Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 19, 2002, 29–38

ASYMPTOTICALLY CRITICAL POINTS AND THEIR MULTIPLICITY

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Abstract. In this paper we study multiplicity results for the critical points of a functional via topological information which ensures multiplicity of critical points for a sequence of approximating functionals. The main statement is quite simple, and it seems it could be usefully compared with a large class of problems. In particular we mention some problems that can be studied in this framework.

1. Introduction: main concepts and aims

There are many variational problems whose solutions spontaneously come out as limit of solutions of approximating problems. This can happen in various situations.

A typical case is the one of studying a problem which is "irregular", for many reasons, in the sense that it lies (a little or a lot) beyond the classical framework of reference in which similar problems are studied (for degeneration of the coefficients, for lack of differentiability, ...). Usually in this situation a sequence of "regular" functionals $(h_n)_n$ comes out in a very natural way. Such a sequence "tends" in some sense to an "irregular" functional h whose "stationary" points, suitably defined, solve the initial problem.

2000 *Mathematics Subject Classification.* 58E05, 58E30.
Key words and phrases. Asymptotically critical points, ∇-compactness.

An analogous situation arises while using approximation methods (of Galerkin type) for a problem (for example a differential equation) in an infinite dimensional space H by a sequence of problems in finite dimensional subspaces H_n. In such a case, if $h\colon H \to \mathbb{R}$ is the functional whose critical points are object of investigation, one can consider the functionals $h_n = h_{|H_n}$. If one wants to consider all h_n defined in the same space H, one can set $h_n(u) = \infty$ if $u \in H \setminus H_n$. The sequence $(h_n)_n$ defined in this way can be studied with the methods of [9], which extend the results of this paper.

An important problem of this type is the one of the bounce trajectories between two given points in a billiard with perfectly elastic walls, in presence, possibly, of a field of conservative forces (see [4] and [10]). If we describe the billiard with the closure of an open subset Ω of $\mathbb{R}^N$ and the potential of the field with a function $V\colon \mathbb{R}^N \to \mathbb{R}$, given two points A and B in Ω, it is natural to consider, for example, the functionals

$$f_n(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2\,dt - \int_0^1 V(\gamma(t))\,dt - n\int_0^1 U(\gamma(t))\,dt$$

defined on the space $X = \big\{\gamma \in H^1([0,1];\mathbb{R}^N) \mid \gamma(0) = A, \gamma(1) = B\big\}$, where $U\colon R^N \to \mathbb{R}$ is a continuous function such that $U(x) = 0$ if $x \in \overline{\Omega}$ and $U(x) > 0$ if $x \in \mathbb{R}^N \setminus \overline{\Omega}$. Under suitable hypotheses, if, for all n, γ_n in X is a critical point of f_n and if $(\gamma_n)_n$ converges in a suitable sense to a curve γ^* of X, then γ^* has image in $\overline{\Omega}$ and it is a bounce trajectory, in the sense that it verifies the "reversed" inequality

$$\int_0^1 \dot\gamma^* \cdot \dot\varphi - \int_0^1 \nabla V(\gamma^*) \cdot \varphi \leq 0 \tag{1}$$

for all φ in $H^1_0([0,1];\mathbb{R}^N)$ such that $\varphi(t) \cdot \nu(\gamma^*(t)) \leq 0$, for all, t such that $\gamma^*(t) \in \partial\Omega$ (where $\nu(x)$ is the outward normal to Ω in x of $\partial\Omega$), with the condition that it preserves its energy:

$$\frac{1}{2}|\dot\gamma^*(t)|^2 + V(\gamma^*(t)) = \text{constant}.$$

These two conditions characterize the bounce curves γ^* in the billiard $\overline{\Omega}$ in a satisfactory way and we can assume them just to define such curves (we observe, *en passant*, that the usual elliptic variational inequalities have the sign "$\geq$" instead of the "$\leq$" of reference). Note that the fact that γ^* solves (1), is equivalent to say that γ^* is a "upper critical point" (see [5], [11]) for the functional $f\colon X \to \mathbb{R} \cup \{-\infty\}$ defined by

$$f(\gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |\dot\gamma(t)|^2\,dt - \int_0^1 V(\gamma(t))\,dt & \text{if } \gamma([0,1]) \subset \overline{\Omega}, \\ -\infty & \text{elsewhere.} \end{cases}$$

Moreover, it should be noted that the conservation of the energy is *not* a consequence of inequality (1), since it follows from the fact that γ_n is a critical point for f_n.

Therefore, in this case it seems unnatural to introduce a functional whose "critical" points (in some sense) solve the bounce problem, while it is spontaneous to refer to all the sequence $(f_n)_n$.

Thus we are led to give the following definitions.

Let (h_n) be a sequence of functionals defined, for example, on a Riemannian manifold M and let us also consider a functional $h: M \to \mathbb{R}$. Assume that h_n, h and M are regular, just for simplicity.

Definition 1.1. We say that u in M is asymptotically critical for the couple $((h_n)_n, h)$, if there exists a strictly increasing sequence of integers $(n_k)_k$ in $\mathbb{N}$ and there exists a sequence $(u_k)_k$ in M such that

$$\nabla h_{n_k}(u_k) \to 0, \quad u_k \to u \quad \text{and} \quad h_{n_k}(u_k) \to h(u).$$

We also say that $h(u)$ is an asymptotically critical value (level) for the couple $((h_n)_n, h)$.

Note that it is not necessary to impose that u is a critical point for h. Indeed we could completely eliminate the function h, from the previous definition, substituting the third limit condition by the assumption that the limit of $h_{n_k}(u_k)$ exists, and we would still call it asymptotically critical value. But in this case we should add the hypothesis that for every u in M satisfying modified definition of asymptotically critical point, the limit of $(h_{n_k}(u_k))_k$ doesn't depend on $(n_k)_k$ and $(u_k)_k$.

In order to obtain multiplicity results related to asymptotically critical points for a couple $((h_n)_n, h)$, we introduce the following definition.

Definition 1.2. Let c be a real number. We say that the couple $((h_n)_n, h)$ is ∇-compact at level c, or that condition $\nabla(h_n, h; c)$ holds, if for every strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and for every $(u_k)_k$ in M such that

$$\nabla h_{n_k}(u_k) \to 0 \quad \text{and} \quad h_{n_k}(u_k) \to c,$$

there exists a strictly increasing sequence $(k_j)_j$ in $\mathbb{N}$ and there exists u in M such that $u_{k_j} \to u$ and $h(u) = c$.

If a and b are real numbers with $a \le b$ and $\nabla(h_n, h; c)$ holds for all c in $[a, b]$, we say that $\nabla(h_n, h; a, b)$ holds.

This condition is the unique connection we will assume between the sequence $(h_n)_n$ and the function h. It has two features: it express a kind of convergence of $(h_n)_n$ to h and a sort of Palais–Smale conditionb for the couple $((h_n)_n, h)$.

In Section 2 we will prove, in particular, the following theorem.

Theorem 1.3. *Suppose M is a complete Riemannian manifold of class $C^{1,1}$ and every h_n is of class C^1; let a and b be in $\mathbb{R}$ with $a \leq b$. If $\nabla(h_n, h; a, b)$ holds, then the number of asymptotically critical points for $((h_n)_n, h)$ with asymptotically critical value in $[a, b]$ is greater than or equal to*

$$\limsup_{n\to\infty} \operatorname{cat}_M(h_n^b, h_n^a).$$

The notion of relative category we used to prove this theorem is quickly recalled in the appendix, and $h_n^c = \{u \in M \mid h_n(u) \leq c\}$.

Therefore the condition $\nabla(h_n, h; a, b)$, which ensures the permanence of the (quasi) critical points on the "limit" function h, is also sufficient, as Theorem 1.3 shows, to evaluate the multiplicity of such points.

Anyway it is interesting to note that the hypotheses of the Theorem don't imply that $(h_n)_n$ converges to h according to any usual notion of convergence (pointwise, uniform, $\Gamma, \dots$).

Remark 1.4. We can give a weaker and simpler version of the condition of ∇–compactness for a couple $((h_n)_n, h)$ at a level c:

$$\text{for all } (u_n)_n \text{ in} M \text{ such that } \nabla h_n(u_n) \to 0 \text{ and } h_n(u_n) \to c,$$

there exists a subsequence $(u_{n_k})_k$ which converges to a point u of M with $h(u) = c$.

This definition is useful, too; in fact, if we replace it to the one given in Definition 1.2, Theorem 1.3 still holds, provided in the thesis we replace $\limsup_{n\to\infty} \operatorname{cat}_M(h_n^b, h_n^a)$ with $\liminf_{n\to\infty} \operatorname{cat}_M(h_n^b, h_n^a)$.

But in Proposition 2.5 and Theorem 2.6 we will use Definition 1.2 in an essential way.

In [9] we will also consider the case in which h_n and h are not differentiable in the classical sense and we will also give a nonsmooth version of this Theorem. In this nonsmooth version the theorem lets us study the following reversed variational inequality: $u \in H_0^1(\Omega) \cap H_0^2(\Omega)$, $u \geq \varphi$ and

$$\int_\Omega \Delta u \Delta(v-u) - c\int_\Omega Du\cdot D(v-u) - \alpha\int_\Omega u(v-u) \leq 0 \tag{2}$$

for all v in $H_0^1(\Omega) \cap H^2(\Omega)$ such that $v \geq \varphi$, where Ω is an open, bounded and regular subset of $\mathbb{R}^N$ with $N \leq 3$ and $\varphi\colon \Omega \to \mathbb{R}$ is a measurable function such that $\sup \varphi < 0$ (see [9]).

Finally we note that the nonsmooth version of Theorem 1.3 covers the approximation methods of Galerkin type described in [2] and [6]. This paper was also inspired by those techniques, further developed in [3].

We are grateful to Prof. C. Saccon for the conversations on the subject, and in particular for proposing to define the critical point, linking it to the sequence

$(h_n)_n$: in this way one already includes in the definition richer information than the one expressed by the only fact that such a point is a critical point for the limit functional.

Added in proof: also in [1] a different notion of critical point associated to a sequence of functionals is introduced. But in that case the authors consider a class of functionals which are even "quadratic-like", with non degenerate critical points, defined on spaces of finite dimension and with index which is uniformly bounded below from a positive integer.

2. Multiplicity of asymptotically critical points

In this Section, in particular, we want to prove Theorem 1.3.

Hence let us consider a complete Riemannian $C^{1,1}$ manifold M and a sequence $(h_n)_n$ of C^1 real functions defined on M.

We premise the following lemmas.

Lemma 2.1. *Let α and β be real numbers such that $\alpha \leq \beta$ and let A and B be open subsets of M such that*

- $A \subset B$,
- $\inf\{d(x,y) \mid x \in A,\ y \in M \setminus B\} = \delta > 0$,
- $\inf_{n\in\mathbb{N}} \inf\{\|\nabla h_n(u)\| \mid u \in M \setminus A,\ \alpha \leq h_n(u) \leq \beta\} > 0$.

Let γ be a number in $[\alpha,\beta]$. Then there exists $\varepsilon > 0$ such that, for every n in $\mathbb{N}$,

- *$h_n^{\gamma-\varepsilon}$ is a strong deformation retract of $(h_n^{\gamma+\varepsilon} \setminus B) \cup h_n^{\gamma-\varepsilon}$ if $\alpha < \gamma < \beta$,*
- *h_n^{γ} is a strong deformation retract of $(h_n^{\gamma+\varepsilon} \setminus B) \cup h_n^{\gamma}$ if $\gamma = \alpha$,*
- *$h_n^{\gamma-\varepsilon}$ is a strong deformation retract of $(h_n^{\gamma} \setminus B) \cup h_n^{\gamma-\varepsilon}$ if $\gamma = \beta$.*

The proof can be easily obtained from the classical one related to a single functional, by the uniformity with respect to n of the hypothesis made here.

Lemma 2.2. *Let a and b be real numbers such that $a \leq b$, let F be a closed subset of M which doesn't contain asymptotically critical points for $((h_n)_n, h)$ with asymptotically critical value in $[a,b]$. Assume $\nabla(h_n, h; a, b)$ holds. Then*

$$\liminf_{n\to\infty} \inf\{\|\nabla h_n(u)\| \mid u \in F,\ a \leq h_n(u) \leq b\} > 0.$$

Proof. If, by contradiction,

$$\liminf_{n\to\infty} \inf\{\|\nabla h_n(u)\| \ \mid u \in F,\ a \leq h_n(u) \leq b\} = 0,$$

there would exist a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ and $(u_k)_k$ in F such that $a \leq h_{n_k}(u_k) \leq b$ and $\nabla h_{n_k}(u_k) \to 0$. By $\nabla(h_n, h; a, b)$ there would exist a subsequence $(u_{k_j})_j$ which converges to a point u in F. Evidently (passing to a subsequence), such a point u would be an asymptotically critical point for $((h_n)_n, h)$ with asymptotically critical level in $[a, b]$. $\square$

From now on, if $c \in \mathbb{R}$, we denote by Z_c the set of asymptotically critical points for $((h_n)_n, h)$ at level c.

Lemma 2.3. *Let a and b be real numbers such that $a \leq b$ and let $c_1, \dots , c_k$ be the unique asymptotically critical levels for $((h_n)_n, h)$ in $[a, b]$ and suppose that $a \leq c_1 < \dots < c_k \leq b$. Let $U_1, \dots , U_k$ be neighbourhoods of radius δ of the sets $Z_{c_1}, \dots , Z_{c_k}$ respectively ($U_i = \{x \in M \mid d(x, Z_{c_i}) < \delta\}$). Suppose $\nabla(h_n, h; a, b)$ holds. Then*

$$\limsup_{n\to\infty} \operatorname{cat}_M(h_n^b, h_n^a) \leq \sum_{i=1}^{k} \operatorname{cat}_M(U_i).$$

Proof. (I) For every $i = 1, \dots , k$ let U_i' be the open neighbourhood of radius $\delta/2$ of Z_{c_i} and let $a_1, \dots , a_{k+1}$ be real numbers such that

$$a_1 = a \leq c_1 < a_2 < \dots < a_i < c_i < a_{i+1} < \dots < c_k \leq a_{k+1} = b.$$

By Lemma 2.2, for every $i = 1, \dots , k$,

$$\liminf_{n\to\infty} \inf\{\|\nabla h_n(u)\| \mid u \in M \setminus U_i', a_i \leq h_n(u) \leq a_{i+1}\} > 0,$$

since in the closed set $M \setminus U_i'$ the couple $((h_n)_n, h)$ has no asymptotically critical points with asymptotically critical value contained in $[a_i, a_{i+1}]$ (it may have some at a level c_j different from c_i).

(II) For every $i = 1, \dots , k$, by Lemma 2.1 (applied with $\gamma = c_i$, $\alpha = a_i$ and $\beta = a_{i+1}$) there exist $\overline{n}$ in $\mathbb{N}$ and $\varepsilon > 0$ (we can assume it is the same for every $i = 1, \dots , k$) such that $c_i + \varepsilon \leq c_{i+1} - \varepsilon$ for every $i = 1, \dots , k - 1$ and for every $n \geq \overline{n}$

- $h_n^{c_i-\varepsilon}$ is a strong deformation retract of $(h_n^{c_i+\varepsilon} \setminus U_i) \cup h_n^{c_i-\varepsilon}$ if $a_i < c_i < a_{i+1}$,
- h_n^a is a strong deformation retract of $(h_n^{a+\varepsilon} \setminus U_1) \cup h_n^a$ if $c_1 = a$,
- $h_n^{b-\varepsilon}$ is a strong deformation retract of $(h_n^b \setminus U_k) \cup h_n^{b-\varepsilon}$ if $c_k = b$.

(III) Finally, if $n \geq \overline{n}$, we get

$$\operatorname{cat}_M(h_n^b, h_n^a) \leq \sum_{i=2}^{k-1} \operatorname{cat}_M(h_n^{c_i+\varepsilon}, h_n^{c_{i-1}+\varepsilon}) + \operatorname{cat}_M(h_n^{c_1+\varepsilon}, h_n^a) + \operatorname{cat}_M(h_n^b, h_n^{c_k+\varepsilon})$$

by (a) of A.3. The right hand side of this inequality is less or equal to

$$\sum_{i=2}^{k-1} \operatorname{cat}_M(h_n^{c_i+\varepsilon}, h_n^{c_i-\varepsilon}) + \operatorname{cat}_M(h_n^{c_1+\varepsilon}, h_n^{a'}) + \operatorname{cat}_M(h_n^{b'}, h_n^{c_k-\varepsilon})$$

(setting $a' = \max\{a, c_1 - \varepsilon\}$ and $b' = \min\{b, c_k + \varepsilon\}$), since, for example, for $i = 2, \dots, k-1$,

$$\mathrm{cat}_M(h_n^{c_i+\varepsilon}, h_n^{c_{i-1}+\varepsilon}) \le \mathrm{cat}_M(h_n^{c_i+\varepsilon}, h_n^{c_i-\varepsilon}) + \mathrm{cat}_M(h_n^{c_i-\varepsilon}, h_n^{c_{i-1}+\varepsilon}),$$

and $\mathrm{cat}_M(h_n^{c_i-\varepsilon}, h_n^{c_{i-1}+\varepsilon}) = 0$ by Lemma 2.1 and Lemma 2.2. Then

$$\begin{aligned}\mathrm{cat}_M(h_n^b, h_n^a) \le \sum_{i=2}^{k-1} &\mathrm{cat}_M((h_n^{c_i+\varepsilon} \setminus U_i) \cup h_n^{c_i-\varepsilon}, h_n^{c_i-\varepsilon}) + \sum_{i=2}^{k-1} \mathrm{cat}_M(U_i) \\ &+ \mathrm{cat}_M((h_n^{c_1+\varepsilon} \setminus U_1) \cup h_n^{a'}, h_n^{a'}) + \mathrm{cat}_M(U_1) \\ &+ \mathrm{cat}_M((h_n^{b'} \setminus U_k) \cup h_n^{c_k-\varepsilon}, h_n^{c_k-\varepsilon}) + \mathrm{cat}_M(U_k) = \sum_{i=1}^{k} \mathrm{cat}_M(U_i)\end{aligned}$$

by Lemma 2.1 and (b) and (c) of A.3. □

Corollary 2.4. *Suppose $a \le b$ in $\mathbb{R}$ and assume:*

(a) *M is connected and $((h_n)_n, h)$ has only a finite number of asymptotically critical points with asymptotically critical value in $[a, b]$,*
(b) *$\nabla(h_n, h; a, b)$ holds.*

Then the number of asymptotically critical values for $((h_n)_n, h)$ in $[a, b]$ is greater or equal to

$$\limsup_{n\to\infty} \mathrm{cat}_M(h_n^b, h_n^a).$$

In place of (a) *we can, for example, assume that M is contractible.*

Proof. We can assume that every neighbourhood U_i of the sets Z_{c_i} of the previous Lemma is contractible. □

By Lemma 2.3 we deduce Theorem 1.3.

Proof of Theorem 1.3. If $((h_n)_n, h)$ has only a finite number of asymptotically critical points with asymptotically critical value in $[a, b]$, denoting by $c_1, \dots, c_k$ the asymptotically critical values for $((h_n)_n, h)$ in $[a, b]$, there exists neighbourhoods $U_i, \dots, U_k$ of $Z_{c_1}, \dots, Z_{c_k}$ respectively such that $\mathrm{cat}_M(U_i) =$ cardinality of Z_{c_i}. By Lemma 2.3 the thesis follows. □

We want to underline the fact that Definition 1.2 (instead of the modified one recalled in Remark 1.4, lets us extend another classical result to our case.

First of all we can prove the following proposition.

Proposition 2.5. *Let c be in $\mathbb{R}$ and suppose that $\nabla(h_n, h; c)$ holds. Then Z_c is compact.*

Proof. Let $(z_i)_i$ be a sequence in Z_c. Then for all i there exists a strictly increasing sequence $(n_k^i)_k$ in $\mathbb{N}$ and there exists $(u_k^i)_k$ in M such that $u_k^i \to z_i$, $\nabla h_{n_k^i}(u_k^i) \to 0$ and $h_{n_k^i}(u_k^i) \to h(z_i) = c$.

Then for all i in $\mathbb{N}$ there exists k_i in $\mathbb{N}$ such that

$$d(u^i_{k_i}, z_i) \leq \frac{1}{i}, \quad \|\nabla h_{n^i_{k_i}}(u^i_{k_i})\| \leq \frac{1}{i}, \quad |h_{n^i_{k_i}}(u^i_{k_i}) - c| \leq \frac{1}{i},$$

where d is the metric on M. Since k_i can be taken as big as desired, we can suppose that $n^i_{k_i} < n^{i+1}_{k_{i+1}}$ for all i in $\mathbb{N}$.

Setting $n^i_{k_i} = m_i$ and $v_i = u^i_{k_i}$, we get:

$$d(v_i, z_i) \leq \frac{1}{i}, \qquad h_{m_i}(v_i) \to c \quad \text{and} \quad \nabla h_{m_i}(v_i) \to 0.$$

By $\nabla(h_n, h; c)$ there exists a strictly increasing sequence $(i_j)_j$ in $\mathbb{N}$ and there exists z in M such that $v_{i_j} \to z$ and $h(z) = c$. Then $z \in Z_c$ and $z_{i_j} \to z$. □

We recall that in every paracompact Banach manifold M one can prove that for every closed subset E of M there exists a neighbourhood U of E such that $\mathrm{cat}_M(U)$=$\mathrm{cat}_M(E)$.

If $k \in \mathbb{N}$ we set $c_k = \inf\{b \in \mathbb{R} \mid \limsup_{n\to\infty} \mathrm{cat}_M(h^b_n) \geq k\}$. Evidently $c_k \leq c_{k+1}$ for all k in $\mathbb{N}$.

Theorem 2.6. *Let M be a complete Riemannian paracompact manifold of class $C^{1,1}$ and suppose that h_n is of class C^1 for all n in $\mathbb{N}$. If there exists i in $\mathbb{N}$ such that $c = c_{k+1} = \ldots = c_{k+i} \in \mathbb{R}$ and if $\nabla(h_n, h; c)$ holds, then* $\mathrm{cat}_M(Z_c) \geq i$.

Proof. Z_c is compact, so there exists a neighbourhood U of Z_c such that $\mathrm{cat}_M(U)$ =$\mathrm{cat}_M(Z_c)$. Moreover, $d(Z_c, M \setminus U) = \delta > 0$.

If U' is the neighbourhood of Z_c of radius $\delta/2$, by $\nabla(h_n, h; c)$ we get that there exists $\varepsilon > 0$ and there exists $\overline{n}$ in $\mathbb{N}$ such that

$$\inf_{n\geq\overline{n}} \inf\{\|\nabla h_n(u)\| \mid u \in M \setminus U',\ c - \varepsilon \leq h_n(u) \leq c + \varepsilon\} > 0.$$

By Lemma 2.1 (possibly taking a smaller ε), we get that for all $n \geq \overline{n}$

$$h^{c-\varepsilon}_n \text{is a strong deformation retract of } (h^{c+\varepsilon}_n \setminus U) \cup h^{c-\varepsilon}_n.$$

Then, for all $n \geq \overline{n}$,

$$\mathrm{cat}_M(h^{c+\varepsilon}_n) \leq \mathrm{cat}_M(h^{c-\varepsilon}_n) + \mathrm{cat}_M(U).$$

But $\mathrm{cat}_M(h^{c+\varepsilon}_n) \geq k + i$ for infinitely many n $(c + \varepsilon > c_{k+1})$, $\mathrm{cat}_M(h^{c-\varepsilon}_n) \leq k$ definitely $(c - \varepsilon < c_{k+1})$ and $\mathrm{cat}_M(U)$ =$\mathrm{cat}_M(Z_c)$. The thesis follows. □

A. Some recalls on relative category

As we said, for the readers' convenience here we recall the version of relative category that we used in the previous sections. For example see [2], [3], [6]–[8] and [12].

Let X be a topological space and let A and B be two subsets of X. We first recall the following definition.

Definition A.1. We say that B is a strong deformation retract of A in X if

- $B \subset A$,
- there exists $h: [0,1] \times A \to X$ continuous and such that $h(0,u) = u$ for all u in A, $h(1,u) \in B$ for all u in A, $h(t,u) = u$ for all (t,u) in $[0,1] \times B$.

Definition A.2. If $B \subset A$, we say that the *relative category of A with respect to B in X* is k, and we write $\mathrm{cat}_X(A,B)$, if k is the least integer such that there exist $k+1$ closed subsets $A_0, \ldots, A_k$ of X such that

- $A \subset \bigcup_{i=0}^k A_i$,
- $A_1, \ldots, A_k$ are contractible in X,
- B is a strong deformation retract of A_0.

If no such integer exists, we set $\mathrm{cat}_X(A,B) = \infty$.

Note that, if $B = \emptyset$, then $\mathrm{cat}_X(A,B) = \mathrm{cat}_X(A)$, where $\mathrm{cat}_X(A)$ denotes the classical category of Lusternik and Schnirelman.

For the proof of Theorem 1.3 we only need the following properties of the relative category.

Properties A.3.

(a) If $C \subset B \subset A \subset X$, then

$$\mathrm{cat}_X(A,C) \leq \mathrm{cat}_X(A,B) + \mathrm{cat}_X(B,C).$$

(b) If $A \subset A' \subset X$, A' is closed and A is a strong deformation retract of A', then $\mathrm{cat}_X(A',A) = 0$.

(c) If $C \subset A \subset X$, $B \subset X$, then

$$\mathrm{cat}_X(A \cup B, C) \leq \mathrm{cat}_X(A,C) + \mathrm{cat}_X(B).$$

It is interesting to note that from (a) of A.3 we get (putting $C = \emptyset$) that, if $B \subset A \subset X$, then $\mathrm{cat}_X(A,B) \geq \mathrm{cat}_X(A) - \mathrm{cat}_X(B)$.

Remark A.4. We also note the following interesting properties.

(a) If $C \subset B \subset A \subset X$, then $\mathrm{cat}_X(B,C) \leq \mathrm{cat}_X(A,C)$. If B is a strong deformation retract of A, then equality holds.

(b) If $C \subset B \subset A \subset X$, C is a strong deformation retract of B, then $\mathrm{cat}_X(A,C) \leq \mathrm{cat}_X(A,B)$.

Concerning the relations between the relative category and the category of Lusternik and Schnirelmann, we also note that if B is a closed subset of X contained in A, then

$$\mathrm{cat}_X(A,B) \leq \mathrm{cat}_X(A \setminus B)\ (\leq \mathrm{cat}_X(A)).$$

It is also well known that by A.3 and the usual variational techniques, one can prove the following Theorem, which is a particular case of Theorem 1.3.

Theorem A.5. *Let M be a complete $C^{1,1}$ manifold and let $f\colon M \to \mathbb{R}$ be a C^1 function. Suppose $a < b$ in $\mathbb{R}$ and f satisfies $(\mathrm{PS})_c$ for every c in $[a,b]$. Then f has at least $\mathrm{cat}_M(f^b, f^a)$ critical points in $f^{-1}([a,b])$.*

Proof. See [8]. □

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Manuscript received May 31, 2001

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