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STATIONARY STATES FOR DISCRETE DYNAMICAL SYSTEMS IN THE PLANE

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Abstract. The existence of a fixed point for maps of the form *Identity* + *Contraction* acting on $\mathbb{R}^2$ is established under quite general conditions. A counterexample is given in $\mathbb{R}^3$.

1. Introduction

A result due to Sharkovskiĭ ([11]) implies that a continuous map f of an interval I into $\mathbb{R}$ which has a periodic orbit of any period $p > 1$, must have a fixed point. In fact, Sharkovskiĭ established the following theorem.

Theorem 1.1. *Let $f: I \to \mathbb{R}$ be continuous. Assume that f has a periodic orbit of period p. Then f has a periodic orbit of any period q which follows p in the ordering,*

$$3 < 5 < 7 < 9 < \ldots < 6 < 10 < 14 < \ldots < 12 < 20 < 28 < \ldots < 2^2 < 2 < 1.$$

Notice that if f has a periodic orbit of period 3 then it has a periodic orbit of every period. This result was later rediscovered by Li–Yorke ([7]) who were unaware of Sharkovskiĭ's theorem. They also proved the existence of an uncountable set S such that for every point $x_0 \in S$ the orbit $O(x_0)$ is aperiodic and unstable. This property of f motivated the title of their paper *Period three implies chaos*.

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It is easily seen that in the case when a real-valued map F is of the form Identity + Contraction, $F(x) = x + K(x)$, where $|K(x) - K(y)| \leq r|x - y|$, with $r \in (0, 1)$, then F has at most one fixed point and no periodic orbit with period $p > 1$. Moreover, if F does not have a fixed point, then every orbit of F is unbounded.

The purpose of this paper is to analyze the behavior of maps of the form Identity+Contraction in $\mathbb{R}^2$ and in higher dimension with respect to the problem of determining what conditions will insure the presence of fixed points.

The problem is not completely new. In fact, at the beginning of last century, Brower ([3]) proved his famous Lemma on Translation Arcs (see below). His result generated several papers (see [2], [6], [12]–[14]) in which different and/or simpler proof of the lemma were provided. In 1984 M. Brown ([4]) gave an elegant and short proof of Brower's Lemma and in a successive paper [5] he observed that one version of Brower's Lemma is that each orientation preserving homeomorphism of the plane with a periodic orbit must have a fixed point.

We shall show later that Identity + Contraction is orientation preserving. Hence, the presence of a periodic orbit of any period $p > 4$ (it cannot be 4 or less) implies the existence of a fixed point. However, given the particular form of this class of orientation preserving maps, we were able to obtain a stronger theorem.

2. Notations, definitions and preliminary results

2.1. Orientation preserving diffeomorphisms. Given a diffeomorphism $F: \mathbb{R}^2 \to \mathbb{R}^2$ we say that F is orientation preserving if $\det(F'(x)) > 0$ for every $x \in \mathbb{R}^2$. In the case when F is only a homeomorphism we say that F is orientation preserving if we can find a circle C of radius $r > 0$ centered at the origin, $C = \{x \in \mathbb{R}^2 : \|x\| = r\}$, such that a counterclockwise parametrization $\gamma(t) = r(\cos(t), \sin(t))$ of C is mapped to a counterclockwise parametrization of its image under F: $\beta(t) = F(\gamma(t))$. It is easy to see that an orientation preserving diffeomorphism is an orientation preserving homeomorphism. In fact, more is true, since the following result holds (see [10]).

Theorem 2.1. *Any orientation preserving diffeomorphism of F of $\mathbb{R}^2$ is smoothly isotopic to the identity.*

In other words, there exists a smooth homotopy $H: \mathbb{R}^2 \times [0, 1] \to \mathbb{R}^2$ such that $H(x, 0) = x$, $H(x, 1) = F(x)$ and $H(\,\cdot\,, t)$ maps $\mathbb{R}^2$ diffeomorphically onto $\mathbb{R}^2$ for every $t \in (0, 1)$.

A fixed point of F is a point x such that $F(x) = x$. We say that a maps is *fixed point free* if $F(x) \neq x$ for every $x \in \mathbb{R}^2$. An arc α starting at x and ending at y is a translation arc for F provided that $F(x) = y$ and $F(\alpha) \cap \alpha = y$. The following result is due to Brower ([3]).

Theorem 2.2. *Let F be a fixed point free orientation preserving homeomorphim of $\mathbb{R}^2$ and let α be a translation arc of F. Then for each integer $n \geq 2$ we have $F^n(\alpha) \cap \alpha = \emptyset$.*

It is clear from the above formulation that an orientation preserving homemorphism with a periodic point cannot be fixed point free.

2.2. Identity + contraction. A map of the form $F(x) = x + K(x)$ where K is a contraction, is an orientation preserving homeomorphism. In fact, it is easy to check that F is one-to-one and onto, and there exists $r > 0$ such that for every $\|x\| \geq r$ we have $\|K(x)\| < \|x\|$ (see [9]). Therefore, the homotopy $h(x, s) = x + sK(x)$ never vanishes on the circle $C = \{x \in \mathbb{R}^2 : \|x\| = r\}$ and the parametrizations $\gamma(t) = r(\cos(t), \sin(t))$ and $\beta(t) = \gamma(t) + sK(\gamma(t))$ are both counterclockwise.

2.3. Winding number. We say that Γ is a closed, simple, oriented Jordan arc if there exists an injective continuous map $c\colon [\alpha, \beta) \to \mathbb{R}^2$ such that c is uniformly continuous in $[\alpha, \beta)$ and $c(\alpha) = c(\beta)$ when c is extended in a continuous manner to the closed interval $[\alpha, \beta]$ and the image of c is the curve Γ. Clearly Γ can be regarded as the equivalence class of all parametrizations, provided that the equivalence relation is limited to those parametrizations for which the change of variable is increasing. Γ is said to be positively oriented if the bounded region surrounded by Γ lies on the left hand side when Γ is traversed according to c. Given a continuous, nowhere vanishing planar vector field $w\colon \Gamma \to \mathbb{R}^2, w(x) = (u(x), v(x))$ along Γ we can define the angle function

$$\begin{aligned} \theta_w(x) &= \operatorname{arctang}\left(\frac{v(x)}{u(x)}\right) \quad \text{if } u(x) \neq 0, \\ \theta_w(x) &= \operatorname{arctang}\left(\frac{u(x)}{v(x)}\right) \quad \text{if } v(x) \neq 0, \end{aligned}$$

together with the additional condition $\theta_w(x) \in [0, 2\pi)$. Given any initial point $x_0 \in \Gamma$ we call the winding number of the vector field w along Γ the growth, divided by 2π, of the angle function $\theta_w(x)$ as x moves from the position x_0 back to it along Γ. It can be shown that the number obtained is an integer and it is independent of the initial point x_0. For more details the interested reader may consult Amann ([1]).

A closed polygonal path P in the plane is a family of vertices $V = \{V_1, \ldots, V_{n+1}\}$ and segments $S = \{s_1, \ldots, s_n\}$ such that $V_{n+1} = V_1$ and s_i joins V_i with V_{i+1}, $i = 1, \ldots, n$. We also require that no other points are shared by any pair of segments except the vertices, with each vertex belonging only to two consecutive segments. P is said to be positively oriented if walking along P according to its orientation we leave the region bounded by P on our left. For each vertex V_i of P we consider the exterior angle $\theta_i \in (-\pi, \pi)$ between the vectors $V_{i-1}V_i$ and

V_iV_{i+1}. The angle is positive (negative) if V_{i+1} is to the left (right) of the half line starting at V_{i-1} and going through V_i when we walk along P following its positive orientation.

3. Results

Before proving our result on the existence of fixed points for maps of the form Identity+Contraction we would like to recall that Marotto ([8]) established a result analogous to the theorem of Li–Yorke for discrete dynamical systems in dimension higher than 1. Moreover, we would like to present an example of a continuous map of the plane into itself with a periodic orbit of period 3 and no periodic orbits of any other period or fixed points. First, we discuss the result of Marotto.

A hyperbolic fixed point x_0 of a differentiable map $F\colon \mathbb{R}^q \to \mathbb{R}^q$ is said to be a snap-back repeller if there exists a point z_0 in the unstable manifold of x_0 and a positive integer m such that

$$F^m(z_0) = x_0 \quad \text{and} \quad \det(F^m_x(z_0)) \neq 0$$

For differentiable maps F of $\mathbb{R}^q$ into itself having a snap-back repeller, Marotto established the existence of uncountably many aperiodic and unstable orbits and of infinitely many periodic orbits of different period.

We now provide an example of a continuous map of $\mathbb{R}^2$ into itself which has a periodic orbit of period 3 and no fixed points or periodic orbits of any other period.

Example 3.1. Recall that there exists exactly one value of $a \in (3,4)$ such that the quadratic map $f(x,a) = ax(1-x)$ has one and only one periodic orbit of period 3. Let us denote this value with r and let $\{x_1, x_2, x_3\}$ be the corresponding periodic orbit of period 3. Recall that $x_i \in (0,1)$, $i = 1,2,3$ and $f(x,r) = x$ implies that either $x_{s1} = 0$ or $x_{s2} = 1 - 1/r$. Notice that both fixed points are in the interval $[0,1]$. Define

$$F(x,y) = (f(x,r), |(x-x_1)(x-x_2)(x-x_3)| \exp(f(x,r)-x) + \exp(y) - 1).$$

First, notice that the orbit $(x_1,0)$, $(x_2,0)$, $(x_3,0)$ is periodic of period 3. It is easy to check that F does not have any fixed point. In fact, since the first component must equal either x_{s1} or x_{s2} the second component must satisfy the equality $s + \exp(y) - 1 = y$ with $s \in (0,1)$. The equation does not have any solution. Moreover, F does not have any periodic orbit of period different from 3. In fact, the presence of such an orbit would require the existence of a value y such that $t + \exp(y) - 1 < y$ with $t > 0$. Since this inequality cannot be satisfied we conclude that F does not have any periodic orbit of period different from 3.

We are now ready to discuss the main theoretical result of this paper.

Lemma 3.2. *Let P be a closed polygonal path and assume that P is positively oriented. Then the sum of its interior angles is 2π.*

Proof. The result is obviously true if the polygonal path is a triangle. For polygonal paths with more vertices the conclusion is obtained using an induction argument together with the property that the sum of the interior angles of a triangle is π. $\square$

Theorem 3.3. *Let $V_1, \dots, V_{n+1}$ be $n+1$ points in the plane forming a closed polygonal path P. Let $v_i = V_iV_{i+1}$, $i = 1, \dots, n$. Define the vector field $W\colon [0, 2n] \to \mathbb{R}^2$ as follows*

(i) *when $t \in (2i-2, 2i-1)$ set $W(t) = v_i$,*
(ii) *when $t \in (2i-1, 2i)$ set $W(t) = (t-2i+1)v_{i+1} + (2i-t)v_i$.*

Then the winding number of W with respect to P is ∓ 1.

Proof. Notice that W follows the direction of the edges and at each vertex turns the corner. Clearly $W(t) \neq 0$ for every $t \in [0, 2n]$. Moreover, by Lemma 3.2 the algebraic sum of the angles described by $W(t)$ as t ranges from 0 to $2n$ is $\mp 2\pi$. Hence, the winding number of W is ∓ 1. $\square$

The proof of the main result of this paper is based on two additional lemmas of a technical nature.

Lemma 3.4. *Assume that the line segment $[y, y+K(y)]$ intersect the line segment $[x, x+K(x)]$. Then the angle θ between the two oriented segments is acute.*

Proof. Let $t, s > 0$ be such that $x + tK(x) = y + sK(y)$. Since

$$\|K(y) - K(x)\|^2 = \|K(x)\|^2 + \|K(y)\|^2 - 2K(x) \cdot K(y)$$

and

$$\|K(x) - K(y)\|^2 \leq r^2\|x-y\|^2 = r^2(t^2\|K(x)\|^2 + s^2\|K(y)\|^2 - 2stK(x) \cdot K(y)$$

we obtain

$$2K(x) \cdot K(y)(1 - r^2st) \geq (1 - r^2t^2)\|K(x)\|^2 + (1 - r^2s^2)\|K(y)\|^2 > 0.$$

Therefore, the angle is acute. $\square$

Lemma 3.5. *Assume that $\|w - x\| \leq \sqrt{1-r^2}\|K(x)\|$. Then for every z in the line segment $[w, x+K(x)]$ the angle between $K(z)$ and $x+K(x)-w$ is acute.*

Proof. Without loss of generality we can assume that $x = 0$. Then $z = (1-t)w + tK(0)$. We have

$$\|K(0) - w - K(z)\|^2 = \|K(0) - w\|^2 + \|K(z)\|^2 - 2K(z) \cdot (K(0) - w).$$

Moreover,

$$\begin{aligned}\|K(0) - w - K(z)\|^2 &= \|K(0) - K(z)\|^2 + \|w\|^2 - 2(K(0) - K(z)) \cdot w \\ &\leq r^2\|z\|^2 + \|w\|^2 - 2(K(0) - K(z)) \cdot w.\end{aligned}$$

Therefore

$$2K(z)\cdot(K(0)-w) \geq \|K(0)-w\|^2+\|K(z)\|^2-r^2\|z\|^2-\|w\|^2+2(K(0)-K(z))\cdot w.$$

Thus, it is enough to show that

$$\|K(0) - w\|^2 + \|K(z)\|^2 > r^2\|z\|^2 + \|w\|^2 + 2(K(z) - K(0)) \cdot w.$$

Since $\|K(0) - w\|^2 = \|K(0)\|^2 + \|w\|^2 - 2K(0) \cdot w$ we must prove that

$$\|K(0)\|^2 + \|K(z)\|^2 > r^2\|z\|^2 + 2K(z) \cdot w.$$

From $z = (1 - t)w + tK(0)$ we derive

$$\begin{aligned}r^2\|z\|2 &= r^2(1 - t)^2\|w\|^2 + r^2t^2\|K(0)\|^2 + 2r^2t(1 - t)w \cdot K(0) \\ &\leq r^2\|K(0)\|^2\bigg(t^2 + \frac{(1 - t)^2}{4} + t(1 - t)\bigg).\end{aligned}$$

Hence, it is enough to have

$$\|K(0)\|^2\bigg(1 - r^2\bigg(\frac{t^2}{4} + (1 - t)^2 + t(1 - t)\bigg) + \|K(z)\|\bigg)^2 > 2\|K(z)\|\|w\|.$$

Adding to both sides $\|w\|^2$ we obtain

$$\|K(0)\|^2\bigg(1 - r^2\bigg(\frac{t^2}{4} + (1 - t)^2 + t(1 - t)\bigg) + \|K(z)\| - \|w\|\bigg)^2 > \|w\|^2.$$

Since

$$\|K(0)\|^2\bigg(1 - r^2\bigg(\frac{t^2}{4} + (1 - t)^2 + t(1 - t)\bigg)\bigg) \geq \|K(0)\|^2(1 - r^2)$$

we conclude that the angle is acute. □

We are now in a position of proving the existence of stationary states of dynamical systems governed by functions F such that $F - I$ is a contraction.

Theorem 3.6. *Let K be a contraction with constant $k \in (0, 1)$ in an open set U of the plane containing a finite sequence of states $\{x_1, \ldots , x_{n+1}\}$ of the system $F(x) = x + K(x)$, together with the line segments joining each state with the next. Then there exists a point y in the convex hull of the sequence such that $K(y) = 0$ provided that one of the following conditions is verified:*

(i) *$x_{n+1} = x_1$, i.e. the sequence of states is a periodic orbit of the system,*
(ii) *the segment $[x_n, x_{n+1}]$ intersects (at w) the segment $[x_1, x_2]$,*

(iii) *there is* $w \in [x_n, x_{n+1}]$ *such that the segment* $[w, x_2] \subset U$ *and*

$$\|w - x_1\| \leq \sqrt{1-k^2}\|K(x_1)\|.$$

Proof. The proof uses the winding number of the vector field $K(x)$ along with the following closed polygonal paths:

(j) the one made of the segments $[x_2, x_3], \dots, [x_n, x_{n+1} = x_1], [x_1, x_2]$ in case (i).

(jj) the one made of the segments $[x_2, x_3], \dots, [x_n, w], [w, x_2]$ in cases (ii) and (iii).

To make the proof simpler let us rename the vertices of the paths in the following way:

$$\begin{aligned} z_1 &= x_2, \dots, z_{n-1} = x_n, \quad z_n = x_{n+1}, \quad &&\text{in case (j)},\\ z_1 &= x_2, \dots, z_{n-1} = x_n, \quad z_n = w, \quad &&\text{in case (jj)}, \end{aligned}$$

and set $v_i = z_{i+1} - z_i$, $i = 1, \dots, n-1$, $v_n = z_1 - z_n$. Define the parametrization $\gamma\colon [0, 2n] \to \mathbb{R}^2$ by

(1) $\gamma(0) = \gamma(2n) = z_1$,

(2) for $i = 1, \dots, n$ and $t \in (2i-2, 2i-1)$ set $\gamma(t) = z_i + tv_i$,

(3) for $i = 1, \dots, n$ and $t \in (2i-1, 2i)$ set $\gamma(t) = z_i$.

Notice that γ describes the paths with velocity v_i along the i-th edge, and stops at each corner for exactly one unit of time.

Consider the vector fields $V(t) + K(\gamma(t))$ and $W\colon [0, 2n] \to \mathbb{R}^2$ defined as in Theorem 3.3. The homotopy $F(s,t) = sV(t) + (1-s)W(t)$ never vanishes. In fact, if $F(s,t) = 0$ for some (s,t) then $V(t)$ and $W(t)$ must be opposite. This can never happen along the edges since, by Lemmas 3.4 and 3.5, $v_i \cdot K(x_i + \tau v_i) > 0$ for every $\tau \in (0,1)$. It cannot happen at any vertex v_i since

$$K(x_i) \cdot ((t - 2i + 1)v_{i+1} + (2i - t)v_i) = (2i - t)\|v_i\|^2 + (t - 2i + 1)v_i \cdot v_{i+1} > 0.$$

Since $F(0,t) = W(t)$ and $F(1,t) = V(t)$ we conclude that V and W have the same winding number. Consequently, the vector field $V(t)$ must vanish in at least one point belonging to the convex hull of the sequence $\{x_0, \dots, x_n\}$. $\square$

We suspect that the inequality $\|w - x_1\| \leq \sqrt{1-k^2}\|K(x_1)\|$ is optimal. However, we have not been able to find examples with orbits as close as desired to the inequality. For the map of Example 3.7 below we located orbits for which $\|w - x_1\|$ if about twice as large as $\sqrt{1-k^2}\|K(x_1)\|$.

Example 3.7. Let $u = (x, y)$ and

$$F(u) = u + K(u) = (x + M(r(1.58 - \arctan(x), ry - \arctan(y))$$

where $r \in (0,1)$, M is a counterclockwise rotation of an angle of t radians with t selected to achieve the desired result. The dynamical system governed by F does not have any stationary states. In fact, the existence of such a state (x,y) would imply

$$\begin{aligned} r(1.58 - \arctan(x))\cos(t) - (ry - \arctan(y))\sin(t) &= 0, \\ r(1.58 - \arctan(x))\sin(t) - (ry - \arctan(y))\cos(t) &= 0. \end{aligned}$$

Squaring and adding the two equalities we obtain

$$r^2(1.58 - \arctan(x))^2 + (ry - \arctan(y))^2 = 0$$

which is clearly false.

From Theorem 3.6 we derive that all orbits of a fixed point free map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form $F(x) = x + K(x)$ where K is a contraction must be unbounded, mimicking the analogous result we found in $\mathbb{R}$. Example 1.3 below shows that in dimension $d \geq 3$ this is no longer true.

Example 3.8. Let $u = (x,y)$ and

$$\begin{aligned} g(x,y,z) &= (1 - x^2 - y^2)(2 + \arctan(z)), & x^2 + y^2 &\leq 1, \\ g(x,y,z) &= 0, & x^2 + y^2 &> 1. \end{aligned}$$

Define $F(u) = u + K(u) = (x,y,z) + r(M(x,y), g(x,y,z))$, where M is the rotation of Example 3.7 with $t = \arccos(-r/2)$. It can be shown that K is a contraction if $r < 0.2$. Take $r = 2\cos(11\pi/21)$. Then the orbit of the point $(1,0,0)$ is periodic of period 42.

Since for $x^2 + y^2 > 0$ the first two coordinates of $F(u)$ are not fixed, all potential fixed points must be on the z-axis. But $z + g(x,y,z) = z$ implies $\arctan(z) = -2$, which is clearly impossible. For example, the orbit starting at $(0.8, 0, 2)$ goes to ∞ while remaining on the cylinder $x^2 + y^2 = 0.64$.

We conclude the paper with a result showing that for maps of the form $F = I + K$, $F(x) = x + K(x)$ such that $K(x_0) \neq 0$ the orbit starting from x_0 can never reach a fixed point in finitely many steps. In other words the map does not have eventually stationary orbits which are not stationary. Thus, in every dimension, a map of this type cannot have a snap-back repeller.

Lemma 3.9. *Assume that $K(x_0) \neq 0$ and x is any point of the segment joining x_0 with $x_0 + K(x_0)$. Then*

$$K(x) \cdot K(x_0) \leq \frac{1}{2}((1 - k^2)\|K(x_0)\|^2 + \|K(x)\|^2).$$

Proof. Obviously $\|x - x_0\|^2 \leq \|K(x_0)\|^2$ and $\|K(x) - K(x_0)\|^2 \leq k^2\|x - x_0\|^2$. Therefore $\|K(x) - K(x_0)\|^2 \leq k^2\|K(x_0)\|^2$. The claimed result follows. □

Theorem 3.10. *Assume that $K(x_0) \neq 0$. Then the orbit starting from x_0 will never reach a stationary state in finitely many steps.*

Proof. From Lemma 3.5 we obtain $K(x_1) \neq 0$ where $x_1 = x_0 + K(x_0)$. Keep going. $\square$

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