Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 18, 2001, 395–408

WELL-POSEDNESS AND POROSITY IN BEST APPROXIMATION PROBLEMS

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Abstract. Given a nonempty closed subset A of a Banach space X and a point $x \in X$, we consider the problem of finding a nearest point to x in A. We define an appropriate complete metric space $\mathcal{M}$ of all pairs (A, x) and construct a subset Ω of $\mathcal{M}$ which is the countable intersection of open everywhere dense sets such that for each pair in Ω this problem is well-posed. As a matter of fact, we show that the complement of Ω is not only of the first category, but also sigma-porous.

Introduction

Given a nonempty closed subset A of a Banach space $(X, ||\cdot||)$ and a point $x \in X$, we consider the minimization problem

$$\text{(P)} \qquad \min\{||x-y|| : y \in A\}.$$

It is well-known that if A is convex and X is reflexive, then problem (P) always has at least one solution. This solution is unique when X is strictly convex.

If A is merely closed but X is uniformly convex, then according to classical results of Stechkin [15] and Edelstein [4] the set of all points in X having a unique nearest point in A is G_δ and dense in X. Since the sixties there has been a lot

2000 *Mathematics Subject Classification.* 41A50, 41A52, 41A65, 54E35, 54E52.

Key words and phrases. Banach space, complete metric space, generic property, hyperbolic space, nearest point, porous set.

of activity in this direction. In particular, it is known [8], [9] that the following properties are equivalent for any Banach space X:

(A) X is reflexive and has a Kadec–Klee norm.
(B) For each closed nonempty subset A of X, the set of points in $X \setminus A$ with nearest points in A is dense in $X \setminus A$.
(C) For each closed nonempty subset A of X, the set of points in $X \setminus A$ with nearest points in A is generic (that is, a dense G_δ subset) in $X \setminus A$.

A more recent result of De Blasi, Myjak and Papini [3] establishes well-posedness of problem (P) for a uniformly convex X, closed A and a generic $x \in X$.

In this connection we recall that the minimization problem (P) is said to be well-posed if it has a unique solution, say a_0, and every minimizing sequence of (P) converges to a_0.

In the generic approach, instead of considering the existence of a solution to problem (P) for a single point $x \in X$, one investigates it for the whole space X and shows that solutions exist for most points in X. Such an approach is common in global analysis and the theory of dynamical systems (see, for example, [2], [10], [13], [14] and the references mentioned there). Recently it has also been used in the study of the structure of extremals of variational and optimal control problems [17], [18].

A more precise formulation of the De Blasi–Myjak–Papini result mentioned above involves the notion of porosity which we now present [1], [3], [14], [16].

Let (Y,d) be a complete metric space. We denote by $B(y,r)$ the closed ball of center $y \in Y$ and radius $r > 0$. A subset $E \subset Y$ is called porous (with respect to the metric d) if there exist $\alpha \in (0,1)$ and $r_0 > 0$ such that for each $r \in (0, r_0]$ and each $y \in Y$ there exists $z \in Y$ for which

$$B(z, \alpha r) \subset B(y,r) \setminus E.$$

A subset of the space Y is called σ-porous (with respect to d) if it is a countable union of porous subsets of Y.

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure 0 and are of the first category. Also, every Banach space contains a set of the first category which is not σ-porous.

To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B(z,s) \subset B(y,r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0,1)$ is a constant which depends only on E.

Using this terminology and denoting by F the set of all points such that the minimization problem (P) is well-posed, we note that De Blasi, Myjak and Papini [3] proved, in fact, that the complement $X \setminus F$ is σ-porous in X.

However, the fundamental restriction in all these results is that they hold only under certain assumptions on the space X. In view of the Lau–Konjagin result mentioned above these assumptions cannot be removed. On the other hand, many generic results in nonlinear functional analysis hold in any Banach space. Therefore the following natural question arises: Can generic results for best approximation problems be obtained in general Banach spaces? In the present paper we answer this question in the affirmative.

To this end, we change our point of view and consider a new framework. The main feature of this new framework is that the set A in problem (P) may also vary. In our first result (Theorem 2.1) we fix x and consider the space $S(X)$ of all nonempty closed subsets of X equipped with an appropriate complete metric, say h. We then show that the collection of all sets $A \in S(X)$ for which problem (P) is well-posed has a σ-porous complement.

In the second result (Theorem 2.2) we consider the space of pairs $S(X) \times X$ with the metric $h(A, B) + ||x - y||$, where $A, B \in S(X)$ and $x, y \in X$. Once again we show that the family of all pairs $(A, x) \in S(X) \times X$ for which problem (P) is well-posed has a σ-porous complement.

In our final result (Theorem 2.3) we show that for any nonempty separable closed subset X_0 of X, there exists a subset $\mathcal{F}$ of $(S(X), h)$ with a σ-porous complement such that any $A \in \mathcal{F}$ has the following property:

There exists a dense G_δ subset F of X_0 such that for any $x \in F$ the minimization problem (P) is well-posed.

The precise statements of these three theorems can be found in Section 2. Section 1 contains more information on porous sets and the class of hyperbolic spaces. Two auxiliary results are presented in Section 3. The proofs of Theorems 2.1–2.3 are given in Section 4.

1. Porous sets and hyperbolic spaces

In this section we provide more information on porous sets and the class of hyperbolic spaces. Let (Y, ρ) be a metric space. We denote by $B_\rho(y, r)$ the closed ball of center $y \in Y$ and radius $r > 0$.

The following simple observation was made in [19].

Proposition 1.1. *Let E be a subset of the metric space (Y, ρ). Assume that there exist $r_0 > 0$ and $\beta \in (0, 1)$ such that the following property holds:*

(P1) *For each $x \in Y$ and each $r \in (0, r_0]$ there exists $z \in Y \setminus E$ such that $\rho(x, z) \le r$ and $B_\rho(z, \beta r) \cap E = \emptyset$.*

Then E is porous with respect to ρ.

Proof. Let $x \in Y$ and $r \in (0, r_0]$. By property (P1) there exists $z \in Y \setminus E$ such that

$$\rho(x, z) \leq r/2 \quad \text{and} \quad B_\rho(z, \beta r/2) \cap E = \emptyset.$$

Hence $B_\rho(z, \beta r/2) \subset B_\rho(x, r) \setminus E$ and Proposition 1.1 is proved. □

As a matter of fact property (P1) can be weakened.

Proposition 1.2. *Let E be a subset of the metric space (Y, ρ). Assume that there exist $r_0 > 0$ and $\beta \in (0, 1)$ such that the following property holds:*

(P2) *For each $x \in E$ and each $r \in (0, r_0]$ there exists $z \in Y \setminus E$ such that $\rho(x, z) \leq r$ and $B_\rho(z, \beta r) \cap E = \emptyset$.*

Then E is porous with respect to ρ.

Proof. We may assume that $\beta < 1/2$. Let $x \in Y$ and $r \in (0, r_0]$. We will show that there exists $z \in Y \setminus E$ such that

$$\rho(x, z) \leq r \quad \text{and} \quad B_\rho(z, \beta r/2) \cap E = \emptyset. \tag{1.1}$$

If $B_\rho(x, r/4) \cap E = \emptyset$, then (1.1) holds with $z = x$. Assume now that $B_\rho(x, r/4) \cap E \neq \emptyset$. Then there exists

$$x_1 \in B_\rho(x, r/4) \cap E. \tag{1.2}$$

By property (P2) there exists $z \in Y \setminus E$ such that

$$\rho(x_1, z) \leq r/2 \quad \text{and} \quad B_\rho(z, \beta r/2) \cap E = \emptyset. \tag{1.3}$$

The relations (1.2) and (1.3) imply that

$$\rho(x, z) \leq \rho(x, x_1) + \rho(x_1, z) \leq 3r/4.$$

Thus there indeed exists $z \in Y \setminus E$ satisfying (1.1). Proposition 1.2 is now seen to follow from Proposition 1.1. □

The following definition was introduced in [19].

Assume that a set Y is equipped with two metrics ρ_1 and ρ_2 such that $\rho_1(x, y) \leq \rho_2(x, y)$ for all $x, y \in Y$ and that the metric spaces (Y, ρ_1) and (Y, ρ_2) are complete.

We say that a set $E \subset Y$ is porous with respect to the pair (ρ_1, ρ_2) if there exist $r_0 > 0$ and $\alpha \in (0, 1)$ such that for each $x \in E$ and each $r \in (0, r_0]$ there exists $z \in Y \setminus E$ such that $\rho_2(z, x) \leq r$ and $B_{\rho_1}(z, \alpha r) \cap E = \emptyset$.

Proposition 1.2 implies that if E is porous with respect to (ρ_1, ρ_2), then it is porous with respect to both ρ_1 and ρ_2.

A set $E \subset Y$ is called σ-porous with respect to (ρ_1, ρ_2) if it is a countable union of sets which are porous with respect to (ρ_1, ρ_2).

As a matter of fact it turns out that our results are true not only for Banach spaces but also for all complete hyperbolic spaces [7], [12]. We now recall the definition of this important class of spaces.

Let (X,ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c: R^1 \to X$ is a metric embedding of R^1 into X if $\rho(c(s),c(t)) = |s-t|$ for all real s and t. The image of R^1 under a metric embedding will be called a metric line. The image of a real interval $[a,b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping will be called a metric segment.

Assume that (X,ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x,y]$. For each $0 \le t \le 1$ there is a unique point z in $[x,y]$ such that

$$\rho(x,z) = t\rho(x,y) \quad \text{and} \quad \rho(z,y) = (1-t)\rho(x,y).$$

This point will be denoted by $(1-t)x \oplus ty$. We will say that X, or more precisely (X,ρ,M), is a hyperbolic space if

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\right) \le \frac{1}{2}\rho(y,z) \tag{1.4}$$

for all x, y and z in X. An equivalent requirement is that

$$\rho\left(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z\right) \le \frac{1}{2}(\rho(x,w) + \rho(y,z)) \tag{1.5}$$

for all x, y, z and w in X.

It is clear that all normed linear spaces are hyperbolic. A discussion of more examples of hyperbolic spaces and in particular of the Hilbert ball can be found, for example, in [5]–[7], [11], [12] and in references therein.

2. Main results

Let (X,ρ,M) be a complete hyperbolic space. For each $x \in X$ and each $A \subset X$ set

$$\rho(x,A) = \inf\{\rho(x,y) : y \in A\}.$$

Denote by $S(X)$ the family of all nonempty closed subsets of X. For each $A, B \in S(X)$ define

$$H(A,B) = \max\{\sup\{\rho(x,B) : x \in A\},\ \sup\{\rho(y,A) : y \in B\}\} \tag{2.1}$$

and

$$\widetilde{H}(A,B) = H(A,B)(1 + H(A,B))^{-1}.$$

Here we use the convention that $\infty/\infty = 1$. It is easy to see that $\widetilde{H}$ is a metric on $S(X)$ and that the space $(S(X),\widetilde{H})$ is complete.

Fix $\theta \in X$. For each natural number n and each $A, B \in S(X)$ we set

$$h_n(A,B) = \sup\{|\rho(x,A) - \rho(x,B)| : x \in X \text{ and } \rho(x,\theta) \le n\} \tag{2.2}$$

and

$$h(A,B) = \sum_{n=1}^{\infty} [2^{-n} h_n(A,B)(1 + h_n(A,B))^{-1}].$$

Once again it is not difficult to see that h is a metric on $S(X)$ and that the metric space $(S(X), h)$ is complete. Clearly

$$\widetilde{H}(A,B) \ge h(A,B) \quad \text{for all } A, B \in S(X).$$

We equip the set $S(X)$ with the pair of metrics $\widetilde{H}$ and h.

We now state our three main results. Their proofs will be given in Section 4.

Theorem 2.1. *Let (X, ρ, M) be a complete hyperbolic space and let $\widetilde{x} \in X$. Then there exists a set $\Omega \subset S(X)$ such that its complement $S(X) \setminus \Omega$ is σ-porous with respect to the pair $(h, \widetilde{H})$ and such that for each $A \in \Omega$ the following property holds:*

- (C1) *There exists a unique $\widetilde{y} \in A$ such that $\rho(\widetilde{x}, \widetilde{y}) = \rho(\widetilde{x}, A)$. Moreover, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in A$ satisfies $\rho(\widetilde{x}, x) \le \rho(\widetilde{x}, A) + \delta$, then $\rho(x, \widetilde{y}) \le \varepsilon$.*

To state our second result we endow the Cartesian product $S(X) \times X$ with the pair of metrics d_1 and d_2 defined by

$$\begin{aligned} d_1((A,x),(B,y)) &= h(A,B) + \rho(x,y), \\ d_2((A,x),(B,y)) &= \widetilde{H}(A,B) + \rho(x,y), \end{aligned}$$

for $x, y \in X$, $A, B \in S(X)$.

Theorem 2.2. *Let (X, ρ, M) be a complete hyperbolic space. There exists a set $\Omega \subset S(X) \times X$ such that its complement $[S(X) \times X] \setminus \Omega$ is σ-porous with respect to the pair (d_1, d_2) and such that for each $(A, \widetilde{x}) \in \Omega$ the following property holds:*

- (C2) *There exists a unique $\widetilde{y} \in A$ such that $\rho(\widetilde{x}, \widetilde{y}) = \rho(\widetilde{x}, A)$. Moreover, for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $z \in X$ satisfies $\rho(\widetilde{x}, z) \le \delta$, $B \in S(X)$ satisfies $h(A,B) \le \delta$, and $y \in B$ satisfies $\rho(y,z) \le \rho(z,B) + \delta$, then $\rho(y, \widetilde{y}) \le \varepsilon$.*

In classical generic results the set A was fixed and x varied in a dense G_δ subset of X. In our first two results the set A is also variable. However, in our final result we show that if X_0 is a nonempty closed separable subset of X, then for every fixed A in a dense G_δ subset of $S(X)$ with a σ-porous complement, the set of all $x \in X_0$ for which problem (P) is well-posed contains a dense G_δ subset of X_0.

Theorem 2.3. *Let (X,ρ,M) be a complete hyperbolic space. Assume that X_0 is a nonempty closed separable subset of X. Then there exists a set $\mathcal{F}\subset S(X)$ such that $S(X)\setminus\mathcal{F}$ is σ-porous with respect to the pair $(h,\widetilde{H})$ and such that for each $A\in\mathcal{F}$ the following property holds:*

(C3) *There exists a set $F\subset X_0$ which is a countable intersection of open everywhere dense subsets of X_0 with the relative topology such that for each $\widetilde{x}\in F$ there exists a unique $\widetilde{y}\in A$ for which $\rho(\widetilde{x},\widetilde{y})=\rho(\widetilde{x},A)$. Moreover, if $\{y_i\}_{i=1}^{\infty}\subset A$ satisfies $\lim_{i\to\infty}\rho(\widetilde{x},y_i)=\rho(\widetilde{x},A)$, then $y_i\to\widetilde{y}$ as $i\to\infty$.*

3. Auxiliary results

Let (X,ρ,M) be a complete hyperbolic space and let $S(X)$ be the family of all nonempty closed subsets of X.

Lemma 3.1. *Let $A\in S(X)$, $\widetilde{x}\in X$ and let $r,\varepsilon\in(0,1)$. Then there exists $\overline{x}\in X$ such that $\rho(\overline{x},A)\le r$ and for the set $\widetilde{A}=A\cup\{\overline{x}\}$ the following properties hold:*

$$\rho(\widetilde{x},\overline{x})=\rho(\widetilde{x},\widetilde{A}),$$
$$\text{if } x\in\widetilde{A} \text{ and } \rho(\widetilde{x},x)\le\rho(\widetilde{x},\widetilde{A})+\varepsilon r/4,\ \text{ then } \rho(\overline{x},x)\le\varepsilon.$$

Proof. If $\rho(\widetilde{x},A)\le r$, then the lemma holds with $\overline{x}=\widetilde{x}$ and $\widetilde{A}=A\cup\{\widetilde{x}\}$. Therefore we may retrict ourselves to the case where

$$\rho(\widetilde{x},A)>r. \tag{3.1}$$

Choose $x_0\in A$ such that

$$\rho(\widetilde{x},x_0)\le\rho(\widetilde{x},A)+r/2. \tag{3.2}$$

There exists

$$\overline{x}\in\{\gamma\widetilde{x}\oplus(1-\gamma)x_0:\gamma\in(0,1)\} \tag{3.3}$$

such that

$$\rho(\overline{x},x_0)=r \text{ and } \rho(\widetilde{x},\overline{x})=\rho(\widetilde{x},x_0)-r. \tag{3.4}$$

Set $\widetilde{A}=A\cup\{\overline{x}\}$. We have by (3.4) and (3.2),

$$\rho(\widetilde{x},\overline{x})=\rho(\widetilde{x},x_0)-r\le\rho(\widetilde{x},A)+r/2-r=\rho(\widetilde{x},A)-r/2.$$

Therefore $\rho(\widetilde{x},\overline{x})=\rho(\widetilde{x},\widetilde{A})$, and if $x\in\widetilde{A}$ and $\rho(\widetilde{x},x)<\rho(\widetilde{x},\widetilde{A})+r/2$, then $x=\overline{x}$. This completes the proof of Lemma 3.1.

Before stating our next lemma we choose, for each $\varepsilon\in(0,1)$ and each natural number n, a number

$$\alpha(\varepsilon,n)\in(0,16^{-n-2}\varepsilon). \tag{3.5}$$

Lemma 3.2. *Let $A \in S(X)$, $\widetilde{x} \in X$ and let $r, \varepsilon \in (0,1)$. Suppose that n is a natural number, let*

$$\alpha = \alpha(\varepsilon, n) \tag{3.6a}$$

and assume that

$$\rho(\widetilde{x}, \theta) \le n \quad \text{and} \quad \{z \in X : \rho(x,\theta) \le n\} \cap A \ne \emptyset. \tag{3.6b}$$

Then there exists $\overline{x} \in X$ such that $\rho(\overline{x}, A) \le r$ and such that the set $\widetilde{A} = A \cup \{\overline{x}\}$ has the following two properties:

$$\rho(\widetilde{x}, \overline{x}) = \rho(\widetilde{x}, \widetilde{A}); \tag{3.7}$$

if

$$\widetilde{y} \in X, \quad \rho(\widetilde{y}, \widetilde{x}) \le \alpha r, \tag{3.8}$$

$$B \in S(X), \quad h(\widetilde{A}, B) \le \alpha r, \tag{3.9}$$

and

$$z \in B, \quad \rho(\widetilde{y}, z) \le \rho(\widetilde{y}, B) + \varepsilon r/16, \tag{3.10}$$

then

$$\rho(z, \overline{x}) \le \varepsilon. \tag{3.11}$$

Proof. By Lemma 3.1 there exists $\overline{x} \in X$ such that

$$\rho(\overline{x}, A) \le r \tag{3.12}$$

and such that for the set $\widetilde{A} = A \cup \{\overline{x}\}$ the equality (3.7) is true and the following property holds:

$$\text{If } x \in \widetilde{A} \text{ and } \rho(\widetilde{x}, x) \le \rho(\widetilde{x}, \widetilde{A}) + \varepsilon r/8, \text{ then } \rho(\overline{x}, x) \le \varepsilon/2. \tag{3.13}$$

Assume that $\widetilde{y} \in X$ satisfies (3.8) and $B \in S(X)$ satisfies (3.9). We will show that

$$\rho(\widetilde{y}, B) < \rho(\widetilde{x}, \widetilde{A}) + 4\alpha r 16^n. \tag{3.14}$$

By (3.8),

$$|\rho(\widetilde{y}, \widetilde{A}) - \rho(\widetilde{x}, \widetilde{A})| \le \alpha r.$$

Combined with (3.7) this implies that

$$|\rho(\widetilde{y}, \widetilde{A}) - \rho(\widetilde{x}, \overline{x})| \le \alpha r. \tag{3.15}$$

The relations (3.7) and (3.6b) imply that

$$\rho(\widetilde{x}, \overline{x}) \le \rho(\widetilde{x}, A) \le 2n \quad \text{and} \quad \rho(\overline{x}, \theta) \le 3n. \tag{3.16}$$

It follows from (2.2) and (3.9) that

$$h_{4n}(\widetilde{A},B)(1+h_{4n}(\widetilde{A},B))^{-1} \le 2^{4n}h(\widetilde{A},B) \le 2^{4n}\alpha r$$

and combined with (3.5) and (3.6a) this implies that

$$h_{4n}(\widetilde{A},B) \le 2^{4n}\alpha r(1-2^{4n}\alpha r)^{-1} < 2^{4n+1}\alpha r. \tag{3.17}$$

Since $\overline{x} \in \widetilde{A}$ it now follows from (3.17), (3.16) and (2.2) that $\rho(\overline{x},B) < 2^{4n+1}\alpha r$ and there exists $\overline{y} \in X$ such that

$$\overline{y} \in B \quad \text{and} \quad \rho(\overline{x},\overline{y}) < 2\alpha r 16^n. \tag{3.18}$$

By (3.18), (3.8) and (3.7),

$$\begin{aligned}\rho(\widetilde{y},B) &\le \rho(\widetilde{y},\overline{y}) \le \rho(\widetilde{y},\overline{x}) + \rho(\overline{x},\overline{y})\\ &< \rho(\widetilde{y},\widetilde{x}) + \rho(\widetilde{x},\overline{x}) + 2\alpha r16^n \le 2\alpha r16^n + \alpha r + \rho(\widetilde{x},\widetilde{A}).\end{aligned}$$

This certainly implies (3.14), as claimed.

Assume now that $z \in B$ satisfies (3.10). It follows from (3.10), (3.14), (3.6a) and (3.5) that

$$\rho(\widetilde{y},z) \le \rho(\widetilde{y},B) + \varepsilon r/16 \le \rho(\widetilde{x},\widetilde{A}) + 4\alpha r16^n + \varepsilon r/16 \le \rho(\widetilde{x},\widetilde{A}) + \varepsilon r/8. \tag{3.19}$$

The relations (3.19), (3.16) and (3.7) imply that

$$\rho(\widetilde{y},z) \le \rho(\widetilde{x},\widetilde{A}) + \varepsilon r/8 \le 2n + r/8. \tag{3.20}$$

By (3.20), (3.8), (3.6a) and (3.6b),

$$\begin{aligned}\rho(z,\theta) &\le \rho(z,\widetilde{y}) + \rho(\widetilde{y},\theta) \le 2n + r/8 + \rho(\widetilde{y},\theta)\\ &\le 2n + r/8 + \rho(\widetilde{y},\widetilde{x}) + \rho(\widetilde{x},\theta) \le 2n + r/8 + \alpha r + n \le 4n.\end{aligned} \tag{3.21}$$

It follows from (3.17), (2.2), (3.10) and (3.21) that

$$\rho(z,\widetilde{A}) = |\rho(z,\widetilde{A}) - \rho(z,B)| \le h_{4n}(\widetilde{A},B) < 2\alpha r16^n.$$

Hence there exists $\widetilde{z} \in X$ such that

$$\widetilde{z} \in \widetilde{A} \quad \text{and} \quad \rho(z,\widetilde{z}) < 2\alpha r16^n. \tag{3.22}$$

By (3.8), (3.22) and (3.10) we have

$$\begin{aligned}\rho(\widetilde{x},\widetilde{z}) &\le \rho(\widetilde{x},\widetilde{y}) + \rho(\widetilde{y},z) + \rho(z,\widetilde{z})\\ &\le \alpha r + \rho(\widetilde{y},z) + 2\alpha r16^n \le \alpha r + 2\alpha r16^n + \rho(\widetilde{y},B) + \varepsilon r/16.\end{aligned}$$

It follows from this inequality, (3.14), (3.6a) and (3.5) that

$$\begin{aligned}\rho(\widetilde{x},\widetilde{z}) &\le \alpha r + 2\alpha r16^n + \varepsilon r/16 + \rho(\widetilde{x},\widetilde{A}) + 4\alpha r16^n\\ &\le \rho(\widetilde{x},\widetilde{A}) + 8\alpha r16^n + \varepsilon r/16 \le \rho(\widetilde{x},\widetilde{A}) + \varepsilon r/8.\end{aligned}$$

Thus

$$\rho(\widetilde{x},\widetilde{z}) \leq \rho(\widetilde{x},\widetilde{A}) + \varepsilon r/8.$$

Using this inequality, (3.22) and (3.13) we see that $\rho(\overline{x},\widetilde{z}) \leq \varepsilon/2$. Combining this fact with (3.22), (3.6a) and (3.5) we conclude that

$$\rho(z,\overline{x}) \leq \rho(z,\widetilde{z}) + \rho(\widetilde{z},\overline{x}) \leq 2\alpha r 16^n + \varepsilon/2 \leq \varepsilon.$$

Thus (3.11) holds and Lemma 3.2 is proved.

4. Proofs of Theorems 2.1–2.3

Proof of Theorem 2.1. For each integer $k \geq 1$ denote by Ω_k the set of all $A \in S(X)$ which have the following property:

(P3) There exist $x_A \in X$ and $\delta_A > 0$ such that if $x \in A$ satisfies $\rho(x,\widetilde{x}) \leq \rho(\widetilde{x},A) + \delta_A$, then $\rho(x,x_A) \leq 1/k$.

Clearly $\Omega_{k+1} \subset \Omega_k$, $k = 1,2,\ldots$ Set

$$\Omega = \bigcap_{k=1}^{\infty} \Omega_k.$$

First we will show that $S(X)\setminus\Omega$ is σ-porous with respect to the pair $(h,\widetilde{H})$. To meet this goal it is sufficient to show that $S(X)\setminus\Omega_k$ is σ-porous with respect to $(h,\widetilde{H})$ for all sufficiently large integers k.

There exists a natural number k_0 such that $\rho(\theta,\widetilde{x}) \leq k_0$. Let $k \geq k_0$ be an integer. We will show that the set $S(X)\setminus\Omega_k$ is σ-porous with respect to $(h,\widetilde{H})$. For each integer $n \geq k_0$ set

$$E_{nk} = \{A \in S(X)\setminus\Omega_k : \{z \in X : \rho(z,\theta) \leq n\} \cap A \neq \emptyset\}.$$

By Lemma 3.2 the set E_{nk} is porous with respect to $(h,\widetilde{H})$ for all integers $n \geq k_0$. Since $S(X)\setminus\Omega_k = \bigcup_{n=k_0}^{\infty} E_{nk}$ we conclude that $S(X)\setminus\Omega_k$ is σ-porous with respect to $(h,\widetilde{H})$. Therefore $S(X)\setminus\Omega$ is also σ-porous with respect to $(h,\widetilde{H})$.

Let $A \in \Omega$. We will show that A has property (C1). By the definition of Ω_k and property (P3), for each integer $k \geq 1$ there exist $x_k \in X$ and $\delta_k > 0$ such that the following property holds:

(P4) If $x \in A$ satisfies $\rho(x,\widetilde{x}) \leq \rho(\widetilde{x},A) + \delta_k$, then $\rho(x,x_k) \leq 1/k$.

Let $\{z_i\}_{i=1}^{\infty} \subset A$ be such that

$$\lim_{i\to\infty} \rho(\widetilde{x},z_i) = \rho(\widetilde{x},A). \tag{4.1}$$

Fix an integer $k \geq 1$. It follows from property (P4) that for all large enough natural numbers i,

$$\rho(\widetilde{x},z_i) \leq \rho(\widetilde{x},A) + \delta_k \quad \text{and} \quad \rho(z_i,x_k) \leq 1/k.$$

Since k is an arbitrary natural number we conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to some $\widetilde{y} \in A$. Clearly $\rho(\widetilde{x}, \widetilde{y}) = \rho(\widetilde{x}, A)$. If the minimizer $\widetilde{y}$ were not unique we would be able to construct nonconvergent minimizing sequence $\{z_i\}_{i=1}^{\infty}$. Thus $\widetilde{y}$ is the unique solution to problem (P) (with $x = \widetilde{x}$) and any sequence $\{z_i\}_{i=1}^{\infty} \subset A$ satisfying (4.1) converges to $\widetilde{y}$. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. For each integer $k \geq 1$ denote by Ω_k the set of all $(A, \widetilde{x}) \in S(X) \times X$ which have the following property:

(P5) There exist $\overline{x} \in X$ and $\overline{\delta} > 0$ such that if $x \in X$ satisfies $\rho(x, \widetilde{x}) \leq \overline{\delta}$, $B \in S(X)$ satisfies $h(A, B) \leq \overline{\delta}$, and $y \in B$ satisfies $\rho(y, x) \leq \rho(x, B) + \overline{\delta}$, then $\rho(y, \overline{x}) \leq 1/k$.

Clearly $\Omega_{k+1} \subset \Omega_k$, $k = 1, 2, \ldots$ Set

$$\Omega = \bigcap_{k=1}^{\infty} \Omega_k.$$

First we will show that $[S(X) \times X] \setminus \Omega$ is σ-porous with respect to the pair (d_1, d_2). For each pair of natural numbers n and k set

$$E_{nk} = \{(A, x) \in [S(X) \times X] \setminus \Omega_k : \rho(x, \theta) \leq n,\ B_\rho(\theta, n) \cap A \neq \emptyset\}.$$

By Lemma 3.2 the set E_{nk} is porous with respect to (d_1, d_2) for all natural numbers n and k. Since

$$[S(X) \times X] \setminus \Omega = \bigcup_{k=1}^{\infty} ([S(X) \times X] \setminus \Omega_k) = \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} E_{nk},$$

the set $[S(X) \times X] \setminus \Omega$ is σ-porous with respect to (d_1, d_2), by definition.

Let $(A, \widetilde{x}) \in \Omega$. We will show that $(A, \widetilde{x})$ has property (C2).

By the definition of Ω_k and property (P5), for each integer $k \geq 1$ there exist $x_k \in X$ and $\delta_k > 0$ with the following property:

(P6) If $x \in X$ satisfies $\rho(x, \widetilde{x}) \leq \delta_k$, $B \in S(X)$ satisfies $h(A, B) \leq \delta_k$, and $y \in B$ satisfies $\rho(y, x) \leq \rho(x, B) + \delta_k$, then $\rho(y, x_k) \leq 1/k$.

Let $\{z_i\}_{i=1}^{\infty} \subset A$ be such that

$$\lim_{i \to \infty} \rho(\widetilde{x}, z_i) = \rho(\widetilde{x}, A). \tag{4.2}$$

Fix an integer $k \geq 1$. It follows from property (P6) that for all large enough natural numbers i,

$$\rho(\widetilde{x}, z_i) \leq \rho(\widetilde{x}, A) + \delta_k \quad \text{and} \quad \rho(z_i, x_k) \leq 1/k.$$

Since k is an arbitrary natural number we conclude that $\{z_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to some $\widetilde{y} \in A$. Clearly $\rho(\widetilde{x}, \widetilde{y}) = \rho(\widetilde{x}, A)$. It is not difficult to see that $\widetilde{y}$ is the unique solution to the minimization problem (P) with $x = \widetilde{x}$.

Let $\varepsilon > 0$. Choose a natural number $k > 4/\min\{1, \varepsilon\}$. By property (P6),

$$\rho(\widetilde{y}, x_k) \le 1/k. \tag{4.3}$$

Assume that $z \in X$ satisfies $\rho(z, \widetilde{x}) \le \delta_k$, $B \in S(X)$ satisfies $h(A, B) \le \delta_k$ and $y \in B$ satisfies $\rho(y, z) \le \rho(z, B) + \delta_k$. Then it follows from property (P6) that $\rho(y, x_k) \le 1/k$. Combined with (4.3) this implies that $\rho(y, \widetilde{y}) \le 2/k < \varepsilon$. This completes the proof of Theorem 2.2. □

Proof of Theorem 2.3. Let $\{x_i\}_{i=1}^{\infty} \subset X_0$ be an everywhere dense subset of X_0. For each natural number p there exists a set $\mathcal{F}_p \subset S(X)$ such that Theorem 2.1 holds with $\widetilde{x} = x_p$ and $\Omega = \mathcal{F}_p$. Set $\mathcal{F} = \bigcap_{p=1}^{\infty} \mathcal{F}_p$. Clearly $S(X) \setminus \mathcal{F}$ is σ-porous with respect to the pair $(h, \widetilde{H})$.

Let $A \in \mathcal{F}$ and let $p \ge 1$ be an integer. By Theorem 2.1, which holds with $\widetilde{x} = x_p$ and $\Omega = \mathcal{F}_p$, there exists a unique $\overline{x}_p \in A$ such that

$$\rho(x_p, \overline{x}_p) = \rho(x_p, A) \tag{4.4}$$

and the following property holds:

(P7) For each integer $k \ge 1$ there exists $\delta(p, k) > 0$ such that if $x \in A$ satisfies $\rho(x, x_p) \le \rho(x_p, A) + 4\delta(p, k)$, then $\rho(x, \overline{x}_p) \le 1/k$.

For each pair of natural numbers p and k set

$$V(p, k) = \{z \in X_0 : \rho(z, x_p) < \delta(p, k)\}.$$

It follows from property (P7) that for each pair of integers $p, k \ge 1$ the following property holds:

(P8) If $x \in A$, $z \in X_0$, $\rho(z, x_p) \le \delta(p, k)$ and $\rho(z, x) \le \rho(z, A) + \delta(p, k)$, then $\rho(x, \overline{x}_p) \le 1/k$.

Set

$$F = \bigcap_{k=1}^{\infty} \Big[\bigcup \{V(p, k) : p = 1, 2, \dots\} \Big].$$

Clearly F is a countable intersection of open everywhere dense subsets of X_0.

Let $x \in F$. Consider a sequence $\{x_i\}_{i=1}^{\infty} \subset A$ such that

$$\lim_{i \to \infty} \rho(x, x_i) = \rho(x, A). \tag{4.5}$$

Let $\varepsilon > 0$. Choose a natural number $k > 8^{-1}/\min\{1, \varepsilon\}$. There exists an integer $p \ge 1$ such that $x \in V(p, k)$. By the definition of $V(p, k)$, $\rho(x, x_p) < \delta(p, k)$. It follows from this inequality and property (P8) that for all sufficiently large integers i, $\rho(x, x_i) \le \rho(x, A) + \delta(p, k)$ and $\rho(x_i, \overline{x}_p) \le 1/k < \varepsilon/2$. Since ε is an arbitrary positive number we conclude that $\{x_i\}_{i=1}^{\infty}$ is a Cauchy sequence which converges to $\widetilde{y} \in A$. Clearly $\widetilde{y}$ is the unique minimizer of the minimization problem $z \to \rho(x,z)$, $z \in A$. Note that we have shown that any sequence $\{x_i\}_{i=1}^{\infty} \subset A$ satisfying (4.5) converges to $\widetilde{y}$. This completes the proof of Theorem 2.3. □

Remark. As a matter of fact, Theorems 2.1–2.3 hold for a wider class of complete metric spaces. This is because the inequality (1.4) (equivalently, (1.5)) was not used in their proofs. We emphasize, however, that all our results are new even in Banach spaces.

Note added in the proof. It has recently come to our attention that F. S. De Blasi and J. Myjak used the product space approach to obtain genericity results pertaining to best approximation problems in Banach spaces. See their papers *On the minimum distance theorem to a closed convex set in Banach space*, Bull. Acad. Polon. Sci. 29 (1981), 373–376, and *On almost well posed problems in the theory of best approximation*, Bull. Math. Soc. Sci. Math. R. S. Roum 28 (1984), 109–117.

Acknowledgments. The work of the first author was partially supported by the Fund for the Promotion of Research at the Technion and by the Technion VPR Fund – E. and M. Mendelson Research Fund.

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Manuscript received July 14, 2000

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