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MINIMAL DISPLACEMENT OF RANDOM VARIABLES UNDER LIPSCHITZ RANDOM MAPS

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Abstract. Let (Ω, Σ) be a measurable space and X be a separable metric space. It is shown that for measurable maps $\zeta, \eta: \Omega \to X$, if a random map $T: \Omega \times X \to X$ satisfies $d(T(\omega, \zeta(\omega)), T(\omega, \eta(\omega))) \leq \alpha d(\zeta(\omega), \eta(\omega)) + \gamma$ then $\inf\{d(\xi(\omega), T(\omega, \xi(\omega)))\} \leq \gamma/(1-\alpha)$, where $\gamma > 0$, $\alpha \in (0,1)$ and inf is taken over all measurable maps $\xi: \Omega \to X$. Several consequences of this result are also obtained.

1. Introduction

Random fixed point theorems for random contraction mappings on Polish spaces were first proved by Spacek ([22]) and Hans ([11], [12]). Subsequently Bharucha–Reid ([5]) has given sufficient conditions for a stochastic analogue of Schauder's fixed point theorem for a random operator. Itoh ([13]) introduced random condensing operators and considerably improved the known results. Recently Sehgal and Water ([20], [21]), Papageorgiou ([17]), Xu ([24]), Beg et al ([1]–[3]), Tan and Yuan ([23]), Lishan ([16]) and many other authors have studied the fixed points of random maps. Klee ([15]) initiated to study the problem of determining, how near certain mappings come to having a fixed point. This problem is called "minimal displacement" and is meaningful in situations where the existence of fixed point of the mapping is not known. Further results in this

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direction have been obtained by Reich ([18], [19]), Goebel et al ([8]–[10]), Furi and Martelli ([7]), Bula ([6]) and Kirk ([14]). The aim of this paper is to study the minimal displacement of a random variable under an α-lipschitz random map. Suppose (Ω, Σ) is a measurable space and X is a separable metric space. For measurable maps $\zeta, \eta \colon \Omega \to X$, if a random map $T \colon \Omega \times X \to X$ satisfies the condition $d(T(\omega, \zeta(\omega)), T(\omega, \eta(\omega))) \leq \alpha d(\zeta(\omega), \eta(\omega)) + \gamma$ then it is proved that $\inf\{d(\xi(\omega), T(\omega, \xi(\omega))\} \leq \gamma/(1-\alpha)$, where $\gamma > 0$, $\alpha \in (0,1)$ and inf is taken over all measurable maps $\xi \colon \Omega \to X$. Several consequences of above result are also given.

2. Preliminaries

Let (Ω, Σ) be a measurable space (Σ – sigma algebra) and K a nonempty subset of a metric space X. A mapping $\xi \colon \Omega \to X$ is *measurable* (*or random variable*) if and only if $\xi^{-1}(U) \in \Sigma$ for each open subset U of X. The mapping $T \colon \Omega \times K \to X$ is a *random map* if and only if for each fixed $x \in K$, the mapping $T(\,\cdot\,, x) \colon \Omega \to X$ is measurable. A measurable mapping $\xi \colon \Omega \to X$ is a *random fixed point* of the random map $T \colon \Omega \times K \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$. The random map $T \colon \Omega \times K \to X$ is called *α-lipschitz* if for each $\omega \in \Omega$, it satisfies $d(T(\omega, x), T(\omega, y)) \leq \alpha d(x, y)$, for all $x, y \in K$. We denote by $M(\Omega, X)$ the set of all measurable functions from Ω into a metric space X. For a subset K of X, let $\operatorname{diam}(K)$ denote the diameter of K, $B(x_0; r)$ the spherical ball centred at x_0 with radius r i.e. $B(x_0; r) = \{x \in X : d(x_0, x) \leq r\}$, $r(K)$ the Chebyshev radius of K i.e. $r(K) = \inf_{x \in K} \sup_{y \in K} d(x, y)$ and $T^n(\omega, x)$ the nth iterate $T(\omega, T(w, T(w, \dots, T(\omega, x) \dots)))$ of T.

For more details and other related results we refer to [1], [4], [5], [10].

3. Minimal displacement

Theorem 3.1. *Let K be a nonempty bounded closed convex subset of a separable Banach space X and $\alpha \geq 1$. If $T \colon \Omega \times K \to K$ is an α-lipschitz random map then*

$$\inf\{\|\xi(w) - T(w, \xi(w))\| : \xi \in M(\Omega, K)\} \leq r(K)\left(1 - \frac{1}{\alpha}\right) \tag{1}$$

for each $\omega \in \Omega$.

Proof. Let $\varepsilon > 0$ and select $\eta \in M(\Omega, K)$ such that $K \subset B(\eta(w); r(K) + \varepsilon)$. Defining $T_\varepsilon \colon \Omega \times K \to K$ by

$$T_\varepsilon(w, x) = \left[1 - \frac{1}{\alpha + \varepsilon}\right]\eta(w) + \frac{1}{\alpha + \varepsilon} T(w, x).$$

Since T is α-lipschitzian. Therefore T_ε is $\alpha/(\alpha + \varepsilon)$-lipschitzian.

Here T_ε is a random contraction mapping which has a unique random fixed point $\xi_\varepsilon \in M(\Omega, k)$. Thus

$$\begin{aligned}
&\|\xi_\varepsilon(w) - T(w, \xi_\varepsilon(w))\| \\
&\quad = \|T_\varepsilon(w, \xi_\varepsilon(w)) - T(w, \xi_\varepsilon(w))\| \\
&\quad = \left\| \left[1 - \frac{1}{\alpha+\varepsilon}\right]\eta(w) + \frac{1}{\alpha+\varepsilon} T(w, \xi_\varepsilon(w)) - T(w, \xi_\varepsilon(w)) \right\| \\
&\quad = \left(1 - \frac{1}{\alpha+\varepsilon}\right)\|\eta(w) - T(w, \xi_\varepsilon(w))\| \le \left(1 - \frac{1}{\alpha+\varepsilon}\right)(r(K) + \varepsilon).
\end{aligned}$$

Since X is a separable Banch space, therefore by letting $\varepsilon \to 0$, we obtain

$$\inf\{\|\xi(w) - T(w, \xi(w))\| : \xi \in M(\Omega, K)\} \le r(K)\left(1 - \frac{1}{\alpha}\right). \qquad \square$$

Corollary 3.2. *Let $B(0;1)$ be the unit ball in a separable Banach space X. If $T\colon \Omega \times B(0;1) \to B(0;1)$, is an α-lipschitz random map then*

$$\operatorname{Inf}\{\|\xi(w) - T(w, \xi(w))\| : \xi \in M(\Omega, k)\} \le \frac{\alpha - 1}{\alpha}.$$

Definition 3.3. Let (X, d) be a metric space. A random map $T\colon \Omega \times X \to X$ is said to be γ-*random contraction* with contractive constant α, $\gamma > 0$ if there exists $\alpha \in (0,1)$ such that, for each $\xi, \eta \in M(\Omega, X)$,

$$d(T(w, \xi(w)), T(w, \eta(w))) \le \alpha d(\xi(w), \eta(w)) + \gamma, \tag{2}$$

for all $\omega \in \Omega$.

If $\alpha \in (0,1)$ and $\ell \in (\alpha, 1)$, then for each $\xi, \eta \in M(\Omega, X)$ and $\omega \in \Omega$,

$$\alpha d(\xi(w), \eta(w)) + \gamma \le \ell d(\xi(w), \eta(w))$$

if and only if

$$d(\xi(w), \eta(w)) \ge \frac{\gamma}{\ell - \alpha}.$$

Example 3.4. Consider R with usual metric and $(\Omega, \Sigma) = R$ with Lebesgue measure. The random Dirichlet map $T\colon \Omega \times R \to R$ defined for all $x \in R$, $\omega \in \Omega$, by

$$T(\omega, x) = \begin{cases} 1 & x \in Q, \\ 0 & \text{otherwise}, \end{cases}$$

is 1-random contraction.

Definition 3.5. Let (X, d) be a metric space. A random map $T\colon \Omega \times X \to X$ is said to be a h_ℓ-*random contractive* if for $h > 0$, $\ell \in (0,1)$ and $\omega \in \Omega$,

$$d(T(w, \xi(w)), T(w, \eta(w)) \le \ell \max\{d(\xi(w), \eta(w)), h\}, \tag{3}$$

where $\xi, \eta \in M(\Omega, X)$.

Theorem 3.6. *Let (X,d) be a separable metric space. If $T:\Omega\times X\to X$ is a γ-random contraction with contractive constant $\alpha\in(0,1)$, then T is h_ℓ random contractive for $h=\gamma/(\ell-\alpha)$ and $\ell\in(\alpha,1)$.*

Proof. If for each $\omega\in\Omega$, $d(\xi(w),\eta(w))\ge h$ then

$$d(T(w,\xi(w)),T(w,\eta(w)))\le \alpha d(\xi(w),\eta(w))+\gamma\le \ell d(\xi(w),\eta(w)).$$

Otherwise,

$$d(T(w,\xi(w)),T(w,\eta(w)))\le \alpha d(\xi(w),\eta(w))+\gamma< \alpha h+\gamma=\ell h. \qquad \square$$

Theorem 3.7. *Let (X,d) be a separable metric space. If $T:\Omega\times X\to X$ is a γ-random contraction with contractive constant α, then*

$$\inf\{d(\xi(w),T(w,\xi(w))):\xi\in M(\Omega,X)\}\le\frac{\gamma}{1-\alpha}, \tag{4}$$

for each $\omega\in\Omega$.

Proof. Let $\ell\in(\alpha,1)$, $h(\ell)=\gamma/(\ell-\alpha)$ and let $\xi\in M(\Omega,X)$. Suppose

$$d(T^n(w,\xi(w)),T^{n+1}(w,\xi(w)))\ge h(\ell),\quad \text{for } n=1,2,\dots$$

Inequality (4), implies that $\{T^n(w,\xi(w))\}$ is a Cauchy sequence and clearly this leads to a contradiction therefore for some n,

$$d(T^n(w,\xi(w)),T^{n+1}(w,\xi(w)))\le h(\ell).$$

Since $\ell\in(\alpha,1)$, and X is a separable metric space, hence for each $\omega\in\Omega$,

$$\inf\{d(\xi(w),T(w,\xi(w))):\xi\in M(\Omega,X)\}\le\frac{\gamma}{1-\alpha}. \qquad \square$$

Definition 3.8. Let (X,d) be a metric space. A random mapping $T:\Omega\times X\to X$ is said to be *h-nonexponsive* if for $\xi,\eta\in M(\Omega,X)$,

$$d(T(w,\xi(w)),T(w,\eta(w)))\le\max\{d(\xi(w),\eta(w)),h\}, \tag{5}$$

for each $\omega\in\Omega$.

Theorem 3.9. *If K is a nonempty bounded closed convex subset of a separable Banach space X and if $T:\Omega\times K\to K$ is h-nonexpansive random mapping, then*

$$\operatorname{Inf}\{\|\xi(w)-T(w,\xi(w))\|:\xi\in M(\Omega,X)\}\le h,$$

for each $\omega\in\Omega$.

Proof. Let $\delta\in M(\Omega,X)$. For $t\in(0,1)$, define

$$T_t(w,\xi(w))=(1-t)\delta(w)+tT(w,\xi(w))$$

for $w\in\Omega$ and $\xi\in M(\Omega,X)$.

Then T_t is h_t-random contractive for each $t \in (0,1)$. It further implies that for each such t there exists $\xi_t \in M(\Omega, X)$ such that

$$\|\xi_t(w) - T_t(w, \xi_t(w))\| \leq th. \tag{6}$$

But

$$\begin{aligned} &\|\xi_t(w) - T(w, \xi_t(w))\| \\ &\qquad = \|\xi_t(w) - tT(w, \xi_t(w) - (1-t)T(w, \xi_t(w))\| \\ &\qquad = \|\xi_t(w) - T_t(w, \xi_t(w)) + (1-t)\delta(w) - (1-t)T(w, \xi_t(w))\| \\ &\qquad \leq \|\xi_t(w) - T_t(w, \xi_t(w))\| + (1-t)\|\delta(w) - T(w, \xi_t(w))\|. \end{aligned}$$

Since K is a bounded subset of a separable Banach space X, therefore letting $t \to 1^-$, we obtain by using inequality (6), for each $w \in \Omega$,

$$\operatorname{Inf}\{\|\xi(w) - T(w, \xi(w))\| : \xi \in M(\Omega, X)\} \leq h. \qquad \square$$

Theorem 3.10. *Suppose K is a nonempty bounded closed convex subset of a separable Banach space X. For $\xi, \eta \in M(\Omega, K)$, suppose $T\colon \Omega \times K \to K$ satisfies*

$$\|T(w, \xi(w)) - T(w, \eta(w))\| \leq h\|\xi(w) - \eta(w)\|^p, \tag{7}$$

for some $h, p \in (0,1)$. Then, for each $\omega \in \Omega$,

$$\operatorname{Inf}\{\|\xi(w) - T(w, \xi(w))\| : \xi \in M(\Omega, K)\} \leq h^{1/(1-p)}.$$

Proof. Since $h \in (0,1)$, Theorem 3.9 implies that there exists $\xi \in M(\Omega, K)$ such that

$$\|\xi(w) - T(w, \xi(w))\| < 1,$$

for each $\omega \in \Omega$. Thus,

$$\|T(w, \xi(w)) - T^2(w, \xi(w))\| \leq h\|\xi(w) - T(w, \xi(w)\|^p < h,$$

for each $\omega \in \Omega$. From inequality (7), we have

$$\|T^2(w, \xi(w)) - T^3(w, \xi(w))\| \leq h\|T(w, \xi(w) - T^2(w, \xi(w))\|^p < h^{1+p},$$

for each $\omega \in \Omega$. By induction, we obtain

$$\|T^{n+1}(w, \xi(w)) - T^{n+2}(w, \xi(w))\| < h^{(1+p+\ldots+p^n)},$$

for each $\omega \in \Omega$. Therefore,

$$\lim_{n\to\infty} \|T^{n+1}(w, \xi(w)) - T^{n+2}(w, \xi(w))\| \leq \lim_{n\to\infty} h^{(1+p+\ldots+p^n)} = h^{1/(1-p)}.$$

Hence, for each $\omega \in \Omega$,

$$\operatorname{Inf}\{\|\xi(w) - T(w, \xi(w))\| : \xi \in M(\Omega, k)\} \leq h^{1/(1-p)}. \qquad \square$$

Remark 3.11. (i) It will be of great interest to find for which sets or spaces is the estimate (1) sharp.

(ii) The condition (7) is random analogue of the famous Holder condition. If $p = 1$ and $h < 1$ then these are random contraction mappings which are well known to have a unique random fixed points (see e.g. [22], [11], [12], [1]). For $p = 1$ and $h = 1$ these are called nonexpansive random maps and they also have an extensive random fixed point theory (see e.g. [24], [2], [16]).

(iii) The condition of h-nonexpansive (5) is much more general than Holder condition (7) for $0 < h, p \leq 1$.

References

[1] I. Beg, *Random fixed points of random operators satisfying semicontractivity conditions*, Math. Japon. **46** (1997), 151–155.

[2] I. Beg and N. Shahzad, *Random fixed point theorems for nonexpansive and contractive type random operators on Banach spaces*, J. Appl. Math. Stochastic. Anal. **7** (1994), 569–580.

[3] ———, *Some random approximation theorem with applications*, Nonlinear Anal. **35** (1999), 609–616.

[4] A. T. Bharucha-Reid, *Random Integral Equations*, Academic Press, New York, 1972.

[5] ———, *Fixed point theorems in probabilistic analysis*, Bull. Amer. Math. Soc. **82** (1976), 641–657.

[6] I. Bula, *On the stability of the Bohl–Brouwer–Schauder theorem*, Nonlinear Anal. **26** (1996), 1859–1868.

[7] M. Furi and M. Martelli, *On the minimal displacement of points under the alpha-Lipschitz maps in normed spaces*, Boll. Un. Mat. Ital. **9** (1975), 791–799.

[8] K. Goebel, *On the minimal displacement of points under lipschitzian mappings*, Pacific J. Math. **45** (1973), 151–163.

[9] K. Goebel and T. Komorowski, *Retracting balls onto spheres and minimal dispacement problems*, Fixed Point Theory and Applications (J. B. Baillon and M. Thera, eds.), Longman Scientific, Essex, 1991, pp. 155–172.

[10] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Univ. Press, Cambridge, 1990.

[11] O. Hans, *Reduzierende zulliällige transformaten*, Czechoslovak Math. J. **7** (1957), 154–158.

[12] ———, *Random operator equations*, Proc. 4th Berkeley Symposium on Mathematical Statistics and Probability, vol. II, Part I, University of California Press, Berkeley, 1961, pp. 185–202.

[13] S. Itoh, *Random fixed point theorems with an application to random differential equations in Banach spaces*, J. Math. Anal. Appl. **67** (1979), 261–273.

[14] W. A. Kirk, *Holder continuity and minimal displacement*, Numer. Funct. Anal. Optim. **19** (1998), 71–79.

[15] V. Klee, *Stability of the fixed point property*, Colloq. Math. **VIII** (1961), 43–46.

[16] L. Liu, *Some random approximations and random fixed point theorems for* 1*-set-contractive random operators*, Proc. Amer. Math. Soc. **125** (1997), 515–521.

[17] N. S. Papageorgiou, *Random fixed point theorems for measurable multifunctions in Banach spaces*, Proc. Amer. Math. Soc. **97** (1986), 507–514.

[18] S. Reich, *Minimal displacement of points under weakly inward pseudo-lipschitzian mappings* I, Atti. Acad. Naz. Linzei Cl. Sci. Fis. Mat. Natur. Rend. Lincei **59** (1975), 40–44.

[19] ———, *Minimal displacement of points under weakly inward pseudo-lipschitzian mappings* II, Atti. Acad. Naz. Linzei Cl. Sci. Fis. Mat. Natur. Rend. Lincei **60** (1976), 95–96.

[20] V. M. Sehgal and C. Waters, *Some random fixed point theorems*, Contemp. Math. **21** (1983), 215–218.

[21] ———, *Some random fixed point theorems for condensing operators*, Proc. Amer. Math. Soc. **90** (1984), 425–429.

[22] A. Spacek, *Zufällige gleichungen*, Czechoslovak Math. J. **5** (1955), 462–466.

[23] K. K. Tan and X. Z. Yuan, *Random fixed point theorems and approximations*, Stochastic Anal. Appl. **15** (1997), 103–123.

[24] H. K. Xu, *Some random fixed point theorems for condensing and nonexpansive operators*, Proc. Amer. Math. Soc. **110** (1990), 395–400.

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