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UPPER AND LOWER SOLUTIONS FOR PROBLEMS WITH SINGULAR SIGN CHANGING NONLINEARITIES AND WITH NONLINEAR BOUNDARY DATA

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Dedicated to Andrzej Granas with admiration

Abstract. An upper and lower solution approach is presented for singular boundary value problems. In particular our nonlinearity may be singular in its dependent variable and is allowed to change sign.

1. Introduction

The boundary value problem

$$\begin{cases} y'' + \left(\dfrac{t^2}{32y^2} - \dfrac{\lambda^2}{8}\right) = 0 & 0 < t < 1, \\ y(0) = 0,\ 2y'(1) - (1+v)y(1) = 0 & 0 < v < 1 \text{ and } \lambda > 0, \end{cases}$$

arises in nonlinear mechanics. The problem models the large deflection membrane response of a spherical cap [3], [6]. Here $S_r = y/t$ is the radial stress at points on the membrane, $d(\rho S_r)/d\rho$ is the circumferential stress ($\rho = t^2$), λ is a load geometry parameter and v is the Poisson ratio.

Motivated by the above example this paper discusses the more general boundary value problem

$$\begin{cases} y'' + q(t)f(t,y) = 0 & \text{for } 0 < t < 1, \\ y(0) = y'(1) + \psi(y(1)) = 0, \end{cases}$$

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where our nonlinearity f is allowed to change sign. Notice that f may not be a Carathéodory function [2], [4] because of the singular behaviour of the y variable i.e. f may be singular at $y = 0$. Examples are

$$f(t,y) = t^{-1}e^{\frac{1}{y}} - (1-t)^{-1} \quad \text{and} \quad f(t,y) = \frac{g(t)}{y^\sigma} - h(t), \quad \sigma > 0$$

which correspond to Emden–Fowler equations; here $g(t) > 0$ for $t \in (0,1)$ and $h(t)$ may change sign. There are two main approaches in the literature to establishing existence for singular problems. The first approach is based on an argument initiated by Habets and Zanolin [5], and the second approach is based on ideas presented by Agarwal, O'Regan and Lakshmikantham [1]. In this paper we combine both approaches to obtain a very general existence theory. The results presented are easy to state and apply in practice. However the proofs involved are quite technical. It is also worth remarking here that other types of boundary data and other types of singular problems could be discussed using the ideas in this paper. To illustrate this we also discuss the problem

$$\begin{cases} \dfrac{1}{p}(py')' + q(t)f(t,y) = 0 & \text{for } 0 < t < 1, \\ \lim\limits_{t\to 0^+} p(t)y'(t) = y(1) = 0, \end{cases}$$

in this paper. Here $p \in C[0,1] \cap C^1(0,1)$ with $p > 0$ on $(0,1)$. We do *not* assume $\int_0^1 (1/p(s))\,ds < \infty$ but rather $\int_0^1 (1/p(s)) \int_0^s p(x)q(x)\,dx\,ds < \infty$.

2. Existence theory

In this section we first discuss the boundary value problem

$$(1) \qquad \begin{cases} y'' + q(t)f(t,y) = 0 & \text{for } 0 < t < 1, \\ y(0) = y'(1) + \psi(y(1)) = 0, \end{cases}$$

where our nonlinearity f may change sign. We begin with our main result.

Theorem 2.1. *Let $n_0 \in \{1, 2, \dots\}$ be fixed and suppose the following conditions are satisfied:*

$$(2) \qquad f\colon [0,1] \times (0,\infty) \to \mathbb{R} \text{ is continuous},$$

$$(3) \qquad q \in C(0,1) \text{ with } q > 0 \text{ on } (0,1) \text{ and } tq \in L^1[0,1],$$

$$(4) \qquad \psi\colon \mathbb{R} \to \mathbb{R} \text{ is continuous}$$

$$(5) \qquad \begin{cases} \text{let } n \in \{n_0, n_0+1, \dots\} \text{ and associated with each } n \text{ we have} \\ \text{a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a nonincreasing sequence} \\ \text{with } \lim_{n\to\infty} \rho_n = 0 \text{ and such that} \\ \text{for } 1/2^{n+1} \le t \le 1 \text{ we have } q(t)f(t,\rho_n) \ge 0, \end{cases}$$

$$
\text{(6)} \qquad \begin{cases} \textit{there exists a function } \alpha \in C[0,1] \cap C^1(0,1] \cap C^2(0,1) \\ \textit{with } \alpha(0) = 0, \alpha'(1) + \psi(\alpha(1)) \le 0,\ \alpha > 0 \textit{ on } (0,1] \\ \textit{such that } q(t)f(t,\alpha(t)) + \alpha''(t) \ge 0 \textit{ for } t \in (0,1), \end{cases}
$$

and

$$
\text{(7)} \qquad \begin{cases} \textit{there exists a function } \beta \in C[0,1] \cap C^1(0,1] \cap C^2(0,1) \\ \textit{with } \beta(t) \ge \alpha(t) \textit{ and } \beta(t) \ge \rho_{n_0} \textit{ for } t \in [0,1], \\ \beta'(1) + \psi(\beta(1)) \ge 0 \textit{ with } q(t)f(t,\beta(t)) + \beta''(t) \le 0 \textit{ for } t \in (0,1) \\ \textit{and } q(t)f(1/2^{n_0+1},\beta(t)) + \beta''(t) \le 0 \textit{ for } t \in (0,1/2^{n_0+1}). \end{cases}
$$

Then (1) *has a solution* $y \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ *with* $y(t) \ge \alpha(t)$ *for* $t \in [0,1]$.

Proof. For $n = n_0, n_0+1, \dots$ let

$$
e_n = \left[\frac{1}{2^{n+1}}, 1\right] \quad \text{and} \quad \theta_n(t) = \max\left\{\frac{1}{2^{n+1}}, t\right\}, \quad 0 \le t \le 1
$$

and

$$
f_n(t,x) = \max\{f(\theta_n(t),x), f(t,x)\}.
$$

Next we define inductively

$$
\begin{aligned} g_{n_0}(t,x) &= f_{n_0}(t,x), \\ g_n(t,x) &= \min\{f_{n_0}(t,x), \dots, f_n(t,x)\}, \quad n = n_0+1, n_0+2, \dots \end{aligned}
$$

Notice

$$
f(t,x) \le \dots \le g_{n+1}(t,x) \le g_n(t,x) \le \dots \le g_{n_0}(t,x)
$$

for $(t,x) \in (0,1) \times (0,\infty)$ and

$$
g_n(t,x) = f(t,x) \quad \text{for } (t,x) \in e_n \times (0,\infty).
$$

Without loss of generality assume $\rho_{n_0} \le \min_{t\in[1/3,1]} \alpha(t)$. Fix $n \in \{n_0, n_0 + 1, \dots\}$. Let $t_n \in [0,1/3]$ be such that

$$
\alpha(t_n) = \rho_n \quad \text{and} \quad \alpha(t) \le \rho_n \quad \text{for } t \in [0,t_n].
$$

Define

$$
\alpha_n(t) = \begin{cases} \rho_n & \text{if } t \in [0,t_n], \\ \alpha(t) & \text{if } t \in (t_n,1]. \end{cases}
$$

We begin with the boundary value problem

$$
\text{(8)} \qquad \begin{cases} y'' + q(t) g^*_{n_0}(t,y) = 0 & \text{for } 0 < t < 1, \\ y(0) = \rho_{n_0},\ y'(1) + \psi^*_{n_0}(y(1)) = 0, \end{cases}
$$

where

$$g^*_{n_0}(t,y) = \begin{cases} g_{n_0}(t,\alpha_{n_0}(t)) + r(\alpha_{n_0}(t) - y) & y \le \alpha_{n_0}(t), \\ g_{n_0}(t,y) & \alpha_{n_0}(t) \le y \le \beta(t), \\ g_{n_0}(t,\beta(t)) + r(\beta(t) - y) & y \ge \beta(t), \end{cases}$$

with

$$\psi^*_{n_0}(z) = \begin{cases} \psi(\beta(1)) & z > \beta(1), \\ \psi(z) & \alpha_{n_0}(1) = \alpha(1) \le z \le \beta(1), \\ \psi(\alpha(1)) & z < \alpha_{n_0}(1) = \alpha(1) \end{cases}$$

and $r\colon \mathbb{R} \to [-1,1]$ the radial retraction defined by

$$r(u) = \begin{cases} u & \text{for } |u| \le 1, \\ u/|u| & \text{for } |u| > 1. \end{cases}$$

From Schauder's fixed point theorem we know [1] that (8) has a solution $y_{n_0} \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$. We first show

$$y_{n_0}(t) \ge \alpha_{n_0}(t), \quad t \in [0,1]. \tag{9}$$

Suppose (9) is not true. Then $y_{n_0} - \alpha_{n_0}$ has a negative absolute minimum at $\tau \in (0,1]$. Now since $y_{n_0}(0) - \alpha_{n_0}(0) = 0$ there exists $\tau_0 \in [0,\tau)$ with

$$y_{n_0}(\tau_0) - \alpha_{n_0}(\tau_0) = 0 \quad \text{and} \quad y_{n_0}(t) - \alpha_{n_0}(t) < 0, t \in (\tau_0,\tau). \tag{10}$$

Now either

$$y_{n_0}(t) - \alpha_{n_0}(t) < 0, \quad t \in (\tau_0, 1] \tag{11}$$

or

$$\begin{cases} \exists \tau_1 > \tau \text{ with } y_{n_0}(t) - \alpha_{n_0}(t) < 0 \text{ for } t \in (\tau_0,\tau_1) \\ \text{and } y_{n_0}(\tau_1) - \alpha_{n_0}(\tau_1) = 0. \end{cases} \tag{12}$$

(note (11) occurs if $\tau = 1$).

Case 1. Suppose (12) occurs. Then

$$y_{n_0}(\tau_0) - \alpha_{n_0}(\tau_0) = y_{n_0}(\tau_1) - \alpha_{n_0}(\tau_1) = 0$$

and

$$y_{n_0}(t) - \alpha_{n_0}(t) < 0, \quad t \in (\tau_0,\tau_1).$$

We now claim

$$(y_{n_0} - \alpha_{n_0})''(t) < 0 \quad \text{for a.e. } t \in (\tau_0,\tau_1). \tag{13}$$

If (13) is true then

$$y_{n_0}(t) - \alpha_{n_0}(t) = -\int_{\tau_0}^{\tau_1} G(t,s)[y''_{n_0}(s) - \alpha''_{n_0}(s)]\,ds \quad \text{for } t \in (\tau_0,\tau_1)$$

with

$$G(t,s) = \begin{cases} \dfrac{(s-\tau_0)(\tau_1-t)}{\tau_1-\tau_0} & \text{for } \tau_0 \le s \le t, \\ \dfrac{(t-\tau_0)(\tau_1-s)}{\tau_1-\tau_0} & \text{for } t \le s \le \tau_1, \end{cases}$$

so we have

$$y_{n_0}(t) - \alpha_{n_0}(t) > 0 \quad \text{for } t \in (\tau_0, \tau_1),$$

a contradiction. As a result if we show that (13) is true then we obtain a contradiction in this case. To see (13) we will show

$$(y_{n_0} - \alpha_{n_0})''(t) < 0 \quad \text{for } t \in (\tau_0, \tau_1) \text{ provided } t \ne t_{n_0}.$$

Fix $t \in (\tau_0, \tau_1)$ and assume $t \ne t_{n_0}$. Then

$$\begin{aligned} (y_{n_0} - \alpha_{n_0})''(t) &= -[q(t)\{g_{n_0}(t, y_{n_0}(t)) + r(\alpha_{n_0}(t) - y_{n_0}(t))\} + \alpha''_{n_0}(t)] \\ &= \begin{cases} -[q(t)\{g_{n_0}(t,\alpha(t)) + r(\alpha(t) - y_{n_0}(t))\} + \alpha''(t)] & \text{if } t \in (t_{n_0}, 1), \\ -[q(t)\{g_{n_0}(t,\rho_{n_0}) + r(\rho_{n_0} - y_{n_0}(t))\}] & \text{if } t \in (0, t_{n_0}). \end{cases} \end{aligned}$$

(a) $t \ge 1/2^{n_0+1}$. Then since $g_{n_0}(t,x) = f(t,x)$ for $x \in (0,\infty)$, from (5) and (6), we have

$$\begin{aligned} &(y_{n_0} - \alpha_{n_0})''(t) \\ &\quad = \begin{cases} -[q(t)\{f(t,\alpha(t)) + r(\alpha(t) - y_{n_0}(t))\} + \alpha''(t)] & \text{if } t \in (t_{n_0}, 1), \\ -[q(t)\{f(t,\rho_{n_0}) + r(\rho_{n_0} - y_{n_0}(t))\}] & \text{if } t \in (0, t_{n_0}), \end{cases} \\ &\quad < 0. \end{aligned}$$

(b) $t \in (0, 1/2^{n_0+1})$. Then since

$$g_{n_0}(t,x) = \max\left\{ f\left(\frac{1}{2^{n_0+1}}, x\right), f(t,x) \right\}$$

we have $g_{n_0}(t,x) \ge f(t,x)$ and $g_{n_0}(t,x) \ge f(1/2^{n_0+1}, x)$ for $x \in (0,\infty)$. From (5) and (6) we have

$$\begin{aligned} &(y_{n_0} - \alpha_{n_0})''(t) \\ &\quad \le \begin{cases} -[q(t)\{f(t,\alpha(t)) + r(\alpha(t) - y_{n_0}(t))\} + \alpha''(t)] & \text{if } t \in (t_{n_0}, 1), \\ -[q(t)\{f(1/2^{n_0+1},\rho_{n_0}) + r(\rho_{n_0} - y_{n_0}(t))\}] & \text{if } t \in (0, t_{n_0}), \end{cases} \\ &\quad < 0. \end{aligned}$$

Now subcase (a) and (b) guarantee that (13) holds, so we obtain a contradiction.

Case 2. Suppose (11) occurs. Then

$$y_{n_0}(\tau_0) - \alpha_{n_0}(\tau_0) = 0 \quad \text{and} \quad y_{n_0}(t) - \alpha_{n_0}(t) < 0, \quad t \in (\tau_0, 1].$$

Essentially the same reasoning as in Case 1 guarantees that

$$(y_{n_0} - \alpha_{n_0})''(t) < 0 \quad \text{for } t \in (\tau_0, \tau_1) \text{ provided } t \ne t_{n_0}. \tag{14}$$

Notice (14) implies $\alpha_{n_0} - y_{n_0}$ is convex on $(\tau_0, 1)$; to see this we need only note that $(\alpha_{n_0} - y_{n_0})'_-(t_{n_0}) \le (\alpha_{n_0} - y_{n_0})'_+(t_{n_0})$ since $\alpha'(t_{n_0}) \ge 0$ (note if $\alpha'(t_{n_0}) < 0$ then there exists $\delta > 0$ with $\alpha(t) > \alpha(t_{n_0}) = \rho_{n_0}$ for $t \in [t_{n_0} - \delta, t_{n_0})$, a contradiction). Thus

$$\begin{aligned}\alpha'_{n_0}(1) - y'_{n_0}(1) &\ge \frac{[\alpha_{n_0}(1) - y_{n_0}(1)] - [\alpha_{n_0}(\tau_0) - y_{n_0}(\tau_0)]}{1 - \tau_0} \\ &\ge \alpha_{n_0}(1) - y_{n_0}(1) = \alpha(1) - y_{n_0}(1).\end{aligned}$$

This inequality could also be obtained using the mean value theorem for integrals on $[t_{n_0}, 1]$ and $[\tau_0, t_{n_0}]$ if $t_{n_0} \in (\tau_0, 1)$ (again noting that $(\alpha_{n_0} - y_{n_0})'_-(t_{n_0}) \le (\alpha_{n_0} - y_{n_0})'_+(t_{n_0})$). Now since $y_{n_0}(1) < \alpha_{n_0}(1)$ we have

$$\begin{aligned}0 &< \alpha_{n_0}(1) - y_{n_0}(1) \le \alpha'_{n_0}(1) - y'_{n_0}(1) = \alpha'(1) + \psi^*_{n_0}(y_{n_0}(1)) \\ &\le -\psi(\alpha(1)) + \psi^*_{n_0}(y_{n_0}(1)) = -\psi(\alpha(1)) + \psi(\alpha(1)) = 0,\end{aligned}$$

a contradiction.

So in both Cases 1 and 2 we have a contradiction. Thus (9) holds. In addition since $\alpha(t) \le \alpha_{n_0}(t)$ for $t \in [0,1]$ we have

$$\alpha(t) \le \alpha_{n_0}(t) \le y_{n_0}(t) \quad \text{for } t \in [0,1]. \tag{15}$$

Next we show

$$y_{n_0}(t) \le \beta(t) \quad \text{for } t \in [0,1]. \tag{16}$$

If (16) is not true then $y_{n_0} - \beta$ would have a positive absolute maximum at say $\tau_0 \in (0,1]$. We first discuss the case $\tau_0 \in (0,1)$, so $(y_{n_0} - \beta)'(\tau_0) = 0$ and $(y_{n_0} - \beta)''(\tau_0) \le 0$. There are two cases to consider, namely $\tau_0 \in [1/2^{n_0+1}, 1)$ and $\tau_0 \in (0, 1/2^{n_0+1})$.

(a) $\tau_0 \in [1/2^{n_0+1}, 1)$. Then $y_{n_0}(\tau_0) > \beta(\tau_0)$ together with $g_{n_0}(\tau_0, x) = f(\tau_0, x)$ for $x \in (0,\infty)$ gives

$$\begin{aligned}(y_{n_0} - \beta)''(\tau_0) &= -q(\tau_0)[g_{n_0}(\tau_0, \beta(\tau_0)) + r(\beta(\tau_0) - y_{n_0}(\tau_0))] - \beta''(\tau_0) \\ &= -q(\tau_0)[f(\tau_0, \beta(\tau_0)) + r(\beta(\tau_0) - y_{n_0}(\tau_0))] - \beta''(\tau_0) > 0\end{aligned}$$

from (7), a contradiction.

(b) $\tau_0 \in (0, 1/2^{n_0+1})$. Then $y_{n_0}(\tau_0) > \beta(\tau_0)$ together with

$$g_{n_0}(\tau_0, x) = \max\left\{f\left(\frac{1}{2^{n_0+1}}, x\right), f(\tau_0, x)\right\}$$

for $x \in (0,\infty)$ gives

$$\begin{aligned}(y_{n_0} - \beta)''(\tau_0) = {} & -q(\tau_0)\left[\max\left\{f\left(\frac{1}{2^{n_0+1}}, \beta(\tau_0)\right), f(\tau_0, \beta(\tau_0))\right\}\right. \\ & \left. + r(\beta(\tau_0) - y_{n_0}(\tau_0))\right] - \beta''(\tau_0) > 0\end{aligned}$$

from (7), a contradiction.

It remains to discuss the case $\tau_0 = 1$. If $\tau_0 = 1$ there exists δ, $0 \le \delta < 1$ with $y_{n_0}(t) - \beta(t) > 0$ for $t \in (\delta, 1]$ and $y_{n_0}(\delta) - \beta(\delta) = 0$. Now for $t \in (\delta, 1)$ we have

$$(y_{n_0} - \beta)''(t) = -q(t)[g_{n_0}(t, \beta(t)) + r(\beta(t) - y_{n_0}(t))] - \beta''(t).$$

Fix $t \in (\delta, 1)$. If $t \in (0, 1/2^{n_0+1})$ then

$$\begin{aligned}(y_{n_0} - \beta)''(t) = &-q(t)\bigg[\max\bigg\{f\bigg(\frac{1}{2^{n_0+1}}, \beta(t)\bigg), f(t, \beta(t))\bigg\}\\ &+ r(\beta(t) - y_{n_0}(t))\bigg] - \beta''(t) > 0,\end{aligned}$$

whereas, if $t \in [1/2^{n_0+1}, 1)$, then

$$(y_{n_0} - \beta)''(t) = -q(t)[f(t, \beta(t)) + r(\beta(t) - y_{n_0}(t))] - \beta''(t) > 0.$$

Thus

$$(y_{n_0} - \beta)''(t) > 0 \quad \text{for } t \in (\delta, 1),$$

so $y_{n_0} - \beta$ is convex on $(\delta, 1)$. As a result $y'_{n_0}(1) - \beta'(1) \ge y_{n_0}(1) - \beta(1)$ and this together with $\beta'(1) \ge -\psi(\beta(1))$ gives

$$\begin{aligned}0 < y_{n_0}(1) - \beta(1) &\le y'_{n_0}(1) - \beta'(1) = -\psi^*_{n_0}(y_{n_0}(1)) - \beta'(1)\\ &\le -\psi^*_{n_0}(y_{n_0}(1)) + \psi(\beta(1)) = -\psi(\beta(1)) + \psi(\beta(1)) = 0,\end{aligned}$$

a contradiction. Thus (16) holds, so we have

$$\alpha(t) \le \alpha_{n_0}(t) \le y_{n_0}(t) \le \beta_{n_0}(t) \quad \text{for } t \in [0, 1]. \tag{17}$$

Also notice that $\psi^*_{n_0}(y_{n_0}(1)) = \psi(y_{n_0}(1))$. Next we consider the boundary value problem

$$\begin{cases} y'' + q(t)g^*_{n_0+1}(t, y) = 0 & \text{for } 0 < t < 1,\\ y(0) = \rho_{n_0+1},\ y'(1) + \psi^*_{n_0+1}(y(1)) = 0, \end{cases} \tag{18}$$

here

$$g^*_{n_0+1}(t, y) = \begin{cases} g_{n_0+1}(t, \alpha_{n_0+1}(t)) + r(\alpha_{n_0+1}(t) - y), & y \le \alpha_{n_0+1}(t),\\ g_{n_0+1}(t, y), & \alpha_{n_0+1}(t) \le y \le y_{n_0}(t),\\ g_{n_0+1}(t, y_{n_0}(t)) + r(y_{n_0}(t) - y), & y \ge y_{n_0}(t), \end{cases}$$

and

$$\psi^*_{n_0+1}(z) = \begin{cases} \psi(y_{n_0}(1)), & z > y_{n_0}(1),\\ \psi(z), & \alpha_{n_0+1}(1) = \alpha(1) \le z \le y_{n_0}(1),\\ \psi(\alpha(1)), & z < \alpha_{n_0+1}(1) = \alpha(1). \end{cases}$$

Now Schauder's fixed point theorem guarantees that (18) has a solution $y_{n_0+1} \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$. We first show

$$y_{n_0+1}(t) \ge \alpha_{n_0+1}(t), \quad t \in [0, 1]. \tag{19}$$

Suppose (19) is not true. Then $y_{n_0+1}-\alpha_{n_0+1}$ has a negative absolute minimum at $\tau\in(0,1]$. There exists $\tau_0\in[0,\tau)$ with

$$y_{n_0+1}(\tau_0)-\alpha_{n_0+1}(\tau_0)=0 \quad\text{and}\quad y_{n_0+1}(t)-\alpha_{n_0+1}(t)<0,\quad t\in(\tau_0,\tau).$$

Now either

$$y_{n_0+1}(t)-\alpha_{n_0+1}(t)<0,\quad t\in(\tau_0,1] \tag{20}$$

or

$$\begin{cases}\exists\tau_1>\tau \text{ with } y_{n_0+1}(t)-\alpha_{n_0+1}(t)<0 \quad \text{for } t\in(\tau_0,\tau_1)\\ \text{and } y_{n_0+1}(\tau_1)-\alpha_{n_0+1}(\tau_1)=0.\end{cases} \tag{21}$$

Case 1. Suppose (21) occurs. If we show

$$(y_{n_0+1}-\alpha_{n_0+1})''(t)<0 \quad \text{for a.e. } t\in(\tau_0,\tau_1),$$

then as before we obtain a contradiction. Fix $t\in(\tau_0,\tau_1)$ and assume $t\neq t_{n_0+1}$. Then

$$\begin{aligned}&(y_{n_0+1}-\alpha_{n_0+1})''(t)\\ &\quad=\begin{cases}-[q(t)\{g_{n_0+1}(t,\alpha(t))+r(\alpha(t)-y_{n_0+1}(t))\}+\alpha''(t)] & \text{if } t\in(t_{n_0+1},1),\\ -[q(t)\{g_{n_0+1}(t,\rho_{n_0+1})+r(\rho_{n_0+1}-y_{n_0+1}(t))\}] & \text{if } t\in(0,t_{n_0+1}).\end{cases}\end{aligned}$$

(a) $t\geq 1/2^{n_0+2}$. Then since $g_{n_0+1}(t,x)=f(t,x)$ for $x\in(0,\infty)$, from (5) and (6), we have

$$\begin{aligned}&(y_{n_0+1}-\alpha_{n_0+1})''(t)\\ &\quad=\begin{cases}-[q(t)\{f(t,\alpha(t))+r(\alpha(t)-y_{n_0+1}(t))\}+\alpha''(t)] & \text{if } t\in(t_{n_0+1},1),\\ -[q(t)\{f(t,\rho_{n_0+1})+r(\rho_{n_0+1}-y_{n_0+1}(t))\}] & \\ & \text{if } t\in(0,t_{n_0+1}),\end{cases}\\ &\quad<0.\end{aligned}$$

(b) $t\in(0,1/2^{n_0+2})$. Then since $g_{n_0+1}(t,x)$ equals

$$\min\left\{\max\left\{f\left(\frac{1}{2^{n_0+1}},x\right),f(t,x)\right\},\max\left\{f\left(\frac{1}{2^{n_0+2}},x\right),f(t,x)\right\}\right\}$$

we have $g_{n_0+1}(t,x)\geq f(t,x)$ and

$$g_{n_0+1}(t,x)\geq\min\left\{f\left(\frac{1}{2^{n_0+1}},x\right),f\left(\frac{1}{2^{n_0+2}},x\right)\right\}$$

for $x \in (0,\infty)$. Thus we have

$$
\begin{aligned}
&(y_{n_0+1}-\alpha_{n_0+1})''(t)\\
&\le \begin{cases} -[q(t)\{f(t,\alpha(t))+r(\alpha(t)-y_{n_0+1}(t))\}+\alpha''(t)] & \text{if } t\in(t_{n_0+1},1),\\ -[q(t)\{\min\{f(1/2^{n_0+1},\rho_{n_0+1}), f(1/2^{n_0+2},\rho_{n_0+1})\} & \\ \quad +r(\rho_{n_0+1}-y_{n_0+1}(t))\}] & \text{if } t\in(0,t_{n_0+1}),\end{cases}\\
&< 0,
\end{aligned}
$$

from (5) and (6) (note $f(1/2^{n_0+1},\rho_{n_0+1}) \ge 0$ since $f(t,\rho_{n_0+1}) \ge 0$ for $t \in [1/2^{n_0+2},1]$ and $1/2^{n_0+1} \in (1/2^{n_0+2},1)$).

Now subcase (a) and (b) guarantee that we have a contradiction.

Case 2. Suppose (20) occurs. Then

$$y_{n_0+1}(\tau_0)-\alpha_{n_0+1}(\tau_0)=0 \quad \text{and} \quad y_{n_0+1}(t)-\alpha_{n_0+1}(t)<0, \quad t\in(\tau_0,1].$$

Essentially the same reasoning as in Case 1 guarantees that

$$(y_{n_0+1}-\alpha_{n_0+1})''(t)<0 \quad \text{for } t\in(\tau_0,\tau_1) \text{ provided } t\ne t_{n_0+1}.$$

As a result $\alpha_{n_0+1}-y_{n_0+1}$ is convex on $(\tau_0,1)$, so $\alpha'_{n_0+1}(1)-y'_{n_0+1}(1) \ge \alpha_{n_0+1}(1)-y_{n_0+1}(1)$. Thus

$$
\begin{aligned}
0 &< \alpha_{n_0+1}(1)-y_{n_0+1}(1) \le \alpha'_{n_0+1}(1)-y'_{n_0+1}(1)\\
&= \alpha'(1)+\psi^*_{n_0+1}(y_{n_0+1}(1)) \le -\psi(\alpha(1))+\psi(\alpha(1))=0,
\end{aligned}
$$

a contradiction.

So in both Cases 1 and 2 we have a contradiction. Thus (19) holds, so

$$\alpha(t)\le\alpha_{n_0+1}(t)\le y_{n_0+1}(t) \quad \text{for } t\in[0,1]. \tag{22}$$

Next we show

$$y_{n_0+1}(t)\le y_{n_0}(t) \quad \text{for } t\in[0,1]. \tag{23}$$

If (23) is not true then $y_{n_0+1}-y_{n_0}$ would have a positive absolute maximum at say $\tau_0\in(0,1]$. Suppose first $\tau_0\in(0,1)$, so

$$(y_{n_0+1}-y_{n_0})'(\tau_0)=0 \quad \text{and} \quad (y_{n_0+1}-y_{n_0})''(\tau_0)\le 0.$$

Then $y_{n_0+1}(\tau_0)>y_{n_0}(\tau_0)$ together with $g_{n_0}(\tau_0,x)\ge g_{n_0+1}(\tau_0,x)$ for $x\in(0,\infty)$ gives

$$
\begin{aligned}
&(y_{n_0+1}-y_{n_0})''(\tau_0)\\
&\quad = -q(\tau_0)[g_{n_0+1}(\tau_0,y_{n_0}(\tau_0))+r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))]-y''_{n_0}(\tau_0)\\
&\quad \ge -q(\tau_0)[g_{n_0}(\tau_0,y_{n_0}(\tau_0))+r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))]-y''_{n_0}(\tau_0)\\
&\quad = -q(\tau_0)[r(y_{n_0}(\tau_0)-y_{n_0+1}(\tau_0))]>0,
\end{aligned}
$$

a contradiction. It remains to discuss the case $\tau_0 = 1$. If $\tau_0 = 1$ then there exists δ, $0 \le \delta < 1$ with $y_{n_0+1}(t) - y_{n_0}(t) > 0$ for $t \in (\delta, 1]$ and $y_{n_0+1}(\delta) - y_{n_0}(\delta) = 0$. Now for $t \in (\delta, 1)$ we have

$$\begin{aligned}(y_{n_0+1} - y_{n_0})''(t) &= -\, q(t)[g_{n_0+1}(t, y_{n_0}(t)) + r(y_{n_0}(t) - y_{n_0+1}(t))] - y''_{n_0}(t)\\ &\ge -\, q(t)[g_{n_0}(t, y_{n_0}(t)) + r(y_{n_0}(t) - y_{n_0+1}(t))] - y''_{n_0}(t)\\ &= -\, q(t)[r(y_{n_0}(t) - y_{n_0+1}(t))] > 0.\end{aligned}$$

Thus $y_{n_0+1} - y_{n_0}$ is convex on $(\delta, 1)$. As a result $y'_{n_0+1}(1) - y'_{n_0}(1) \ge y_{n_0+1}(1) - y_{n_0}(1)$ and so

$$\begin{aligned}0 &< y_{n_0+1}(1) - y_{n_0}(1) \le y'_{n_0+1}(1) - y'_{n_0}(1)\\ &= -\psi^*_{n_0+1}(y_{n_0+1}(1)) + \psi(y_{n_0}(1)) = -\psi(y_{n_0}(1)) + \psi(y_{n_0}(1)) = 0,\end{aligned}$$

a contradiction. Thus (23) holds.

Now proceed inductively to construct $y_{n_0+2}, y_{n_0+3}, \dots$. as follows. Suppose we have y_k for some $k \in \{n_0 + 1, n_0 + 2, \dots\}$ with $\alpha_k(t) \le y_k(t) \le y_{k-1}(t)$ for $t \in [0, 1]$. Then consider the boundary value problem

$$\text{(24)} \qquad \begin{cases} y'' + q(t) g^*_{k+1}(t, y) = 0, & 0 < t < 1,\\ y(0) = \rho_{k+1},\ y'(1) + \psi^*_{k+1}(y(1)) = 0; \end{cases}$$

here

$$g^*_{k+1}(t, y) = \begin{cases} g_{k+1}(t, \alpha_{k+1}(t)) + r(\alpha_{k+1}(t) - y) & y \le \alpha_{k+1}(t),\\ g_{k+1}(t, y) & \alpha_{k+1}(t) \le y \le y_k(t),\\ g_{k+1}(t, y_k(t)) + r(y_k(t) - y) & y \ge y_k(t), \end{cases}$$

and

$$\psi^*_{k+1}(z) = \begin{cases} \psi(y_k(1)) & z > y_k(1),\\ \psi(z) & \alpha_{k+1}(1) = \alpha(1) \le z \le y_k(1),\\ \psi(\alpha(1)) & z < \alpha_{k+1}(1) = \alpha(1). \end{cases}$$

Now Schauder's fixed point theorem guarantees that (24) has a solution $y_{k+1} \in C[0, 1] \cap C^1(0, 1] \cap C^2(0, 1)$, and essentially the same reasoning as above yields

$$\text{(25)} \qquad \alpha(t) \le \alpha_{k+1}(t) \le y_{k+1}(t) \le y_k(t) \quad \text{for } t \in [0, 1].$$

Thus for each $n \in \{n_0, n_0 + 1, \dots .\}$ we have

$$\text{(26)} \qquad \alpha(t) \le y_n(t) \le y_{n-1}(t) \le \dots \le y_{n_0}(t) \le \beta(t) \quad \text{for } t \in [0, 1].$$

Lets look at the interval $[1/2^{n_0+1}, 1]$. Let

$$R_{n_0} = \sup\{|q(x) f(x, y)| : x \in [1/2^{n_0+1}, 1] \text{ and } \alpha(x) \le y \le y_{n_0}(x)\}.$$

Now, since $y'_n(1) = -\psi^*_n(y_n(1))$, we have

$$|y'_n(1)| \le \sup_{z \in [\alpha(1), \beta(1)]} |\psi(z)| \equiv K_0$$

and so

$$|y_n'(t)| \le K_0 + R_{n_0} \int_{1/2^{n_0+1}}^1 q(x)\,dx \quad \text{for } t \in [1/2^{n_0+1}, 1].$$

As a result

(27) $\{y_n\}_{n=n_0+1}^{\infty}$ is a bounded, equicontinuous family on $[1/2^{n_0+1}, 1]$.

The Arzela–Ascoli theorem guarantees the existence of a subsequence N_{n_0} of integers and a function $z_{n_0} \in C[1/2^{n_0+1}, 1]$ with y_n converging uniformly to z_{n_0} on $[1/2^{n_0+1}, 1]$ as $n \to \infty$ through N_{n_0}. Similarly

$$\{y_n\}_{n=n_0+1}^{\infty} \text{ is a bounded, equicontinuous family on } [1/2^{n_0+2}, 1],$$

so there is a subsequence N_{n_0+1} of N_{n_0} and a function $z_{n_0+1} \in C[1/2^{n_0+2}, 1]$ with y_n converging uniformly to z_{n_0+1} on $[1/2^{n_0+2}, 1]$ as $n \to \infty$ through N_{n_0+1}. Note $z_{n_0+1} = z_{n_0}$ on $[1/2^{n_0+1}, 1]$ since $N_{n_0+1} \subseteq N_{n_0}$. Proceed inductively to obtain subsequences of integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \ldots \supseteq N_k \supseteq \ldots$$

and functions $z_k \in C[1/2^{k+1}, 1]$ with y_n converging uniformly to z_k on $[1/2^{k+1}, 1]$ as $n \to \infty$ through N_k, and $z_k = z_{k-1}$ on $[1/2^k, 1]$.

Define a function $y: [0,1] \to [0,\infty)$ by $y(x) = z_k(x)$ on $[1/2^{k+1}, 1]$ and $y(0) = 0$. Notice y is well defined and $\alpha(t) \le y(t) \le y_{n_0}(t) (\le \beta(t))$ for $t \in (0,1)$. Next fix $t \in (0,1)$ and let $m \in \{n_0, n_0+1, \ldots\}$ be such that $1/2^{m+1} < t < 1$. Let $N_m^* = \{n \in N_m : n \ge m\}$. Now $y_n, n \in N_m^*$, satisfies the integral equation

$$y_n(t) = y_n(1) + \psi(y_n(1))(1-t) - \int_t^1 (x-t) q(x) f(x, y_n(x))\,dx.$$

Let $n \to \infty$ through N_m^* to obtain

$$z_m(t) = z_m(1) + \psi(z_m(1))(1-t) - \int_t^1 (x-t) q(x) f(x, z_m(x))\,dx.$$

That is

$$y(t) = y(1) + \psi(y(1))(1-t) - \int_t^1 (x-t) q(x) f(x, y(x))\,dx.$$

We can do this argument for each $t \in (0,1)$, so $y''(t) + q(t) f(t, y(t)) = 0$ for $t \in (0,1)$ and $y'(1) = -\psi(y(1))$. It remains to show y is continuous at 0. Let $\epsilon > 0$ be given. Now since $\lim_{n\to\infty} y_n(0) = 0$ there exists $n_1 \in \{n_0, n_0+1, \ldots\}$ with $y_{n_1}(0) < \varepsilon/2$. Since $y_{n_1} \in C[0,1]$ there exists $\delta_{n_1} > 0$ with

$$y_{n_1}(t) < \varepsilon/2 \quad \text{for } t \in [0, \delta_{n_1}].$$

Now for $n \ge n_1$ we have, since $\{y_n(t)\}$ is nonincreasing for each $t \in [0,1]$,

$$\alpha(t) \le y_n(t) \le y_{n_1}(t) < \varepsilon/2 \quad \text{for } t \in [0, \delta_{n_1}].$$

Consequently

$$\alpha(t) \le y(t) \le \varepsilon/2 < \varepsilon \quad \text{for } t \in (0, \delta_{n_1}]$$

and so y is continuous at 0. □

Suppose (2)–(6) hold and in addition asume the following conditions are satisfied:

$$(28) \qquad q(t)f(t,y) + \alpha''(t) > 0 \quad \text{for } (t,y) \in (0,1) \times \{y \in (0,\infty) : y < \alpha(t)\}$$

and

$$(29) \qquad \begin{cases} \text{there exists a function } \beta \in C[0,1] \cap C^1(0,1] \cap C^2(0,1) \\ \text{with } \beta(t) \ge \rho_{n_0} \text{ for } t \in [0,1], \beta'(1) + \psi(\beta(1)) \ge 0 \\ \text{with } q(t)f(t,\beta(t)) + \beta''(t) \le 0 \text{ for } t \in (0,1) \text{ and} \\ q(t)f(1/2^{n_0+1}, \beta(t)) + \beta''(t) \le 0 \text{ for } t \in (0, 1/2^{n_0+1}). \end{cases}$$

Also assume *either*

$$(30a) \qquad \beta(t) \ge \alpha(1)$$

or

$$(30b) \qquad \text{if } x > 0,\ y > 0 \text{ with } x - y \le \psi(y) - \psi(x), \text{ then } x - y \le 0$$

occur. Then the result in Theorem 2.1 is again true. This follows immediately from Theorem 2.1 once we show (7) holds i.e. once we show $\beta(t) \ge \alpha(t)$ for $t \in [0,1]$. Suppose it is false. Then $\alpha - \beta$ would have a positive absolute maximum at say $\tau_0 \in (0,1)$ (if (30a) occurs) or $\tau_0 \in (0,1]$ (if (30b) occurs). Suppose $\tau_0 \in (0,1)$ for the moment, so $(\alpha-\beta)'(\tau_0) = 0$ and $(\alpha-\beta)''(\tau_0) \le 0$. Now $\alpha(\tau_0) > \beta(\tau_0)$ and (28) implies

$$q(\tau_0)f(\tau_0, \beta(\tau_0)) + \alpha''(\tau_0) > 0.$$

This together with (29) yields

$$(\alpha-\beta)''(\tau_0) = \alpha''(\tau_0) - \beta''(\tau_0) \ge \alpha''(\tau_0) + q(\tau_0)f(\tau_0, \beta(\tau_0)) > 0,$$

a contradiction. It remains to discuss $\tau_0 = 1$ (only if (30b) occurs). In this case there exists δ, $0 \le \delta < 1$, with $\alpha(t) - \beta(t) > 0$ for $t \in (\delta, 1]$ and $\alpha(\delta) - \beta(\delta) = 0$. In addition for $t \in (\delta, 1)$ we have (as above), $(\alpha-\beta)''(t) \ge 0$ so $\alpha - \beta$ is convex on $(\delta, 1)$. As a result $\alpha'(1) - \beta'(1) \ge \alpha(1) - \beta(1)$ so

$$0 < \alpha(1) - \beta(1) \le \alpha'(1) - \beta'(1) \le -\psi(\alpha(1)) + \psi(\beta(1)).$$

From (30b) we have $\alpha(1) - \beta(1) \le 0$, a contradiction.

COROLLARY 2.2. *Let* $n_0 \in \{1, 2, \dots\}$ *be fixed and suppose* (2)–(6), (28) *and* (29) *hold. Also assume either* (30a) *or* (30b) *occurs. Then* (1) *has a solution* $y \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ *with* $y(t) \geq \alpha(t)$ *for* $t \in [0,1]$.

REMARK 2.1. Assumption (30b) will play a crucial role when we discuss the membrane response of a spherical cap. For example if $\psi(x) = -a_0 x$ for $a_0 < 1$, then clearly (30b) holds.

REMARK 2.2. In (5) one could replace $1/2^{n+1} \leq t \leq 1$ with

(a) $0 \leq t \leq 1 - 1/2^{n+1}$,
(b) $1/2^{n+1} \leq t \leq 1 - 1/2^{n+1}$, or
(c) $0 \leq t \leq 1$,

provided (7) is appropriately adjusted. For example if case (b) occurs then (7) is replaced by

$$\begin{cases} \text{there exists a function } \beta \in C[0,1] \cap C^1(0,1] \cap C^2(0,1) \\ \text{with } \beta(t) \geq \alpha(t) \text{ and } \beta(t) \geq \rho_{n_0} \text{ for } t \in [0,1], \\ \beta'(1) + \psi(\beta(1)) \geq 0 \text{ with } q(t)f(t,\beta(t)) + \beta''(t) \leq 0 \text{ for } t \in (0,1) \\ \text{and } q(t)f(1/2^{n_0+1},\beta(t)) + \beta''(t) \leq 0 \text{ for } t \in (0, 1/2^{n_0+1}) \\ \text{and } q(t)f(1 - 1/2^{n_0+1},\beta(t)) + \beta''(t) \leq 0 \text{ for } t \in (1 - 1/2^{n_0+1}, 1). \end{cases}$$

Next we discuss how to construct the lower solution α in (6) and (28). Suppose the following conditions are satisfied:

$$\text{(31)} \qquad \begin{cases} \text{let } n \in \{n_0, n_0+1, \dots\} \text{ and associated with each } n \\ \text{we have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a decreasing sequence} \\ \text{with } \lim_{n\to\infty} \rho_n = 0 \text{ and there exists aconstant } k_0 > 0 \\ \text{such that for } 1/2^{n+1} \leq t \leq 1 \text{ and } 0 < y \leq \rho_n \\ \text{we have } q(t)f(t,y) \geq k_0, \end{cases}$$

and

$$\text{(32)} \qquad \begin{cases} \psi(u) = -a_0 u, 0 \leq a_0 < 1, \text{ and there exists } \tau \in (0,1) \\ \text{with } f(t,y) > 0 \text{ for } t \in [\tau, 1) \text{ and } 0 < y \leq \rho_{n_0}/1 - a_0(1-\tau). \end{cases}$$

Then the result in O'Regan ([7]) guarantees that there exists a α satisfying (6) and (28) with $\alpha(t) \leq \rho_{n_0}$ for $t \in [0,1]$.

Combining the above with Corollary 2.2 gives the following existence result.

COROLLARY 2.3. *Let* $n_0 \in \{1, 2, \dots\}$ *be fixed and suppose* (2)–(4), (29), (31) *and* (32) *hold. Then* (1) *has a solution* $y \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ *with* $y(t) > 0$ *for* $t \in (0,1]$.

Next we present an example which illustrates how easily the theory can be applied in practice.

Example (Membrane response of a spherical cap). The boundary value problem

$$\text{(33)}\qquad \begin{cases} y'' + \left(\dfrac{t^2}{32y^2} - \dfrac{\lambda^2}{8}\right) = 0 & \text{for } 0 < t < 1, \\ y(0) = 0,\ y'(1) - \dfrac{(1+v)}{2} y(1) = 0 & \text{for } 0 < v < 1,\ \lambda > 0, \end{cases}$$

has a solution $y \in C[0,1] \cap C^1(0,1] \cap C^2(0,1)$ with $y(t) > 0$ for $t \in (0,1]$.

To see this we will apply Corollary 2.3 with

$$q \equiv 1, \quad \psi(z) = \frac{-(1+v)z}{2} \quad \text{and} \quad a_0 = \frac{(1+v)}{2}.$$

Choose and fix $n_0 \in \{1,2,\dots\}$ with

$$\text{(34)}\qquad \frac{1}{2^{n_0/2}} \le \frac{(1-v)(8+\lambda^2)^{1/2}}{2\lambda(1+v)} \quad \text{and} \quad \frac{1}{2^{n_0+2}} \le \frac{4\lambda^2}{8+\lambda^2}.$$

Let

$$\rho_n = \frac{1}{2^{n+2}(8+\lambda^2)^{1/2}} \quad \text{and} \quad k_0 = 1.$$

Clearly (2), (3) and (4) hold. Also notice for $n \in \{1,2,\dots\}$, $1/2^{n+1} \le t \le 1$ and $0 < y \le \rho_n$ that we have

$$q(t)f(t,y) \ge \frac{1}{(32)2^{2n+2}y^2} - \frac{\lambda^2}{8} \ge \frac{1}{(32)2^{2n+2}\rho_n^2} - \frac{\lambda^2}{8} = \frac{(8+\lambda^2)}{8} - \frac{\lambda^2}{8} = 1,$$

so (31) is satisfied. Now let

$$\beta(t) = \frac{t}{2\lambda} + 2^{(n_0+4)/2}\rho_{n_0} = \frac{t}{2\lambda} + \frac{1}{2^{n_0/2}(8+\lambda^2)^{1/2}}.$$

Clearly $\beta(t) \ge \rho_{n_0}$, for $t \in [0,1]$, and

$$\begin{aligned} \beta'(1) + \psi(\beta(1)) &= \frac{1}{2\lambda} - \frac{(1+v)}{2}\left(\frac{1}{2\lambda} + \frac{1}{2^{n_0/2}(8+\lambda^2)^{1/2}}\right) \\ &= \frac{(1-v)}{4\lambda} - \frac{(1+v)}{2}\frac{1}{2^{n_0/2}(8+\lambda^2)^{1/2}} \ge 0 \end{aligned}$$

from (34). Also for $t \in (0,1)$ we have

$$\begin{aligned} \beta''(t) + q(t)f(t,\beta(t)) &= \frac{t^2}{32\left(\dfrac{t}{2\lambda} + \dfrac{1}{2^{n_0/2}(8+\lambda^2)^{1/2}}\right)^2} - \frac{\lambda^2}{8} \\ &\le \frac{t^2(4\lambda^2)}{32t^2} - \frac{\lambda^2}{8} = 0, \end{aligned}$$

whereas, for $t \in (0, 1/2^{n_0+1})$, we have

$$\begin{aligned}\beta''(t) + q(t)f\left(\frac{1}{2^{n_0+1}}, \beta(t)\right) &= \frac{t^2}{(32)2^{2n_0+2}\left(\dfrac{t}{2\lambda} + \dfrac{1}{2^{n_0/2}(8+\lambda^2)^{1/2}}\right)^2} - \frac{\lambda^2}{8} \\ &\leq \frac{(8+\lambda^2)}{(32)2^{n_0+2}} - \frac{\lambda^2}{8} \leq 0\end{aligned}$$

from (34). Thus (29) holds. It remains to check (32). Let $\tau = 1/(1-a_0)$. Now if $t \in [\tau, 1)$ and $0 < y \leq \rho_{n_0}/1 - a_0(1-\tau)$ then we have

$$\begin{aligned}f(t,y) &\geq \frac{\tau^2}{32y^2} - \frac{\lambda^2}{8} \geq \frac{\tau^2}{32}\frac{[1-a_0(1-\tau)]^2}{\rho_{n_0}^2} - \frac{\lambda^2}{8} \\ &\geq \frac{\tau^2(1-a_0)^2}{32\rho_{n_0}^2} - \frac{\lambda^2}{8} = \frac{\tau^2(1-a_0)^2(8+\lambda^2)2^{2n_0+4}}{32} - \frac{\lambda^2}{8} \\ &\geq \tau^2(1-a_0)^2\left(1+\frac{\lambda^2}{8}\right) - \frac{\lambda^2}{8} = \left(1+\frac{\lambda^2}{8}\right) - \frac{\lambda^2}{8} = 1.\end{aligned}$$

Thus (29) holds. Existence of a solution to (33) is now guaranteed from Corollary 2.3.

Many other types of boundary data and other types of singular problems could be discussed using the ideas in this paper. To illustrate this we consider the boundary value problem

$$\begin{cases} (1/p)(py')' + q(t)f(t,y) = 0 & 0 < t < 1, \\ \lim_{t\to 0^+} p(t)y'(t) = y(1) = 0. \end{cases} \tag{35}$$

A slight modification of the arguments in Theorem 2.1 together with the ideas in Theorem 3.1 of [1] yields the following result. The details are left to the reader.

Theorem 2.4. *Let $n_0 \in \{1, 2, \dots\}$ be fixed and suppose the following conditions are satisfied:*

$$f\colon [0,1] \times (0,\infty) \to \mathbb{R} \text{ is continuous}, \tag{36}$$

$$p \in C[0,1] \cap C^1(0,1) \quad \text{with } p > 0 \text{ on } (0,1), \tag{37}$$

$$q \in C(0,1) \quad \text{with } q > 0 \text{ on } (0,1), \tag{38}$$

$$\int_0^1 p(s)q(s)\,ds < \infty \quad \text{and} \quad \int_0^1 \frac{1}{p(t)}\int_0^t p(s)q(s)\,ds < \infty, \tag{39}$$

$$\begin{cases} \text{let } n \in \{n_0, n_0+1, \dots\} \text{ and associated with each } n \\ \text{we have a constant } \rho_n \text{ such that } \{\rho_n\} \text{ is a nonincreasing} \\ \text{sequence with } \lim_{n\to\infty} \rho_n = 0 \text{ and such that} \\ \text{for } 1/2^{n+1} \leq t \leq 1 \text{ we have } p(t)q(t)f(t,\rho_n) \geq 0, \end{cases} \tag{40}$$

$$(41)\qquad \begin{cases} \exists \ a \ function \ \alpha \in C[0,1] \cap C^2(0,1) \ with \ p\alpha' \in AC[0,1], \\ \lim_{t \to 0^+} p(t)\alpha'(t) \geq 0, \alpha(1) = 0, \alpha > 0 \ on \ [0,1) \\ such \ that \ p(t)q(t)f(t,\alpha(t)) + (p(t)\alpha'(t))' \geq 0 \ for \ t \in (0,1), \end{cases}$$

and

$$(42)\qquad \begin{cases} there \ exists \ a \ function \ \beta \in C[0,1] \cap C^2(0,1), \ \ p\beta' \in AC[0,1], \\ with \ \beta(t) \geq \alpha(t) \ and \ \beta(t) \geq \rho_{n_0} \ for \ t \in [0,1], \\ \lim_{t \to 0^+} p(t)\beta'(t) \leq 0 \ with \ p(t)q(t)f(t,\beta(t)) + (p(t)\beta'(t))' \leq 0 \\ for \ t \in (0,1) \ and \ p(t)q(t)f(1/2^{n_0+1},\beta(t)) + (p(t)\beta'(t))' \leq 0 \\ for \ t \in (0,1/2^{n_0+1}). \end{cases}$$

Then (35) *has a solution* $y \in C[0,1] \cap C^2(0,1)$ *with* $py' \in AC[0,1]$ *and* $y(t) \geq \alpha(t)$ *for* $t \in [0,1]$.

References

[1] R. P. Agarwal, D. O'Regan and V. Lakshmikantham, *Existence criteria for singular boundary data modelling the membrane response of a spherical cap*, Nonlinear Anal. (to appear).

[2] L. E. Bobisud, J. E. Calvert and W. D. Royalty, *Some existence results for singular boundary value problems*, Differential Integral Equations **6** (1993), 553–571.

[3] R. W. Dickey, *Rotationally symmetric solutions for shallow membrane caps*, Quart. Appl. Math. **47** (1989), 571–581.

[4] A. Granas, R. B. Guenther and J. W. Lee, *Nonlinear boundary value problems for ordinary differential equations*, Dissertationes Math. **244** (1985).

[5] P. Habets and F. Zanolin, *Upper and lower solutions for a generalized Emden–Fowler equation*, J. Math. Anal. Appl. **181** (1994), 684–700.

[6] T. Y. Na, *Computational Methods in Engineering Boundary Value Problems*, Academic Press New York, 1979.

[7] D. O'Regan, *Boundary value problems singular in the solution variable with nonlinear boundary data*, Proc. Edinburgh Math. Soc. **39** (1996), 505–523.

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