THE PASCAL THEOREM AND SOME ITS GENERALIZATIONS

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Dedicated to Professor Andrzej Granas

Abstract. We present two generalizations of the famous Pascal theorem to the case of algebraic curves of degree 3.

1. Introduction

The 350 years old theorem of B. Pascal [9] says, that if a hexagon is inscribed in a conic, then the opposite sides of the hexagon meet in three collinear points. The dual version of this result is called the Brianchon theorem and says that if a conic is inscribed in a hexagon, then the diagonals of the hexagon intersect at one point. These theorems remain true in some degenerate cases, e.g. when the hexagon degenerates to a pentagon. There exist essentially two proofs of the Pascal theorem, one uses projective geometry methods and the cross-ratio invariant (see Section 2), while the other one relies on the Cayley–Bacharach theorem (see Section 3). It seems that such a beautiful results should have generalizations. For example, the projective proof of the Pascal theorem uses the fact that a conic is a (projective) rational curve. There exist rational curves of higher degrees, e.g. a cubic with one point of self-intersection. There are, however, only few works in this direction. Probably the most interesting is the

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paper [4] by D. Eisenbud, M. Green and J. Harris devoted to generalization of the Cayley–Bacharach theorem to higher dimensions.

In this paper we prove two generalizations of the Pascal theorem (Theorems 4.4 and 5.1 below) which are in the same style as Pascal’s result, i.e. that some points, obtained as results of intersections of algebraic curves lie on a straight line. Theorem 4.4 deals with a general cubic intersected by three lines in nine points. One constructs a conic through five of them and two lines through the remaining four. One obtains three additional points that turn out to lie on straight line. The proof of this result is a standard application of the Cayley–Bacharach theorem. Theorem 5.1 is more subtle. It deals with a rational cubic (i.e. a cubic with a double point) with 8 generic points. One constructs two pairs of conics, each of them through four of these points and the double point. The two quartics defined in this way (each is a sum of two conics) define 4 additional points in their intersection. It turns out that these 4 points lie on a straight line. The proof is analytic and uses the notion of multi-dimensional residuum, applied in a non-trivial case. (In fact, we did not expected such result; it has surprised us a little). We prove also a generalization of the Brianchon theorem (Theorem 5.3), the dual version of Theorem 5.1. It is restricted to simply connected cubics, with cusp singularity and one inflection point. For a configuration of 8 lines tangent to such cubic one constructs two pairs of conics, each tangent to 4 of the lines and to the line tangent at the inflection point. One obtains 4 additional lines tangent to the both corresponding quartics. These lines turn out intersect at one point.

Now we know, how to generalize Theorem 4.4 to curves of higher degrees. Probably there exists also a generalization of Theorem 5.1 to rational curves of higher degrees. It seems that, using Theorem 5.1 or some kind of its inverse, one could provide a geometrical construction of 12 different rational cubics through 8 points in $\mathbb{C}P^2$ in general position (see [7]). The subject seems to be highly interesting. We intend to continue investigations in future papers.

The plan of the article is following: in Section 2 we present the classical Pascal’s proof of Pascal theorem. In Section 3 we introduce the notion of multi-dimensional residuum and prove the Cayley–Bacharach theorem. In Section 4 we present the analytic proof of Pascal theorem, of its inverse and of Theorem 4.4. Section 5 contains the proof of Theorems 5.1 and 5.3.

2. The cross-ratio and conics

The cross-ratio of a quadruple of different points $a_1, \ldots, a_4 \in \mathbb{C}P^1$ is defined as

$$(2.1) \quad cr(a_1, \ldots, a_4) = \frac{(a_1 - a_2)(a_3 - a_4)}{(a_1 - a_4)(a_2 - a_3)}.$$
if all points lie on the affine part $\mathbb{C} = \mathbb{C}P^1 \setminus \{\infty\}$, and

$$\text{cr}(a_1, a_2, a_3, \infty) = \frac{a_1 - a_2}{a_2 - a_3}. \tag{2.2}$$

It is, of course, the limit of (2.1).

**Lemma 2.1.** The cross-ratio is invariant under the action of $\text{PSL}(2, \mathbb{C})$ — the group of automorphisms of $\mathbb{C}P^1$.

**Proof.** If $\sigma : z \to (\alpha z + \beta)/(\gamma z + \delta)$, $\alpha \delta - \beta \gamma = 1$ is an automorphism of $\mathbb{C}P^1$, then

$$\sigma(a) - \sigma(b) = \frac{a - b}{(\gamma a + \delta)(\gamma b + \delta)}. \tag{2.3}$$

The following proposition will be used in the geometrical proof of the Pascal theorem.

Let $A$ and $B$ be two different projective lines on a projective plane $\mathbb{C}P^2$, let us denote by $o$ their unique intersection point. Choose points $a_1, a_2, a_3$ on $A$, and $b_1, b_2, b_3$ on $B$. Define the lines $A_1, A_2$ and $A_3$, where $A_j$ passes through $a_j$ and $b_j$ (see Figure 1).

![Figure 1](image)

**Proposition 2.2.** The lines $A_1$, $A_2$ and $A_3$ intersect at one point if and only if the cross-ratios $\text{cr}(a_1, a_2, a_3, o)$ and $\text{cr}(b_1, b_2, b_3, o)$ are equal.

**Proof.** We choose a projective chart such that the line passing through $o$ and $A_1 \cap A_2$ is the line at infinity. The the affine lines $A^0 = A \cap \mathbb{C}^2$ and $B^0 = B \cap \mathbb{C}^2$ are parallel, similarly parallel are the lines $A_1^0$ and $A_2^0$. The property that the projective lines $A_1, A_2$ and $A_3$ intersect at one point is equivalent to
the property that the affine lines $A^0_0, A^0_2, A^0_3$ are parallel. This is true iff the following condition is fulfilled:

$$\frac{a_1 - a_2}{a_2 - a_3} = \frac{b_1 - b_2}{b_2 - b_3},$$

which exactly means that the cross-ratios $cr(a_1, a_2, a_3, o)$ and $cr(b_1, b_2, b_3, o)$ are equal. (Here $o = \infty$, see (2.2)).

The geometrical proof of the Pascal theorem uses also the following result about 4 points in a projective conic.

Let $C \subset \mathbb{C}P^2$ be a smooth conic, i.e. an algebraic curve of degree two, which is not a sum of two lines. Any point $m \in C$ defines a pencil $m^*$ of projective lines through $m$. The pencil $m^* \simeq \mathbb{C}P^1$ is a projective line in the dual projective space $(\mathbb{C}P^2)^* \simeq \mathbb{C}P^2$.

We are given a map

$$\pi_m: C \rightarrow m^*,$$

which associates with a point $c \in C$ the line $\pi_m(c)$ passing through $m$ and $c$. $\pi_m(m)$ is the line tangent to $C$ at $m$. The map (2.4) defines a biholomorphism between the conic $C$ and $\mathbb{C}P^1 \equiv m^*$. Thus, any smooth conic is a rational curve.

Given a map $\pi_m$ we are able to define a cross-ratio of a quadruple of points $a_1, a_2, a_3, a_4$ on $C$. We define it to be the cross-ratio of the points $\pi_m(a_1), \ldots, \pi_m(a_4)$ in $m^*$. In fact,

$$cr(a_1, \ldots, a_4) = cr(b_1, \ldots, b_4),$$

where $b_j = \pi_m(c_j) \cap L$, for some fixed line $L$ in $\mathbb{C}P^2$.

**Proposition 2.3.** The number $cr(a_1, a_2, a_3, a_4)$ is well defined. It does not depend neither on $m \in C$ nor on the line $L$.

**Proof.** A change of the point $m$ or the line $L$ results as an automorphism of $\mathbb{C}P^1$. The thesis follows from Lemma 2.1. $\square$

Let us recall Pascal’s result. Let $C \subset \mathbb{C}P^2$ be a smooth conic. Let $a_1, \ldots, a_6$ be different points on $C$. We define the lines $A_1 = a_1a_2$, i.e. the line that passes through $a_1$ and $a_2$, $A_2 = a_3a_4$, $A_3 = a_5a_6$, $B_1 = a_2a_5$, $B_2 = a_3a_6$, $B_3 = a_4a_1$. The curves $A = A_1 + A_2 + A_3 \equiv A_1 \cup A_2 \cup A_3$, and $B = B_1 + B_2 + B_3$ intersect at all $6$ points $a_1, \ldots, a_6$, and, besides, at the points $d_1 = A_1 \cap B_2$, $d_2 = A_2 \cap B_1$, $d_3 = A_3 \cap B_3$ laying outside $C$ (see Figure 2).

**Theorem 2.4** (Pascal theorem). The points $d_1, d_2$ and $d_3$ lie on a straight line.

**First Proof.** (We are following [2]). Let $x = B_1 \cap B_2$, $y = A_2 \cap A_3$ (see Figure 2) The points $d_1, x, a_4, a_5$ lie on the line $B_1$. The map $\pi_{a_2}$ associates
with each of them a line in $a_1^*$, namely lines $A_1, B_2, a_2a_4$ and $a_2a_5$. These lines intersect the conic $C$ in the points $a_1, a_3, a_4$ and $a_5$, respectively.

On the other hand, the points $d_3, a_3, a_4, y$ lie on the line $A_3$. They define four lines in $a_1^*$. These intersect $C$ at $a_1, a_3, a_4$ and $a_5$. By Proposition 2.3 we have

$$\text{cr}(d_1, x, a_4, a_5) = \text{cr}(a_1, a_3, a_4, a_5) = \text{cr}(d_3, a_3, a_4, y).$$

Now we apply Proposition 2.2 to the lines $A = A_3$, and $B = B_1$, intersecting at $a_4 = o$. Thus the lines $d_3d_1, a_3x = B_2$ and $ya_5 = A_2$ intersect at a single point $d_2$. □

![Diagram showing the Pappus configuration with points $A_1, A_2, A_3, B_1, B_2, B_3, x, y, d_1, d_2, d_3$ and lines intersecting at these points, forming a hexagon around the conic $C$.](image)

**Figure 2**

**Remark 2.5.** The above proof can be repeated in case when the conic $C = C' + C''$ is a union of two lines. It is then called the Pappus theorem:

Let $C', C''$ be two different lines and $a_1, a_2, a_3 \in C'$, whereas $b_1, b_2, b_3 \in C''$. Let us define the lines $A_1 = a_1b_1$, $A_2 = a_2b_3$, $A_3 = a_3b_2$, $B_1 = b_1a_3$, $B_2 = b_2a_1$, $B_3 = b_1a_2$ and the cubics $A = A_1 + A_2 + A_3$, $B = B_1 + B_2 + B_3$. The cubics provide us then with three additional, i.e. laying outside $C$, intersection points $d_1, d_2$ and $d_3$. Then these points are colinear.

There exists also another proof of the Pappus theorem. We can assume the lines $A_2$ and $A_3$ to be parallel, as well as lines $A_3$ and $B_3$. Let us also assume that $C' \cap C'' = o$ is a finite point. By the Tales theorem there exist homotheties $f$ and $g$ with centre at $o$ such that $B_2 = f(A_2)$ and $B_3 = g(A_3)$. The homotheties commute and their composition is again a homothety, that sends the line $A_1$ to $B_1$. Therefore the latter lines are parallel. In case $o$ lies on the line at infinity, we use translations instead of homotheties.
3. Local residuum and the Cayley–Bacharach theorem

Let us begin with recalling the Cauchy integral formula:

\[
\frac{1}{2\pi i} \int_{|z-a|=\varepsilon} g(z) \, dz = c_{-1} \overset{\text{def}}{=} \text{res}_a g(z),
\]

where \( g(z) \) is a meromorphic function, that expands in a Laurent series \( \sum_{j>j_0} c_j (z-a)^j \) at \( a \) and \( \varepsilon \) is sufficiently small. However, in higher dimensions there is no similar formula.

**Example 3.1** ([10]). Consider a rational function

\[
g(z, w) = \frac{h(z, w)}{zw(z-w)},
\]

where \( h \) is a polynomial. The three lines \( z = 0, w = 0 \) and \( z = w \) correspond to the point \( a \) in one-dimensional case. Let us integrate \( g \) over the following two-dimensional cycles:

\[
\Gamma_1 = \{|z| = \varepsilon_1, \ |w| = \varepsilon_2 > \varepsilon_1\}, \quad \Gamma_2 = \{|z| = \varepsilon_1, \ |w| = \varepsilon_2 < \varepsilon_1\}.
\]

We expand \( g \) at the cycles \( \Gamma_{1,2} \) in the Laurent series:

\[
g|_{\Gamma_1} = \frac{-h}{zw^2} \sum_{k\geq 0} \left( \frac{z}{w} \right)^k, \quad g|_{\Gamma_2} = \frac{h}{z^2w} \sum_{k\geq 0} \left( \frac{w}{z} \right)^k.
\]

Both series are uniformly convergent on the cycles. After integrating them, we obtain:

\[
\frac{1}{(2\pi i)^2} \int_{\Gamma_1} g \, dz \wedge dw = -\frac{\partial h}{\partial w}(0,0),
\]

\[
\frac{1}{(2\pi i)^2} \int_{\Gamma_2} g \, dz \wedge dw = \frac{\partial h}{\partial z}(0,0).
\]

The difference, we have just observed, results from a fact that the cycles \( \Gamma_1 \) and \( \Gamma_2 \) are not homologous in \( \mathbb{C}^2 \setminus \{zw(z-w) = 0\} \).

The higher dimensional approach to residues consists on the notion of local residuum at a point \( a \) of a meromorphic form of type:

\[
\omega = \frac{h(z)}{f_1(z) \cdots f_n(z)} \, dz_1 \wedge \cdots \wedge dz_n,
\]

where \( h \) and \( f_i \)’s are holomorphic functions.

The definition given below agrees with one given by A. Grothendieck as \( \text{Res} \left[ \frac{h \, dz}{f_1 \cdots f_n} \right] \) (cf. [6], [8]).
Definition 3.2. Let $a \in \mathbb{C}^n$ be an isolated zero of the holomorphic map $(f_1, \ldots, f_n)$. The local residue of the $n$-form (3.2) is the integral:

$$\text{res}_a \omega = \frac{1}{(2\pi i)^n} \int_{\Gamma(\varepsilon)} \omega,$$

where $\Gamma(\varepsilon) = \{ z : |f_j(z)| = \varepsilon_j \}$ and $\varepsilon_j > 0$ are small numbers such that $\Gamma(\varepsilon) \subset \{|z| < \varepsilon_0\}$ is a compact non-singular cycle oriented in such a way that $d \text{ arg } f_1 \wedge \cdots \wedge d \text{ arg } f_n > 0$.

In general, it is not easy to compute the local residue of a given non-trivial form. Below some calculations to be used in the sequel are presented.

Example 3.3. If

$$J(a) = \det \left\{ \frac{\partial f_i}{\partial z_j}(a) \right\} \neq 0,$$

i.e. the hypersurfaces $\{ f_i = 0 \}$ intersect transversely at $a$, then

$$\text{res}_a \omega = \frac{h(a)}{J(a)}.$$

This follows directly from the Cauchy formula (3.1), after changing coordinates from $(x_1, \ldots, x_n)$ to $(f_1, \ldots, f_n)$.

Example 3.4. Let $f_1 = P(z,w) + \ldots, f_2 = Q(z,w) + \ldots, h(z,w) = R(z,w) + \ldots$, where $P, Q, R$ are homogeneous polynomials of degrees $p, q, r$ respectively, and the dots denote higher order terms. Assume also that

$$P = \prod_{i=1}^p (z - a_i w), \quad Q = \prod_{i=1}^q (z - b_i w), \quad R = \prod_{i=1}^r (z - c_i w),$$

where

$$a_i \neq a_j \ (i \neq j), \quad b_i \neq b_j \ (i \neq j), \quad a_i \neq b_j,$$

$$r + 2 = p + q.$$

Then we have

$$\text{res}_0 \frac{h dz \wedge dw}{f_1 f_2} = \sum_{i=1}^p \text{res}_{a_i} \frac{\bar{R}(u)}{\bar{P}(u)\bar{Q}(u)} = - \sum_{j=1}^q \text{res}_{b_j} \frac{\bar{R}(u)}{\bar{P}(u)\bar{Q}(u)},$$

where $\bar{P}(u) = \prod(u - a_i), \bar{Q}(u) = \prod(u - b_i), \bar{R}(u) = \prod(u - c_i)$.

Proof. By the assumptions (3.7) and (3.8) it suffices to consider the integral

$$\frac{1}{(2\pi i)^2} \int_{\{P = \varepsilon_1\}} \int_{\{Q = \varepsilon_2\}} \frac{R dz \wedge dw}{PQ}.$$
Putting $z = uw$, which corresponds to the blow-up at $0$, we obtain the integral
\begin{equation}
\frac{1}{(2\pi i)^2} \iint \frac{\tilde{R}(u) \, du \, dw}{P(u)Q(u)w}.
\end{equation}
along the 2-cycle
\[ \tilde{\Gamma}(\varepsilon) = \{|w|^p \cdot |\tilde{P}(u)| = \varepsilon_1, \; |w|^q \cdot |\tilde{Q}(u)| = \varepsilon_2 \}. \]
The projection of the $\tilde{\Gamma}(\varepsilon)$ onto the $u$-plane gives the curve (1-cycle)
\begin{equation}
\Delta(\delta) = \{|\tilde{P}(u)/CT_q(u)\tilde{Q}(u)/CT_p(u)| = \delta \},
\end{equation}
where $\delta = \varepsilon_1 \varepsilon_2^{-1} = \text{const}$, and $\tilde{p} = p/\gcd(p,q)$, $\tilde{q} = q/\gcd(p,q)$. It is then clear that (3.10) equals
\begin{equation}
\pm \frac{1}{2\pi i} \int_{\Delta(\delta)} \frac{\tilde{R} \, du}{PQ}.
\end{equation}
Here the sign and the orientation of $\Delta(\delta)$ should be properly chosen. We take $\delta$ positive and small, such that $\Delta(\delta)$ is an union of small cycles around $a_i$, i.e. $\Delta_i(\delta) \approx \{|u - a_i| = \text{const}\}$. The 2-cycle $\tilde{\Gamma}(\varepsilon)$ becomes sum of cycles $\tilde{\Gamma}_i(\varepsilon)$ that approximate tori $\Delta_i(\delta) \times \{|w| = \text{const}\}$. The orientation is given by $d \arg P \wedge d \arg Q = d \arg u \wedge d \arg w$, provided that $\Delta_i(\delta)$ and $\{|w| = \text{const}\}$ are oriented in the standard way; (it is because $Q \approx \text{const} \cdot w^q$). Therefore
\begin{equation}
\frac{1}{(2\pi i)^2} \iint_{\tilde{\Gamma}_i(\varepsilon)} \omega = \text{res}_{a_i} \frac{\tilde{R}}{PQ} = \frac{\tilde{R}(a_i)}{Q(a_i) \cdot \prod_{j \neq i} (a_i - a_j)}.
\end{equation}
Note, that when we choose $\delta \to \infty$ in (3.11), we obtain $\Gamma(\varepsilon)$ as a union of tori around the points $(b_i, 0)$, but with reversed orientation. This agrees with the formula $\sum \text{res} \tilde{R}/(PQ) = 0$.

The formula (3.9) holds also in case when some of the points $a_i$ coincide, as well as when some of $b_j$’s do. However, it becomes (in general) false, when some $a_i$ equals $b_j$. \hfill $\square$

The next result is fundamental in our paper. Its proof is rather long and can be found in [6] and [10].

**Theorem 3.5 (Residue theorem).** Let $M$ be an analytic complex manifold without boundary and $X_1, \ldots, X_n$ be a system of effective divisors on $M$ with a discrete intersection $Y = X_1 \cap \ldots \cap X_n$. Then, for any meromorphic $n$-form $\omega$ with poles along $X_1, \ldots, X_n$ we have
\[ \sum_{a \in Y} \text{res}_{a} \omega = 0. \]

In this theorem we have used a notion of *divisor*, i.e. a finite formal sum $\sum n_a V_a$, $n_a \in \mathbb{Z}$, of hypersurfaces $V_a$, and of *effective divisor*, i.e. a divisor with
all \( n_\alpha \geq 0 \). In fact, in some charts (e.g. near infinity) we can have \( f_i = \prod g_\alpha^{n_\alpha} \), where \( g_\alpha \) are reduced functions defining \( V_\alpha \).

The following theorem is a very important application of the residue theorem.

**Theorem 3.6 (Cayley [3], Bacharach [1]).** Let \( A \) and \( B \) be two algebraic curves in \( \mathbb{C}P^2 \) of degrees \( p \) and \( q \) which intersect at \( p \cdot q \) different points. Let \( E \subset \mathbb{C}P^2 \) be a curve of degree \( r \leq p + q - 3 \) passing through \( pq - 1 \) points of \( A \cap B \). Then \( E \) passes also through the last point of intersection.

**Proof.** Let \( A = \{ f_1 = 0 \} \), \( B = \{ f_2 = 0 \} \), \( E = \{ h = 0 \} \). Let us consider the form \( \omega = h dx \wedge dy/f_1 f_2 \). The condition imposed on the degrees guarantees that \( \omega \) has no poles on the line at infinity. In fact, near infinity we have \( x = 1/z, \ y = u/z \) and \( dx \wedge dy \sim z^{-3}, \ h \sim z^{-r}, \ f_1 \sim z^{-p}, \ f_2 \sim z^{-q}; \ so \ \omega \sim z^{p+q-r-3} \).

Therefore all the possible residual points are finite, and the formula (3.5) holds.

For any \( a_j \in A \cap B \) we have \( \text{res}_{a_j} \omega = h(a_j)/J(a_j), \) and \( J(a_j) \neq 0 \). Thus if \( a_j \in E \) then \( \text{res}_{a_j} \omega = 0 \); and conversely, if \( \text{res}_{a_j} \omega = 0 \) then \( a_j \in E \). By assumption, \( a_1, \ldots, a_{pq-1} \in E \). Since \( 0 = \sum_{a_j} \text{res}_{a_j} \omega = \text{res}_{a_{pq}} \omega \), we obtain that \( a_{pq} \in E \). \( \square \)

### 4. Three applications of the Cayley–Bacharach theorem

#### 4.1. Second proof of the Pascal theorem.

Let \( A = A_1 + A_2 + A_3 \) and \( B = B_1 + B_2 + B_3 \) be the unions of three lines from the Pascal theorem, \( D = d_1 d_2 \) be the line through \( d_1 \) and \( d_2 \) and let \( E = C + D \). We know that \( \deg E = 3 = \deg A + \deg B - 3 \) and \( A \cap B = \{ a_1, a_6, d_1, d_2, d_3 \} \), where \( a_i \in C \). The curve \( E \) passes through all points of \( A \cap B \) possibly but one, \( d_3 \).

By the Cayley–Bacharach theorem \( E \) must pass also through \( d_3 \). But the point \( d_3 \) cannot lie on the conic \( C \), in which case \( A_3 \) would intersect \( C \) at three points. Therefore \( d_3 \in D \). \( \square \)

**Remark 4.2.** The reader can observe that in the above proof one could choose \( A \) and \( B \) as arbitrary cubics passing through 6 points \( a_1, \ldots, a_6 \). They do not have to be unions of three lines. In this approach the uniqueness of \( A \) and \( B \) is lost, as well as the simple geometrical meaning.

**Theorem 4.3 (Inverse Pascal theorem).** If a hexagon has the property that its opposite sides intersect at three collinear points \( d_1, d_2 \) and \( d_3 \), then its vertices lie on a conic.

**Proof.** Let \( H = A_1 A_2 A_3 B_1 B_2 B_3 \) be the hexagon, where \( A_i, B_j \)'s are the lines containing sides of \( H \). Denote \( a_1 = A_1 \cap B_3, \ a_2 = A_1 \cap A_2, \ a_3 = A_2 \cap A_3, \ a_4 = A_3 \cap B_1, \ a_5 = B_1 \cap B_2, \ a_6 = B_2 \cap B_3 \). Let \( C \) be the (unique) conic through 5 points \( a_1, \ldots, a_5 \), and \( D \) – the line through \( d_1, d_2 \) and \( d_3 \).

As before, we put \( A = A_1 + A_2 + A_3, \ B = B_1 + B_2 + B_3 \) and \( E = C + D \). Theorem 3.5 implies then that \( E \) contains \( a_6 \). Hence \( a_6 \in C \). \( \square \)
The first new result in our work is the following theorem.

**Theorem 4.4.** Let $C \subset \mathbb{CP}^2$ be a general cubic. Take three general lines $A_1$, $A_2$, $A_3$ intersecting $C$ at points $a_1$, $a_2$, $a_3$, $b_1$, $b_2$, $b_3$ and $c_1$, $c_2$, $c_3$, respectively. Let $B_1$ be a conic through $a_1$, $a_2$, $b_1$, $c_1$, $c_2$, $B_2$ a line through $a_3$ and $b_2$ and $B_3$ a line through $b_3$ and $c_3$. Define $A = A_1 + A_2 + A_3$, $B = B_1 + B_2 + B_3$. Then the additional 3 points of the intersection $A$ with $B$ (i.e. $d_1 = A_2 \cap B_1 \setminus \{b_1\}$, $d_2 = A_3 \cap B_2$, $d_3 = A_1 \cap B_3$) are collinear.

**Proof.** Denote by $D$ the line through $d_1$ and $d_2$ and define the quartic $E = C + D$. Then the assumption of Theorem 3.5 are satisfied, $\deg E = 4 = 3 + 4 - 3 = \deg A + \deg B - 3$. Hence $d_3 \in E$. Since $d_3 \notin C$, we conclude that $d_3 \in D$, i.e. $d_1, d_2, d_3$ are collinear. \(\square\)

**Remark 4.5.** As we have already mentioned in Introduction, it is possible to formulate other theorems similar to the Theorem 4.4. Since at the moment we do not have a complete classification of the cases, where the Cayley–Bacharach theorem can be directly applied, we postpone presentation of those results to further publication.

**Remark 4.6.** In the proofs of Theorems 2.4, 4.3 and 4.4 three properties of constructed curves were important:

(i) the curves $A$ and $B$ have regular intersections,
(ii) there is only one line passing through 2 different points,
(iii) there is only one conic passing through 5 points in general position.

In fact, the condition (i) can be weakened. The case when $A$ and $B$ have tangency point is the limit of regular cases. One of the additional points $d_i$ tends to a point $a_j \in C$. In the residuum integral the order of pole increases, but the line $D$ passes through $d_i = a_j$. So the residuum still remains zero.

The property (ii) does not hold when the two points coincide. In that case either the line is fixed by the tangency (e.g. to $C$ at $a_i = a_j$) or, when choosing $D = d_i d_j$ one has a possibility to fix the pair $(d_i, d_j)$.

The condition (iii) does not hold only when 4 points (e.g. $e_1, \ldots, e_4$ of $e_1, \ldots, e_5$) lie on one line $L$. If the latter point $e_5$ does not belong to $L$, we have a pencil of conics with the base set $L \cup \{e_5\}$. If all $e_1, \ldots, e_5 \in L$, then we are given a net of conics with the base $L$. In the above application we have not encountered such degeneracies. But in the next section this phenomenon will play a crucial role.

## 5. Eight points on a rational cubic

A rational algebraic curve $C$ is a curve which admits a parametrization $\mathbb{CP}^1 \to C$, which doesn't have to be one-to-one. It means that the normalization
of \( C \) is diffeomorphic to the projective line. A rational cubic, for instance, must have a singular points (otherwise it would be an elliptic curve of genus 1). It is isomorphic either to the quasi-homogeneous curve

\[ y^2 = x^3. \tag{5.1} \]

or to the curve

\[ y^2 = x^3 + x^2, \tag{5.2} \]

that has one simple double point. In the latter case the parametrization is given by \( x = -1 + t^2, y = t(t^2 - 1)^2 \). The curves (5.1) and (5.2) have only one singular point, namely \( o = (0, 0) \).

Let \( C \) be a rational cubic with a singular point \( o \). Take 8 points \( a_1, \ldots, a_8 \) on \( C \) in general position. Let \( A_1 \) be the unique conic through \( o, a_1, \ldots, a_4 \) and let \( A_2 \) be the conic through \( o, a_5, \ldots, a_8 \). We shall denote by \( A \) the sum \( A_1 + A_2 \).

We define a curve \( B = B_1 + B_2 \) in two ways. Either

\( a) \) \( B_1 \) is the conic through \( o, a_1, a_2, a_5, a_6 \) and \( B_2 \) the conic through \( o, a_3, a_4, a_7, a_8 \),

or

\( b) \) \( B_1 \) is the conic that passes through \( o, a_1, a_2, a_3, a_5 \) and \( B_2 \) the conic through \( o, a_4, a_6, a_7, a_8 \).

Other possibilities are provided by permutations of the set \( \{a_1, \ldots, a_8\} \). We shall not treat them as different.

The curves \( A \) and \( B \) intersect in the points \( a_1, \ldots, a_8 \) with multiplicity 1, in \( o \) with multiplicity 4 and in 4 additional points \( d_1, d_2, d_3, d_4 \).

The following result is the just generalization of the Pascal theorem to the case of rational cubic curves.

**Theorem 5.1.** The points \( d_1, d_2, d_3, d_4 \) lie on one line.

**Proof.** The cases \( a) \) and \( b) \) are particular cases of the following situation: \( A \) is a quartic passing through \( a_1, \ldots, a_8 \) with a double point at \( o \), while \( B \) is another quartic with the same properties. We prove the theorem first in that situation. The proof of Theorem 5.1 follows, since the property that 4 points are colinear is closed.

From now on we shall assume that both \( A \) and \( B \) are generic in the linear system \( \mathcal{L} \) of quartic passing through \( a_1, \ldots, a_8 \) that have double point at \( o \). The following lemma will be proved later.

**Lemma 5.2.** If the points \( a_1, \ldots, a_8 \in C \) are in general position and the quartics \( A, B \in \mathcal{L} \) are typical, then:

\( i) \) the intersections of \( A \) and \( B \) at \( a_1, \ldots, a_8, d_1, d_2, d_3, d_4 \) are non-degenerate,
(ii) the 4 tangent directions of \( A \cup B \) at \( o \) are different.

Consider the 2-form

\[ \omega = \frac{gh \, dx \wedge dy}{f_1 f_2}, \]

where \( f_1(x, y) \) and \( f_2(x, y) \) define respectively quartics \( A \) and \( B \), whereas \( g(x, y) \) defines the cubic \( C \) and \( h(x, y) \) is a quadratic polynomial. Lemma 5.2 implies that the local residua of \( \omega \) can be calculated using the formulae (3.5) and (3.9) (from Examples 3.3 and 3.4). In particular, if \( h(d_i) = 0 \) then \( \text{res}_{d_i} \omega = 0 \). Analogously, if \( h(o) = 0 \) then \( \text{res}_o \omega = 0 \). Let us denote also by \( \omega_0 \) the form \( g \, dx \wedge dy/f_1 f_2 \). We have two possibilities:

(\( \alpha \)) \( \text{res}_o \omega_0 \neq 0 \),

(\( \beta \)) \( \text{res}_o \omega_0 = 0 \).

We claim that there may hold only (\( \beta \)). Suppose conversely, i.e. \( \text{res}_o \omega_0 \neq 0 \). We assume in (5.3) \( h \) to vanish at \( d_1, \ldots, d_4 \). We have then \( \text{res}_{d_i} \omega = 0 \), \( \text{res}_d \omega = 0 \) and \( \text{res}_o \omega = h(o) \cdot \text{res}_o \omega_0 \). By virtue of the residue Theorem 3.4 the equality \( h(o) = 0 \) must hold for any quadratic polynomial vanishing at \( d_1, \ldots, d_4 \). This implies that three of the points \( d_1, \ldots, d_4 \) lie on one line \( L \) passing through \( o \). Let us suppose that these are \( d_1, d_2, d_3 \). Choose now \( h \) to be a linear polynomial vanishing at \( d_1, d_2, d_3, o \). Applying the residue theorem, we obtain that \( h \) vanishes also at \( d_4 \). It would mean that all five points \( d_1, d_2, d_3, d_4, o \) lie on \( L \). But than \( L \) would intersect the quartic \( A \) at five points, what contradicts the genericity. So the case (\( \alpha \)) has been excluded. In particular, \( \text{res}_o \omega = 0 \), whatever \( h \) is.

Let us now take \( h \) quadratic and vanishing at three points of \( d_1, d_2, d_3, d_4 \). By residue theorem \( h \) has to be zero also at the fourth one. The only outcome from this seemingly tangible situation is the fact that all four points \( d_1, d_2, d_3, d_4 \) lie on one line \( D \); it, of course, does not pass through \( o \). The geometrical picture is presented at Figure 3.

(It is worth to mention that Figure 3 was made using the computer programm PASCAL, which uses the inverse Pascal theorem in construction of a conic through 5 points.)

The proof of Theorem 5.1 has been completed. \( \square \)

**Proof of the Lemma 5.2.** It is enough to find two quartics \( A \) and \( B \) that satisfy (i) and (ii), without specifying a priori the 8–ple \( a_1, \ldots, a_8 \). Take \( A = A_1 + A_2 \), where \( A_1 \) and \( A_2 \) are two conics through \( o \) that intersect transversely \( C \setminus \{o\} \) at 8 different points, and with different tangent directions at \( o \) and not tangent to any branch of \( C \) at \( o \). As \( B \) we shall take \( C + M \), where \( M \) is some line avoiding \( o, a_1, \ldots, a_8 \). Since the conditions (i) and (ii) are (Zariski) open, they hold for generic 8-pole \( (a_1, \ldots, a_8) \) and generic quartics \( A, B \in \mathcal{L} \).
Note that in the latter example the points $d_1, \ldots, d_4$ lie on $M$.  

We finish this section by proving a theorem dual to Theorem 5.1. Since the dual curve to a generic cubic curve is a curve of higher degree, we restrict our considerations to the case of the quasi-homogeneous cubic (5.1). This curve is simply connected, has exactly one singular point and exactly one inflection point $\infty$ (at infinity). We denote by $L_\infty$ the line tangent to $C$ at $\infty$.

**Theorem 5.3.** Let $C \subset \mathbb{CP}^2$ be a simply connected cubic with the inflection point $\infty$. Let $a_1, \ldots, a_8 \in C \setminus \infty$ and let $L_1, \ldots, L_8$ be the lines tangent to $C$ at $a_i$. Define $A_1$ as the conic tangent to $L_\infty$, $L_1, L_2, L_3, L_4$, $A_2$ – the conic tangent to $L_\infty$, $L_5, L_6, L_7, L_8$, $B_1$ – the conic tangent to $L_\infty$, $L_1, L_2, L_5, L_6$ and $B_2$ – the conic tangent to $L_\infty$, $L_3, L_4, L_7, L_8$. Denote $A = A_1 + A_2$ and $B = B_1 + B_2$. There exist 4 additional lines $M_1, M_2, M_3, M_4$ which are tangent to $A$ and to $B$. Then the lines $M_j$ intersect at one point. The same statement...
holds when we replace $B_1$ by the conic tangent to $L_\infty$, $L_1$, $L_2$, $L_3$, $L_5$ and $B_2$
by the conic tangent to $L_\infty$, $L_4$, $L_6$, $L_7$, $L_8$.

PROOF. The dual curve to the quasi-homogeneous curve $y^2 = x^3$ is the cubic $27q + 4p^3 = 0$; (where $y = px + q$ is the equation for lines tangent to $C$). The cusp point $x = y = 0$ corresponds to the inflection point $\infty : p = q = 0$.

The dual to a conic is a conic. The dual to a point is a line (of lines through it). In particular, the point of intersection of two curves corresponds to a line tangent to the two dual curves.

Finally, the dual to a line (e.g. the line $D$ from the proof of Theorem 5.1) is a point (i.e. the common point of the lines $M_j$).

□

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