n^b = \{u \in M \mid h_n(u) \leq b\}$ with respect to the set $h_n^a = \{u \in M \mid h_n(u) \leq a\}$. For the notion of relative category, for the properties and remarks related to Theorem 2.5 see [10]. A more general version of this Theorem is given in Appendix A.

We will use Theorem 2.5 in order to prove the following Theorem 2.7, which is fundamental for the results of Section 4. In the proof we will use some technical Lemmas which we postpone in Section 5.

We premise the following definition.

Definition 2.6. Let X be a closed subspace of H and $\mathfrak{c} \in \mathbb{R}$.

(a) We will say that $\nabla(h_n, h; X; \mathfrak{c})$ holds if for every strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and for every $(u_k)_k$ in M such that

$$d(u_k, X) \to 0, \quad h_{n_k}(u_k) \to \mathfrak{c}, \quad P_{X+u_k}\mathrm{grad}_M h_{n_k}(u_k) \to 0,$$

there exist a subsequence $(u_{k_j})_j$ and u in X such that $u_{k_j} \to u$ and $h(u) = \mathfrak{c}$ (here P_{X+z} denotes the orthogonal projection on $X+\mathrm{Span}\,(z)$).

(b) We will say that $\nabla_0(h_n, h; X; \mathfrak{c})$ holds if in (a) we can state that $0 \in \partial^+ h_{|X}(u)$.

Let us now assume that there exist three closed subspaces of H, X_1, X_2 and X_3 such that $H = X_1 \oplus X_2 \oplus X_3$.

Theorem 2.7. *Suppose that*

(a) *there exist* a, b, ϱ *and* R *in* $\mathbb{R}$ *such that* $0 < \varrho < R$,

$$\begin{aligned} &\sup h_n(T) < a < \inf h_n(S) && \text{for all } n \in \mathbb{N}, \\ &\sup h_n(\Delta) \le b && \text{for all } n \in \mathbb{N}, \end{aligned}$$

where

$$\begin{aligned} T &= \{u \in X_1 \oplus X_2 \mid \|u\| = R\} \cup \{u \in X_1 \mid \|u\| \le R\}, \\ S &= \{u \in X_2 \oplus X_3 \mid \|u\| = \varrho\}, \\ \Delta &= \{u \in X_1 \oplus X_2 \mid \|u\| \le R\}, \end{aligned}$$

(b) $\nabla(h_n, h; \mathfrak{c})$ *and* $\nabla_0(h_n, h; X_1 \oplus X_3; \mathfrak{c})$ *hold for all* $\mathfrak{c}$ *in* $[a, b]$,

(c) $h_{|X_1 \oplus X_3}$ *hasn't upper critical points with value in* $[a, b]$,

(d) $\dim(X_1 \oplus X_2) < \infty$, $\dim X_2 \ge 1$.

Then $((h_n)_n, h)$ *has at least* 2 *asymptotically critical points with asymptotically critical value in* $[a, b]$.

Proof. *Step 1.* Let $\Phi\colon H \setminus (X_1 \oplus X_3) \to H$ be defined as follows:

$$\Phi(z) = z + \frac{P(z) - z}{\|P(z) - z\|},$$

where P is the orthogonal projection on $X_1 \oplus X_3$ and set $\mathcal{C} = \{z \in H \mid \|P(z) - z\| \ge 1\}$. We also set $G_n = h_n \circ \Phi_{|\mathcal{C}}$ and $G = h \circ \Phi_{|\mathcal{C}}$, defined on the manifold with boundary $\mathcal{C}$.

We will first prove the theorem for $((G_n)_n, G)$ and then we will deduce it for $((h_n)_n, h)$. We will use the fact that the sublevels of G_n are topologically richer than the ones of h_n.

Step 2. Set $\widetilde{T} = \{z \in \mathcal{C} \mid \Phi(z) \in T\}$, $\widetilde{S} = \{z \in \mathcal{C} \mid \Phi(z) \in S\}$ and $\widetilde{\Delta} = \{z \in \mathcal{C} \mid \Phi(z) \in \Delta\}$. It is clear that

$$\begin{aligned} &\sup G_n(\widetilde{T}) < a < \inf G_n(\widetilde{S}) && \text{for all } n \in \mathbb{N}, \\ &\sup G_n(\widetilde{\Delta}) \le b && \text{for all } n \in \mathbb{N}. \end{aligned}$$

Then, by Theorem 2.2 of [13], $\operatorname{cat}_{\mathcal{C}}(G_n^b, G_n^a) \ge 2$ ($\widetilde{T}$ is a strong deformation retract of G_n^a in $\mathcal{C}$ and then $\operatorname{cat}_{\mathcal{C}}(G_n^b, \widetilde{T}) \le \operatorname{cat}_{\mathcal{C}}(G_n^b, G_n^a)$).

Step 3. By (b) and by Theorem 5.2, $\nabla(G_n, G; a, b)$ holds. Then, by Theorem 2.5, $((G_n)_n, G)$ has at least 2 distinct asymptotically critical points z_1 and z_2 such that $G(z_i) \in [a, b]$, $i = 1, 2$.

We note that $z_i \in \mathring{\mathcal{C}}$. Otherwise, by Proposition 5.3, $\Phi(z_i)$ would be an upper critical point for $h_{|X_1 \oplus X_3}$ with $h(z_i)$ in $[a, b]$, but these fact contradicts (c). Then it is easy to see that $\Phi(z_i)$ are asymptotically critical points for $((h_n)_n, h)$. $\square$

3. The approximating functionals

Let Ω be an open, bounded, connected and smooth subset of $\mathbb{R}^N$, $N \ge 1$, and let $\phi\colon \Omega \to \mathbb{R}$ be a measurable function with $\phi \le 0$.

Set $H = H_0^1(\Omega) \cap H^2(\Omega)$ and consider $f_\omega\colon \Omega \to \mathbb{R}$ defined as

$$f_\omega(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \frac{c}{2} \int_\Omega |Du|^2 \, dx - \frac{\alpha}{2} \int_\Omega u^2 \, dx - \frac{\omega}{k} \int_\Omega ((u - \phi)^-)^k \, dx,$$

where ω, c, α and k are real numbers such that $\omega \ge \omega_0 > 0$, $k > 2$ and, if $N \ge 5$, $k < 2N/(N-4)$.

We recall that the critical points of f_ω solve (P_ω) (see Section 1).

In this section we will show some lemmas which describe the topological properties of the family of functionals f_ω. By these lemmas in Section 4 we will prove the main results of this paper using the technical results of Section 5 and those of [10].

Notations 3.1. We introduce some notations which will be used throughout this paper.

(a) If $u \in H$ we set

$$Q(u) = \frac{1}{2} \int_\Omega |\Delta u|^2 \, dx - \frac{c}{2} \int_\Omega |Du|^2 \, dx - \frac{\alpha}{2} \int_\Omega u^2 \, dx.$$

(b) We respectively denote by $(\Lambda_k)_{k \in \mathbb{N}^*}$ ($\Lambda_1 \le \Lambda_2 \le \ldots$) and by $(E_k)_{k \in \mathbb{N}^*}$ the eigenvalues and the corresponding eigenfunctions of the problem

$$\begin{cases} \Delta^2 u + c\Delta u = \Lambda u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

and with $(\lambda_n)_{n\in\mathbb{N}^*}$ ($\lambda_1 < \lambda_2 \leq \dots$) the eigenvalues of the problem

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

The eigenfunction e_1 corresponding to λ_1 can be chosen strictly positive in Ω.

(c) We denote by H_l the subspace spanned by the eigenfunctions corresponding to the eigenvalues $\Lambda_1, \dots, \Lambda_l$ and by $H_l^\perp$ its orthogonal space in H.

(d) If $e \in H_s^\perp$ and $s \in \mathbb{N}^*$, we set

$$\begin{aligned} \Sigma_R(H_s, e) &= \{v \in H_s \mid \|v\| \leq R\} \\ &\quad \cup \{v - \sigma e \mid v \in H_s, \sigma \geq 0,\ \|v - \sigma e\| = R\}, \\ \Delta_R(H_s, e) &= \{v - \sigma e \mid v \in H_s,\ \sigma \geq 0, \|z\| \leq R\}, \\ T_{l,s}(R) &= \{u \in H_l \mid \|u\| \leq R\} \\ &\quad \cup \{v + z \mid v \in H_l,\ z \in \operatorname{Span}(E_l, \dots, E_s), \|v + z\| = R\}, \\ \Delta_{l,s}(R) &= \{v + z \mid v \in H_l,\ z \in \operatorname{Span}(E_l, \dots, E_s), \|v + z\| \leq R\}, \\ S_s^+(\varrho) &= \{v \in H_s^\perp \mid \|v\| = \varrho\}. \end{aligned}$$

REMARK 3.2. (i) It is clear that $\{\Lambda_k \mid k \in \mathbb{N}^*\} = \{\lambda_n^2 - c\lambda_n \mid n \in \mathbb{N}^*\}$.

(ii) The eigenfunction corresponding to $\lambda_1^2 - c\lambda_1$ is e_1, the first eigenfunction of problem (3.2).

Now we recall some inequalities and some properties of f_ω which can be found in [15].

REMARK 3.3. $\int_\Omega ((u - \phi)^-)^k\, dx = O(\|u\|^k)$, as it is clear.

In the following two lemmas we will assume that there exists l in $\mathbb{N}^*$ such that $\Lambda_l \leq \alpha < \Lambda_{l+1} < \lambda_1^2 - c\lambda_1$.

LEMMMA 3.5. *Suppose that for given l and s in $\mathbb{N}^*$ $\Lambda_l < \Lambda_{l+1} \leq \dots \leq \Lambda_s < \Lambda_{s+1} \leq \lambda_1^2 - c\lambda_1$ and $\Lambda_l \leq \alpha < \Lambda_{l+1}$. Then*

(a) *there exist ϱ' and R' with $0 < \varrho' < R'$ and such that*

$$\sup f_\omega(T_{l,s}(R')) < \inf f_\omega(S_l^+(\varrho')). \tag{3.3}$$

R' doesn't depend on ω and can be taken as big as desired. Moreover,

$$\sup_\omega \sup f_\omega(\Delta_{l,s}(R)) < \infty.$$

(b) *If, moreover, $\sup\phi < 0$ and $N \leq 3$, then ϱ' and $\inf f_\omega(S_l^+(\varrho'))$ do not depend on ω.*

In the proof of this lemma, Remark 3.3 and the following facts play a role:

$$\lim_{\substack{u \in H_s \\ \|u\| \to \infty}} f_\omega(u) = -\infty, \tag{3.4}$$

since, if $u \in H_s$, then $u^- \neq 0$.

(3.5) Under the assumptions of case (b), there exists $\varrho > 0$

such that, if $\|u\| \leq \varrho$, then $f_\omega(u) = Q(u)$.

Lemma 3.5. *Suppose that for a given s in $\mathbb{N}^*$ $\Lambda_s < \Lambda_{s+1} \leq \lambda_1^2 - c\lambda_1$. Let e be in $H_s^\perp$ be such that, set $X = \{v + te \mid v \in H_s, t \geq 0\}$, one has*

$$u \in X,\ u \neq 0 \Rightarrow u^- \neq 0$$

(for example $e = -e_1$). Then

(a) *there exists $\tau > 0$ such that, if $\Lambda_s - \tau < \alpha < \Lambda_{s+1}$, then there exist ϱ'' and R'' with $0 < \varrho'' < R''$ such that*

$$\sup f_\omega(\Sigma_{R''}(H_s, e)) < \inf f_\omega(S_s^+(\varrho'')). \tag{3.6}$$

R'' doesn't depend on ω and can be taken as big as desired.

(b) *If, moreover, $\sup \phi < 0$ and $N \leq 3$, then τ, ϱ'' and $\inf f_\omega(S_s^+(\varrho''))$ do not depend on ω.*

In the proof of this lemma a fundamental role is played by the following fact:

$$\lim_{\alpha \to \Lambda_s} \sup f_\omega(H_s) = 0 \qquad \text{uniformly w.r.t. } \omega. \tag{3.7}$$

In the following two lemmas we will assume that there exists l in $\mathbb{N}^*$ such that $\lambda_1^2 - c\lambda_1 \leq \Lambda_l \leq \alpha < \Lambda_{l+1}$ and $\alpha \neq \lambda_1^2 - c\lambda_1$.

We need the following definition.

Definition 3.6. Suppose that $\lambda_1^2 - c\lambda_1 \leq \Lambda_s$ for a given s in $\mathbb{N}^*$. We set

$$\Lambda_s^* = \inf\{\alpha \in \mathbb{R} \mid Q(u) \leq 0 \text{ for all } u \text{ in } H_s \text{ such that } u \geq 0\},$$

that is

$$\Lambda_s^* = \sup\left\{ \int_\Omega |\Delta u|^2\,dx - c\int_\Omega |Du|^2\,dx \,\middle|\, u \in H_s,\ u \geq 0,\ \int_\Omega u^2\,dx = 1 \right\}.$$

It is clear that $\Lambda_s^* \leq \Lambda_s$ and $\Lambda_s^* < \Lambda_s$ if $\lambda_1^2 - c\lambda_1 < \Lambda_s$.

Lemma 3.7. *Suppose l and s in $\mathbb{N}^*$ are such that $\lambda_1^2 - c\lambda_1 \leq \Lambda_l < \Lambda_{l+1} \leq \ldots \leq \Lambda_s < \Lambda_{s+1}$ and $\Lambda_s^* < \alpha < \Lambda_{l+1}$. Then the thesis of Lemma 3.4 holds.*

Lemma 3.8. *Let s in $\mathbb{N}^*$ be such that $\lambda_1^2 - c\lambda_1 < \Lambda_s < \Lambda_{s+1}$. Let e in $H_s^\perp$ be such that, set $X = \{v + te \mid v \in H_s,\ t \geq 0\}$, if $u \in X \setminus H_s$, then $u^- \neq 0$. Then the thesis of Lemma* 3.5 *holds.*

We note that if $N \geq 4$, it will be enough to take e in H such that $\inf_\Omega e = -\infty$, since functions in H_s are bounded. On the other hand, if $N = 2$ or $N = 3$, one can take, for example, a function e in H such that $\sup_{\partial\Omega} \partial e/\partial\nu = +\infty$, where ν is the unit normal outward to $\partial\Omega$.

4. Multiplicity of solutions

In this section we want to exhibit and prove the main results we got concerning the number of solutions of problem (P) introduced in Section 1. We will use the notations of Section 3.

Let us consider the functional $f\colon H \to \mathbb{R} \cup \{-\infty\}$ defined as

$$f(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\Delta u|^2\,dx - \dfrac{c}{2}\displaystyle\int_\Omega |Du|^2\,dx - \dfrac{\alpha}{2}\displaystyle\int_\Omega u^2\,dx & \text{if } u \geq \phi, \\ -\infty & \text{otherwise,} \end{cases}$$

and set $K_\phi = \{u \in H \mid u \geq \phi\}$.

It is clear that, if $u \in K_\phi$, $0 \in \partial^+ f(u)$ (see Definition 2.1) if and only if u solves problem (P).

The following proposition states that the asymptotically critical points for (f_ω, f) (see Section 2), are solutions of (P). Nevertheless we don't know if the *vice versa* is true. Indeed, in general, we would need the following property, at least in a neighbourhood U of the point u such that $0 \in \partial^+ f(u)$: if $u_\omega \in U$ and $\inf f_\omega(u_\omega) > -\infty$, then (u_ω) admits a converging subsequence (see [5]).

In this section we will study the multiplicity of the asymptotically critical points for (f_ω, f).

Proposition 4.1. *Let $(\omega_n)_n$ be a strictly increasing sequence in $\mathbb{R}$ such that $\omega_n \to \infty$ and let u be in K_ϕ. If u is an asymptotically critical point for $((f_{\omega_n})_n, f)$, then $0 \in \partial^+ f(u)$.*

Proof. Let $(n_k)_k$ in $\mathbb{N}$ be a strictly increasing sequence and let $(u_k)_k$ in H be such that $u_k \to u$, $\nabla f_{\omega_{n_k}}(u_k) \to 0$. Then, for all v in H,

$$\int_\Omega \Delta u_k \Delta(v - u_k)\,dx - c\int_\Omega Du_k \cdot D(v - u_k)\,dx - \alpha\int_\Omega u_k(v - u_k)\,dx \\ + \omega_{n_k}\int_\Omega ((u_k - \phi)^-)^{k-1}(v - u_k)\,dx \to 0.$$

If $v \in K_\phi$, then

$$\int_\Omega ((u_k - \phi)^-)^{k-1}(v - u_k)\,dx = \int_\Omega ((u_k - \phi)^-)^{k-1}(v - \phi)\,dx + \int_\Omega ((u_k - \phi)^-)^{k}\,dx,$$

so that

$$\int_\Omega \Delta u \Delta(v-u)\,dx - c\int_\Omega Du\cdot D(v-u)\,dx - \alpha\int_\Omega u(v-u)\,dx \le 0$$

for every v in K_ϕ. □

For the results of this section we need some other properties concerning f_ω and f.

Proposition 4.2. *Suppose $N\le 3$, $\sup\phi<0$, $\alpha\ne\lambda_1^2-c\lambda_1$. If $(\omega_n)_n$ is a strictly increasing sequence in $\mathbb{R}$ which diverges to ∞, then for all $\mathfrak{c}$ in $\mathbb{R}$ and for all $(u_n)_n$ in H such that $f_{\omega_n}(u_n)\to\mathfrak{c}$, $\nabla f_{\omega_n}(u_n)\to 0$, there exist a subsequence $(u_{n_k})_k$ and u in H such that u_{n_k} converges to u, $u\ge\phi$, $f(u)=\mathfrak{c}$ and $0\in\partial^+ f(u)$. In particular, $\nabla(f_{\omega_n},f;\mathfrak{c})$ holds for every $\mathfrak{c}$ in $\mathbb{R}$ (see Definition 2.4).*

Proof. *Step 1.* Let us show that $(u_n)_n$ is bounded. Suppose by contradiction that, up to a subsequence, $\|u_n\|\to\infty$ as $n\to\infty$. Up to a subsequence, we can suppose that there exists u in H such that $u_n/\|u_n\|\rightharpoonup u$ weakly in H. Since

$$\begin{aligned}\frac{f'_{\omega_n}(u_n)(u_n)}{\|u_n\|} &= \frac{2f_{\omega_n}(u_n)}{\|u_n\|}\\ &\quad+\left(\frac{2}{k}-1\right)\frac{\omega_n\displaystyle\int_\Omega((u_n-\phi)^-)^k\,dx}{\|u_n\|}+\frac{\omega_n\displaystyle\int_\Omega((u_n-\phi)^-)^{k-1}\phi\,dx}{\|u_n\|},\end{aligned}$$

and since $f'_{\omega_n}(u_n)(u_n)/\|u_n\|\to 0$, we get

$$\frac{\omega_n\displaystyle\int_\Omega((u_n-\phi)^-)^k\,dx}{\|u_n\|}\to 0\quad\text{and}\quad\frac{\omega_n\displaystyle\int_\Omega((u_n-\phi)^-)^{k-1}\phi\,dx}{\|u_n\|}\to 0,$$

for $k>2$ and $\phi\le 0$. Therefore

$$\frac{\omega_n\displaystyle\int_\Omega((u_n-\phi)^-)^{k-1}\,dx}{\|u_n\|}\to 0, \tag{4.1}$$

since $\sup\phi<0$. Moreover,

$$\begin{aligned}\frac{f'_{\omega_n}(u_n)(u_n)}{\|u_n\|^2} &= 1-\frac{c\displaystyle\int_\Omega|Du_n|^2\,dx}{\|u_n\|^2}-\frac{\alpha\displaystyle\int_\Omega u_n^2\,dx}{\|u_n\|^2}\\ &\quad+\frac{\omega_n\displaystyle\int_\Omega((u_n-\phi)^-)^{k-1}u_n\,dx}{\|u_n\|^2},\end{aligned}\tag{$*$}$$

$$(**)\qquad \frac{f_{\omega_n}(u_n)}{\omega_n\|u_n\|^k} = \frac{Q(u_n)}{\omega_n\|u_n\|^k} - \frac{\displaystyle\int_\Omega ((u_n-\phi)^-)^k\,dx}{\|u_n\|^k}$$

$$(***)\qquad \frac{f'_{\omega_n}(u_n)(e_1)}{\|u_n\|} = (\lambda_1^2 - c\lambda_1 - \alpha)\frac{\displaystyle\int_\Omega u_n e_1\,dx}{\|u_n\|} + \frac{\omega_n\displaystyle\int_\Omega ((u_n-\phi)^-)^{k-1} e_1\,dx}{\|u_n\|}$$

tend to 0. By $(*)$ we get

$$1 = c\int_\Omega |Du|^2\,dx + \alpha\int_\Omega u^2\,dx,$$

since

$$\int_\Omega ((u_n-\phi)^-)^{k-1} u_n\,dx = -\int_\Omega ((u_n-\phi)^-)^k\,dx + \int_\Omega ((u_n-\phi)^-)^{k-1}\phi\,dx.$$

By $(**)$ we get $\int_\Omega (u^-)^k\,dx = 0$, that is $u \geq 0$. Therefore $u \geq 0$ and $u \not\equiv 0$. But by $(***)$ and (4.1) we get

$$(\lambda_1^2 - c\lambda_1 - \alpha)\int_\Omega ue_1\,dx = 0,$$

which is impossible, since $\alpha \neq \lambda_1^2 - c\lambda_1$.

Step 2. Now we can suppose that u_n converges weakly and pointwise to a function u of H. Then $u \geq \phi$, since $\omega_n \displaystyle\int_\Omega ((u_n-\phi)^-)^k\,dx$ is bounded. Moreover,

$$(4.2)\qquad \omega_n\int_\Omega ((u_n-\phi)^-)^{k-1}\,dx \quad \text{is bounded.}$$

In fact

$$f'_{\omega_n}(u_n)(u_n) = 2f_{\omega_n}(u_n) + \left(\frac{2}{k}-1\right)\omega_n\int_\Omega ((u_n-\phi)^-)^k\,dx + \omega_n\int_\Omega ((u_n-\phi)^-)^{k-1}\phi\,dx \to 0,$$

which implies that $\omega_n\int_\Omega ((u_n-\phi)^-)^{k-1}\phi\,dx$ is bounded; the thesis follows by the fact that $\sup\phi < 0$.

Step 3. If $v \in H$, $f'_{\omega_n}(u_n)(v-u_n) \to 0$, that is

$$(4.3)\qquad \int_\Omega \Delta u_n \Delta(v-u_n)\,dx - c\int_\Omega Du_n\cdot D(v-u_n)\,dx - \alpha\int_\Omega u_n(v-u_n)\,dx + \omega_n\int_\Omega ((u_n-\phi)^-)^{k-1}(v-u_n)\,dx \to 0.$$

But $N \leq 3$, so $u_n \to u$ uniformly and then

$$\omega_n \int_\Omega ((u_\omega - \phi)^-)^{k-1}(u - u_n)\, dx \to 0.$$

Putting $v = u$ in (4.3), we get $\int_\Omega \Delta u_n \Delta(u - u_n)\, dx \to 0$, and then $u_n \to u$ strongly in H.

Step 4. By (4.2) and by the fact that $(u_n - \phi)^- \to 0$ in $L^\infty(\Omega)$, we get

$$\omega_n \int_\Omega ((u_n - \phi)^-)^k\, dx \to 0.$$

Then $f_{\omega_n}(u_n) \to Q(u) = f(u)$. By Proposition 4.1 we finally get that $0 \in \partial^+ f(u)$. □

The following proposition expresses the second property we need to prove the theorems of multiplicity. If X is a closed subspace of H and $u \in H$, we denote by $P_{X+(u)}$ the orthogonal projection on the space $X + \{tu \mid t \in \mathbb{R}\}$.

Proposition 4.3. *Suppose $N \leq 3$, $\alpha \neq \lambda_1^2 - c\lambda_1$, $\sup \phi < 0$. Let X be a closed subspace of H with finite codimension such that $e_1 \in X$ and let $(\omega_n)_n$ be a strictly increasing sequence in $\mathbb{R}$ with $\omega_n \to \infty$. Then for all $(u_n)_n$ in H and for all $\mathfrak{c}$ in $\mathbb{R}$ such that $f_{\omega_n}(u_n) \to \mathfrak{c}$, $d(u_n, X) \to 0$, $P_{X+(u_n)}(\nabla f_{\omega_n}(u_n)) \to 0$, there exists a subsequence $(u_{n_k})_k$ which converges to a point u of X such that $u \geq \phi$, $f(u) = \mathfrak{c}$ and $0 \in \partial^+ f(u)$. In other words, $\nabla_0(f_\omega, f; X; \mathfrak{c})$ holds for every $\mathfrak{c}$ in $\mathbb{R}$ (see Definition 2.6).*

The proof is equal to the proof of the previous proposition.

Finally we need the following result. We write f_α in place of f to emphasize the dependence of f on α.

Proposition 4.4. *Suppose $N \leq 3$, $\sup \phi < 0$ and for some l and s in $\mathbb{N}^*$ $\Lambda_l < \Lambda_{s+1}$ and set $X_l^s = H_l \oplus H_s^\perp$. Then, for every $\delta > 0$,*

$$\inf\{f_\alpha(u) \mid u \in K_\phi \cap X_l^s,\ f_\alpha(u) > 0,\\ 0 \in \partial^+ f_{\alpha|X_l^s}(u),\ \Lambda_l + \delta \leq \alpha \leq \Lambda_{s+1} - \delta\} > 0.$$

Proof. *Step 1.* There exists $\varrho > 0$ such that, if $u \in K_\phi \cap X_l^s$, $\|u\| < \varrho$, $u \neq 0$ and $\Lambda_l < \alpha < \Lambda_{s+1}$, then u is not an upper critical point for f_α on X_l^s.

In fact if ϱ is small enough, then $B(0, \varrho) \subset K_\phi$ (since $H \hookrightarrow C_0^0(\Omega)$ and $\sup \phi < 0$). On the other hand the unique upper critical point for Q on X_l^s is 0, since $\Lambda_l < \alpha < \Lambda_{s+1}$.

Step 2. Fix $\delta > 0$. Suppose by contradiction that there exist α_n in $[\Lambda_l + \delta, \Lambda_{s+1} - \delta]$ and $(u_n)_n$ in $K_\phi \cap X_l^s$ is such that $f_{\alpha_n}(u_n) > 0$, $f_{\alpha_n}(u_n) \to 0$ and $0 \in \partial^+ (f_{\alpha_n})_{|X_l^s}$.

We can suppose that $\alpha_n \to \alpha$ in $[\Lambda_l + \delta, \Lambda_s - \delta]$. First of all we prove that $(u_n)_n$ is bounded. Suppose by contradiction that $\|u_n\| \to \infty$ and $u_n/\|u_n\| \rightharpoonup u$ in $K_\phi \cap X_l^s$. By the following inequality

$$
\begin{aligned}
(4.4)\qquad f'_{\alpha_n}(u_n)(v) - 2f_{\alpha_n}(u_n) = &\int_\Omega \Delta u_n \Delta(v-u_n)\,dx \\
&- c\int_\Omega Du_n \cdot D(v-u_n)\,dx - \alpha_n \int_\Omega u_n(v-u_n)\,dx \le 0,
\end{aligned}
$$

for all v in $K_\phi \cap X_l^s$, we get that $f'_\alpha(u)(v) \le 0$ for all v in $K_\phi \cap X_l^s$. But $B(0,\varrho) \subset K_\phi$, so $f'_\alpha(u)(v) = 0$ for all v in X_l^s. Then $u \equiv 0$, since $u \in X_l^s$ and $\Lambda_l + \delta \le \alpha \le \Lambda_{s+1} - \delta$.

On the other hand $f_{\alpha_n}(u_n)/\|u_n\|^2$ tends to 0, but also to

$$\frac{1}{2} - \frac{c}{2}\int_\Omega |Du|^2\,dx - \frac{\alpha}{2}\int_\Omega u^2\,dx.$$

In this way $u \not\equiv 0$ and a contradiction arises.

Step 3. We can now suppose that $u_n \rightharpoonup u$ in H and $u \in K_\phi \cap X_l^s$. By (4.4) we get again $f'_\alpha(u)(v) \le 0$ for all v in $K_\phi \cap X_l^s$, since $f_{\alpha_n}(u_n) \to 0$. Then $u \equiv 0$.

Since $f_{\alpha_n}(u_n) \to 0$, that is

$$\frac{1}{2}\|u_n\|^2 - \frac{c}{2}\int_\Omega |Du_n|^2\,dx - \frac{\alpha}{2}\int_\Omega u_n^2\,dx \to 0,$$

we get that $\|u_n\| \to 0$. By Step 1 we get $u_n = 0$, which contradicts the hypothesis $f_{\alpha_n}(u_n) > 0$. □

We can now state and prove the main theorems of this paper.

Theorem 4.5. *Suppose $N \le 3$ and $\sup\phi < 0$. Suppose that for a certain s in $\mathbb{N}^*$ one has $\Lambda_s < \Lambda_{s+1} \le \lambda_1^2 - c\lambda_1$. Then there exists $\tau > 0$ such that, if $\Lambda_s - \tau < \alpha < \Lambda_s$, problem* (P) *has at least* 3 *nontrivial solutions, since the couple $((f_\omega)_\omega, f)$ has at least* 3 *nontrivial asymptotically critical points* (*see* (4.1)) *at levels different from* 0.

Proof. *Step 1.* Let Λ_l be the eigenvalue of (3.1) such that $\Lambda_l < \Lambda_{l+1} = \ldots = \Lambda_s$ and set $X_1 = H_l$, $X_2 = H_l^\perp \cap H_s$ and $X_3 = H_s^\perp$.

If $\Lambda_l < \alpha < \Lambda_s$, by Lemma 3.4, by Propositions 4.2 and 4.3, then (a) and (b) of Theorem 2.7 hold, for example with $b = \sup f_{\omega_0}(H_s)$ and $a = \inf Q(S_l^+(\varrho'))$. By (3.7) and by Proposition 4.4 also (c) of Theorem 2.7 holds.

Then there exist 2 asymptotically critical points u_1 and u_2 for the couple $((f_\omega)_\omega, f)$ such that $0 < a \le f(u_i) \le b$.

Step 2. By Lemma 3.5 and Theorem 2.5 it easily follows that there exists an upper critical point u_3 for f such that $f(u_3) > b$, provided τ is small enough. □

With an analogous proof, by Lemmas 3.7 and 3.8, one gets the following theorem.

Theorem 4.6. *Suppose $N = 2$ or $N = 3$ and $\sup \phi < 0$. Suppose that for a certain s in $\mathbb{N}^*$ one has $\lambda_1^2 - c\lambda_1 < \Lambda_s < \Lambda_{s+1}$. Then the thesis of Theorem 4.5 holds.*

We note that the existence of the point u_3 was already proved in [15], showing that, by Lemma 3.5, every f_ω has a critical point v_ω such that $\inf_{\omega \geq \omega_0} f_\omega(v_\omega) > b$, ((PS) holds for every f_ω) and thus a subsequence of (v_ω) converges to u_3. Moreover, in [15] it was proved, in an analogous way, that there exists a solution u_1 which is limit of solutions of problems (P_ω) with $f(u_1) \leq b$.

We can therefore conclude with the following remark.

Remark 4.7. In the previous theorems we can state that there exist 3 non-trivial solutions u_1, u_2, u_3 of (P) such that u_1 and u_3 are limits of solutions of problems (P_ω), while u_2 is, to our knowledge, only an asymptotically critical point for $((f_\omega)_\omega, f)$.

Moreover, the functions u_i satisfy only the inequality (P) and not the corresponding equation, due to the fact that α is not an eigenvalue of the quadratic form

$$\frac{1}{2}\int_\Omega |\Delta u|^2\,dx - \frac{c}{2}\int_\Omega |Du|^2\,dx.$$

5. Blow up

Here we want to examine and prove some technical results which we used in Section 2.

Let H be a Hilbert space. Let us consider a functional $F: H \to \mathbb{R} \cup \{-\infty\}$ and a closed subspace X of H. We will denote by $\mathcal{D}(F)$ the set $\{u \in H \mid F(u) \in \mathbb{R}\}$.

Following the idea of [13], we now want X to be a "barrier" for F. For this purpose we introduce a functional G in the following way.

If $P: H \to X$ is the orthogonal projection of H onto X and $Q = I - P$, we set

$$\mathcal{C} = \{z \in H \mid \|Q(z)\| \geq 1\}.$$

Let us consider the map $\Phi: H \setminus X \to H$ defined as

$$\Phi(z) = z - \frac{Q(z)}{\|Q(z)\|} \qquad \text{for all } z \in H \setminus X.$$

It is clear that Φ induces a diffeomorphism between $\mathring{\mathcal{C}}$ and $H \setminus X$.

Finally we set $G = F \circ \Phi_{|\mathcal{C}}: \mathcal{C} \to \mathbb{R} \cup \{-\infty\}$.

Remark 5.1.

(a) $$d\Phi(z) = P_{X+z} + \left(1 - \frac{1}{\|Q(z)\|}\right) P_{X_z} \quad \text{for all } z \in H \setminus X,$$

where P_{X+z} is the orthogonal projection on $X + \mathrm{Span}\,(z)$ and $P_{X_z} = I - P_{X+z}$.

(b) If $z \in \mathcal{C}$, $\Phi(z) \in \overset{\circ}{\mathcal{D}}(F)$ and F is differentiable in $\Phi(z)$, then

- if $z \in \overset{\circ}{\mathcal{C}}$, then G is differentiable in z and

$$\operatorname{grad} G(z) = d\Phi(z)(\operatorname{grad} F(\Phi(z))),$$

- if $z \in \partial\mathcal{C}$, then $\partial^+ G(z) = \{d\Phi(z)(\operatorname{grad} F(\Phi(z))) + \lambda\, Q(z) \mid \lambda \geq 0\}$.

(c)
- if $z \in \mathcal{C}$, then $\partial^+ G(z) \supset d\Phi(z)(\partial^+ F(\Phi(z)))$,
- if $z \in \overset{\circ}{\mathcal{C}}$, then $\partial^+ G(z) = d\Phi(z)(\partial^+ F(\Phi(z)))$.

(d) If $z \in \partial\mathcal{C}$, the following implications hold:

$$0 \in \partial^+ G(z) \Rightarrow 0 \in \partial^+ G_{|\partial\mathcal{C}}(z) \Leftrightarrow 0 \in \partial^+ F_{|X}(\Phi(z)).$$

Proof. For (a)–(c) see [13]. Concerning (d), we observe that the equivalence holds since, if $w \in X$, then $z + w \in \partial\mathcal{C}$ and therefore $\Phi(z+w) = \Phi(z) + w$. Thus

$$\frac{F(\Phi(z)+w) - F(\Phi(z))}{\|w\|} = \frac{G(z+w) - G(z)}{\|w\|}.$$

The rest is obvious. □

Now let us consider a sequence of regular functionals $(h_n)_n$ defined on H and a functional $h\colon H \to \mathbb{R} \cup \{-\infty\}$. We want to give a condition on $(h_n)_n$ and h which ensures that, setting $G_n = h_n \circ \Phi_{|\mathcal{C}}$ and $G = h \circ \Phi_{|\mathcal{C}}$, the condition $\nabla(G_n, G; \mathfrak{c})$ holds ($\mathfrak{c}$ in $\mathbb{R}$).

Teorem 5.2. *Let X be a closed subspace of finite codimension and $\mathfrak{c}$ be a real number. If $\nabla(h_n, h; \mathfrak{c})$ and $\nabla(h_n, h; X; \mathfrak{c})$ hold, then $\nabla(G_n, G; \mathfrak{c})$ holds.*

Proof. *Step 1.* Let $(z_n)_n$ be a sequence in $\mathcal{C}$ such that $G_n(z_n) \to \mathfrak{c}$ and $\operatorname{grad}_{\mathcal{C}} G_n(z_n) \to 0$.

Step 2. Suppose that $\inf_n \|Q(z_n)\| > 1$; then $z_n \in \overset{\circ}{\mathcal{C}}$ and by (b) of Remark 5.1, $\operatorname{grad}_{\mathcal{C}} G_n(z_n) = d\Phi(z_n)(\operatorname{grad} h_n(\Phi(z_n)))$. By (a) of Remark 5.1

$$P_{X+z_n}(\operatorname{grad} h_n(\Phi(z_n))) \to 0 \quad \text{and also} \quad P_{X_{z_n}}(\operatorname{grad} h_n(\Phi(z_n))) \to 0.$$

Then $\operatorname{grad} h_n(\Phi(z_n)) \to 0$ and $h_n(\Phi(z_n)) \to \mathfrak{c}$. By $\nabla(h_n, h; \mathfrak{c})$ there exist a strictly increasing sequence $(n_k)_k \in \mathbb{N}$ and $u \in H$ such that $\Phi(z_{n_k}) \to u$ and $h(u) = \mathfrak{c}$. Moreover, $u \notin X$. Then $z_{n_k} \to z = \Phi^{-1}(u)$ in $\overset{\circ}{\mathcal{C}}$ and $h(u) = \mathfrak{c}$.

Step 3. Now let us assume that $\|Q(z_n)\| > 1$ for all n and $\lim_n \|Q(z_n)\| = 1$. Then $z_n \in \overset{\circ}{\mathcal{C}}$ and $P_{X+z_n} \operatorname{grad} h_n(\Phi(z_n)) \to 0$, by (a) and (b) of Remark 5.1. Moreover, $P_{X+z_n} = P_{X+\Phi(z_n)}$, since $z_n \in \overset{\circ}{\mathcal{C}}$, and then $P_{X+\Phi(z_n)} \operatorname{grad} h_n(\Phi(z_n)) \to 0$.

By $\nabla(h_n, h; X; \mathfrak{c})$ there exist a strictly increasing sequence $(n_k)_k \in \mathbb{N}$ and $u \in X$ such that $\Phi(z_{n_k}) \to u$ and $h(u) = \mathfrak{c}$. Since X has finite codimension, we can assume that $z_{n_k} \to z$, $z \in \partial\mathcal{C}$. Then $\Phi(z) = u$ and $G(z) = \mathfrak{c}$.

Step 4. In the case $z_n \in \partial\mathcal{C}$ for all n, in particular, by (a) and (b) of Remark 5.1, we get that $P\operatorname{grad} h_n(\Phi(z_n)) \to 0$ $(PP_{X+z_n} = P)$ and $\Phi(z_n) \in X$.

By $\nabla(h_n, h; X; \mathfrak{c})$ there exist a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and u in X such that $\Phi(z_{n_k}) \to u$ and $h(u) = \mathfrak{c}$. Since X has finite codimension, we can assume that $z_{n_k} \to z$, $z \in \partial\mathcal{C}$ and $\Phi(z) = u$. Then $G(z) = \mathfrak{c}$.

By Steps 1–4 the thesis follows, as it is clear. □

We will also need the following statement.

Proposition 5.3. *Suppose that $((h_n)_n, h)$ satisfies the following condition:*

- *if $(n_k)_k$ is a strictly increasing sequence in $\mathbb{N}$, if $u_{n_k} \to u$, $u \in X$, $P_{X+u_{n_k}}\operatorname{grad} h_{n_k}(u_{n_k}) \to 0$ and $h_{n_k}(u_{n_k}) \to \mathfrak{c}$ in $\mathbb{R}$, then $0 \in \partial^+ h_{|X}(u)$ (for example if $\nabla_0(h_n, h; X; \mathfrak{c})$ holds).*

($P_{X+u_{n_k}}$ is the orthogonal projection on $X+\mathrm{Span}\,(u_{n_k})$.) Then, if a point z on $\partial\mathcal{C}$ is asymptotically critical for $((G_n)_n, G)$, then $\Phi(z) \in X$ and $0 \in \partial^+ h_{|X}(\Phi(z))$.

Proof. The proof can be easily obtained by (a) and (b) of Remark 5.1. □

6. Some open problems

(6.1) Are solutions of (P) regular?

(6.2) What can be said about the set $\{x \in \Omega \mid u(x) = \phi(x)\}$, if u is a solution of (P)?

(6.3) In the hypotheses of Theorems 4.5 and 4.6, are there 3 nontrivial solutions such that anyone is limit of *solutions* of problems (P_ω)? Or do solutions of (P) which are limit of solutions of (P_ω) have particular properties (such as, for example, in the case of problem (1.2))?

(6.4) Is it possible for problem (P) to have continuous families of solutions, as the second order problem (1.6), in some cases, has?

(6.5) Is it possible for problem (P) to have several nontrivial solutions for the values of α not considered in Theorems 4.5 and Theorem 4.6?

(6.6) What happens if $N \geq 4$? What happens in Theorem 4.6 if $N = 1$?

Appendix A. Nonsmooth version of the multiplicity theorem

Here we want to give a more general version of Theorem 2.5 which covers the method (of Galerkin type) introduced in [7] (for a unique functional), which inspired us in formulating, for example, Theorem 2.5, as we already said.

Let H be a Hilbert space. Given $g\colon H \to \mathbb{R} \cup \{\pm\infty\}$, we set $\mathcal{D}(g) = \{u \in H \mid g(u) \in \mathbb{R}\}$.

Now let us consider a sequence $(g_n)_n$ of functions such that $g_n\colon H \to \mathbb{R} \cup \{+\infty\}$.

Definition A.1. We say that $u \in \mathcal{D}(g)$ is asymptotically critical for $((g_n)_n, g)$ at level $g(u)$ if there exists a strictly increasing sequence $(n_k)_k \in \mathbb{N}$, there exists $u_k \in \mathcal{D}(g_{n_k})$ and there exists $\eta_k \in \partial^- g_{n_k}(u_k)$ such that $u_k \to u$, $g_{n_k}(u_k) \to g(u)$ and $\eta_k \to 0$.

Definition A.2. Let $\mathfrak{c}$ be a real number. We say that $\nabla(g_n, g; \mathfrak{c})$ holds if

- for all $(n_k)_k$ strictly increasing in $\mathbb{N}$, for all $(u_k)_k$ in H, for all $(\eta_k)_k$ in H, if $u_k \in \mathcal{D}(g_{n_k})$, $\eta_k \in \partial^- g_{n_k}(u_k)$, $\eta_k \to 0$, $g_{n_k}(u_k) \to \mathfrak{c}$,

there exist a subsequence $(u_{k_j})_j$ and u in H such that $u_{k_j} \to u$ and $g(u) = \mathfrak{c}$.

We now want to introduce a class of functions which has the two following properties:

(a) for every function in this class there exists a regular flow of curves of steepest descent,

(b) the class contains the functions of the form $\ell_0 + \ell_1 + \mathbf{I}_M$, where ℓ_0 is a convex function, ℓ_1 is a $C^{1,1}$-function, M is a closed submanifold of class $C^{1,1}$ and $\mathbf{I}_M: H \to \{0, +\infty\}$ is the indicatrix function of M, that is $\mathbf{I}_M(u) = 0$ if $u \in M$ and $\mathbf{I}_M(u) = +\infty$ if $u \notin M$.

(See [3], [6], [14]).

Now let us consider a function $\ell: H \to \mathbb{R} \cup \{+\infty\}$.

Definition A.3. Let $\varphi: \mathcal{D}(\ell)^2 \times \mathbb{R}^3 \to \mathbb{R}^+$ be a continuous function. We say that ℓ is *φ-convex* if

$$\ell(v) \geq \ell(u) + \langle \eta, v - u \rangle - \varphi(u, v, \ell(u), \ell(v), \|\eta\|)\|v - u\|^2$$

for all u, v in $\mathcal{D}(\ell)$ and for all η in $\partial^- \ell(u)$.

Note that nothing is required (at least explicitly) if $\partial^- \ell(u) = \emptyset$.

We say that such an ℓ is *φ-convex of order* r ($r \geq 1$) if

$$\varphi(u, v, \ell(u), \ell(v), \|\eta\|) = \varphi_0(u, v, \ell(u), \ell(v))(1 + \|\eta\|^r),$$

where $\varphi_0: \mathcal{D}(\ell)^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

According to the results about φ-convex functions obtained in the papers above, one can easily prove the following version of Theorem 2.5.

Theorem A.4. *Let a ad b be real numbers with $a \leq b$. Assume that*

- *every function g_n is φ_n-convex of order 2 for some φ_n,*
- *$\nabla(g_n, g; \mathfrak{c})$ holds for all $\mathfrak{c} \in [a, b]$.*

Then $((g_n)_n, g)$ has at least $\limsup_{n\to\infty} \operatorname{cat}_H(g_n^b, g_n^a)$ asymptotically critical points with levels in $[a, b]$.

Also Theorem 2.6 of [10] can be evidently extended to this class of functions.

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