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ALMOST-PERIODICITY PROBLEM AS A FIXED-POINT PROBLEM FOR EVOLUTION INCLUSIONS

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Abstract. Existence of almost-periodic solutions to quasi-linear evolution inclusions under a Stepanov almost-periodic forcing is nontraditionally examined by means of the Banach-like and the Schauder–Tikhonov-like fixed-point theorems. These multivalued fixed-point principles concern condensing operators in almost-periodic function spaces or their suitable closed subsets. The Bohr–Neugebauer-type theorem jointly with the Bochner transform are employed, besides another, for this purpose. Obstructions related to possible generalizations are discussed.

1. Introduction (fixed-point theorems)

In [19], A. M. Fink devotes the whole Chapter 8 to application of fixed-point methods for obtaining almost-periodic solutions of differential equations. More precisely, the applications of the Banach contraction principle and the Schauder fixed-point theorem are there discussed. This approach is rather rare, but efficient (see e.g. [4], [6], [9], [11], [18], [20]). As we have already pointed out in [1], [2], [7], [4], the Schauder–Tikhonov theorem is however more appropriate than the Schauder theorem, because a suitable closed subset of an almost-periodic

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function space jointly with the compactness of related operators must be guaranteed at the same time.

In this paper, we would like to employ the same approach, but for differential inclusions in a Banach space (i.e. for evolution inclusions). Therefore, we need more general fixed-point principles.

The Banach-like fixed-point theorem for multivalued contractions is due to H. Covitz and S. B. Nadler [14].

Theorem 1 ([14]). *If X is a complete metric space and $F\colon X \to 2^X \setminus \{\emptyset\}$ is a* (*multivalued*) *contraction with nonempty closed values, namely*

$$d_H(F(x), F(y)) \leq L d(x,y) \quad \text{for all } x, y \in X,$$

where $L \in [0,1)$ and d_H stands for the Hausdorff metric, then F has a fixed-point, i.e. there exists $\widehat{x} \in X$ such that $\widehat{x} \in F(\widehat{x})$.

Since a closed subset of a complete metric space is complete, Theorem 1 can be immediately reformulated as follows.

Theorem 1'. *If X is a closed subset of a complete metric space and $F\colon X \to 2^X \setminus \{\emptyset\}$ is a contraction with nonempty closed values, then F has a fixed-point.*

The following Schauder–Tikhonov-like fixed-point theorem for condensing multivalued mappings in Fréchet spaces represents a particular case of a more general statement in [3] (cf. also [28], [29]).

Let $\mathcal{M}$ be a class of subsets of a Fréchet space E such that if $\Omega \in \mathcal{M}$, then also $\overline{\mathrm{co}}\,\Omega \in \mathcal{M}$ (where $\overline{\mathrm{co}}$ stands for the closed convex hull). Let $K = (K, \geq)$ be a cone of some vector space with the natural partial ordering (i.e. $x \leq y$, whenever $y - x \in K$). We say that $\beta\colon \mathcal{M} \to K$ is a *measure of noncompactness* in E (see [24], [30]) if $\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega)$, for every $\Omega \in \mathcal{M}$. β is called:

(i) *monotone* if $\Omega_0, \Omega_1 \in \mathcal{M}$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$,

(ii) *nonsingular* if $\{a\}, \Omega \in \mathcal{M}$ implies $\{a\} \cup \Omega \in \mathcal{M}$ and $\beta(\{a\} \cup \Omega) = \beta(\Omega)$.

As a particular case of a measure of noncompactness, satisfying (i), (ii), which is available in any locally convex topological vector space (e.g. a Fréchet space), we point out the *Hausdorff measure of noncompactness* $\gamma\colon \mathcal{M} \to K$ defined by

$$\begin{aligned}\gamma(\Omega)(p) := \inf\{d > 0 : \Omega &\text{ is the union of finitely many balls}\\ &\text{with radius (w.r.t. a seminorm } p\text{) less than } d\}.\end{aligned}$$

Here $\mathcal{M}$ denotes the class of all bounded subsets of E and K is a cone in the vector space of real-valued functions k on a family of seminorms P, generating the locally convex topology, i.e. $k\colon P \to [0, \infty)$.

An upper-semi-continuous mapping $F: E \supset D \to 2^E \setminus \{\emptyset\}$ is said to be *β-condensing* (or, in particular, *γ-condensing*) if $\Omega \subset D$ implies that $\Omega, F(\Omega) \in \mathcal{M}$ and if Ω satisfying the inequality

$$\beta(F(\Omega)) \geq \beta(\Omega) \quad (\text{or } \gamma(F(\Omega)) \geq \gamma(\Omega))$$

implies that Ω is relatively compact.

Theorem 2 ([3]). *Let X be a closed convex subset of a Fréchet space E and let $F: X \to 2^X \setminus \{\emptyset\}$ be an R_δ-mapping (i.e. upper-semi-continuous mapping with R_δ-values) which is β-condensing w.r.t. a monotone, nonsingular measure of noncompactness β on E. Then F has a fixed-point, i.e. there exists $\widehat{x} \in X$ such that $\widehat{x} \in F(\widehat{x})$.*

Let us note that a slightly weaker version of Theorem 2 can be also deduced from our fixed-point theorems in [2], [5], on the basis of the statement in [26]. More precisely, the existence of a nonempty, compact, convex subset $X_0 \subset X$ is implied such that $F(X_0) \subset X_0$, provided the measure of noncompactness is still regular (i.e. $\beta(\Omega) = 0$ if and only if Ω is relatively compact, for every $\Omega \in \mathcal{M}$) and semiadditive (i.e. $\beta(\Omega_0 \cup \Omega_1) = \max\{\beta(\omega_0), \beta(\Omega_1)\}$ for all $\Omega_0, \Omega_1 \in \mathcal{M}$).

Assuming that $F: X \to 2^X \setminus \{\emptyset\}$ is additionally a contraction (w.r.t. all seminorms of E which, however, does not automatically mean the contraction in a metric of E, see [21]) on a closed bounded subset X of a Fréchet space E, it follows (cf. e.g. [24], [30]) that F is γ-condensing w.r.t. the Hausdorff measure of noncompactness γ. So, we can still give

Theorem 2'. *Let X be a closed, bounded and convex subset of a Fréchet space. Let $F: X \to 2^X \setminus \{\emptyset\}$ be a contraction with R_δ-values (i.e., in particular, with nonempty, compact and connected values). Then F has a fixed-point.*

Comparing Theorem 2' to Theorem 1', the assumptions of Theorem 2' might seem to be (in spite of the fact that, in Theorem 1', a contraction is w.r.t. the metric of X) rather restrictive and partially superfluous. Moreover, in order to apply Theorem 2 without further restrictions (like contractivity), verifying that F is mapped into a suitable closed subset of a complete space of almost-periodic functions is rather difficult (for more details, see [4]). On the other hand, there are situations, when Theorem 2' applies, but not Theorem 1'. Let us also note that, for a single-valued F, Theorem 2' is a particular case of a fixed-point theorem in [13].

In our paper, before applying Theorem 1 in Chapter 3 and Theorem 2 in Chapter 4, some further auxiliary results, definitions, notations, etc., are presented in Chapter 2. Several concluding remarks are added in Chapter 5.

2. Auxiliary results (almost-periodic functions)

The notion of almost-periodicity is understood here in the sense of V. V. Stepanov. We say that a locally Bochner integrable function $f \in L_{\mathrm{loc}}(\mathbb{R}, B)$, where B is a real separable Banach space, is *Stepanov almost-periodic* (S-a.p.) if the following is true:

- for all $\varepsilon > 0$ there exists $k > 0$ and for all $a \in \mathbb{R}$ there exists $\tau \in [a, a+k]$ such that
$$D_S(f(t+\tau), f(t)) < \varepsilon,$$
where $D_S(f,g) := \sup_{a\in\mathbb{R}} \int_a^{a+1} |f(t) - g(t)|\, dt$ stands for the Stepanov metric and $|\cdot|$ is the norm in B.

It is well-known (see e.g. [8], [27]) that the space S of S-a.p. functions is Banach and that, for a uniformly continuous $f \in C(\mathbb{R}, B)$, S-a.p. means *uniform almost-periodicity* (a.p.), namely

- for all $\varepsilon > 0$ there exists $k > 0$ and for all $a \in \mathbb{R}$ there exists $\tau \in [a, a+k]$ such that
$$\|f(t+\tau) - f(t)\| < \varepsilon,$$
where $\|\cdot\| := \sup_{t\in\mathbb{R}} \|\cdot\|$.

Denoting $\|f\|_S = D_S(f, 0)$, let us still consider the real Banach space $BS = \{f \in L_{\mathrm{loc}}(\mathbb{R}, B) \mid \|f\|_S < \infty\}$. Obviously, $S \subset BS$.

The *Bochner transform* (see e.g. [8], [27])
$$f^b(t) := f(t+\eta), \quad \eta \in [0,1], \quad t \in \mathbb{R},$$
associates to each $t \in \mathbb{R}$ a function defined on $[0,1]$ and
$$f^b \in L_{\mathrm{loc}}(\mathbb{R}, L([0,1])), \quad \text{whenever } f \in L_{\mathrm{loc}}(\mathbb{R}, B).$$
Thus, $BS = \{f \in L_{\mathrm{loc}}(\mathbb{R}, B) \mid f^b \in L^\infty(\mathbb{R}, L([0,1]))\}$, because
$$\|f\|_S = \|f^b\|_{L^\infty} = \sup\,\mathrm{ess}_{t\in\mathbb{R}} \|f^b(t)\|_{L([0,1])} = \sup\,\mathrm{ess}_{t\in\mathbb{R}} \int_0^1 |f(t+\eta)|\, d\eta.$$

Since still (see again e.g. [8], [27])
$$f^b \in C(\mathbb{R}, L([0,1])), \quad \text{where } f \in L_{\mathrm{loc}}(\mathbb{R}, B),$$
we arrive at
$$BS = \{f \in L_{\mathrm{loc}}(\mathbb{R}, B) \mid f^b \in BC(\mathbb{R}, L([0,1]))\},$$
where BC denotes the space of bounded and continuous functions.

S. Bochner has shown for the space S of S-a.p. functions that (see e.g. [8, pp. 76–78])
$$S = \{f \in L_{\mathrm{loc}}(\mathbb{R}, B) \mid f^b \in C_{\mathrm{ap}}(\mathbb{R}, L([0,1]))\},$$

where C_{ap} means the space of uniformly almost-periodic functions. This important property, jointly with obvious relations

$$\|f\|_S = \|f^b\|_{BC(\mathbb{R},L([0,1]))}$$

and $f_n \xrightarrow{S} f$ inf and only if $f_n^b \xrightarrow{BC} f^b$, will play an important role in the sequel.

Defining, for a given S-a.p. function $f \in S$, the sets

$$\begin{aligned}\Omega_f &:= \{(\varepsilon,k,a,\tau) \in \mathbb{R}^4 \mid \tau \in [a,a+k] \text{ and } \|f(t+\tau)-f(t)\|_S < \varepsilon\},\\ M_f &:= \{g \in C(\mathbb{R},B) \mid (\varepsilon,k,a,\tau) \in \Omega_f \ \Rightarrow \ \text{for all } t \in \mathbb{R} \ |g(t+\tau)-g(t)| < \varepsilon\},\end{aligned}$$

we can state the following lemma (observe that if $g \in M_f$, then $g \in C_{\mathrm{ap}}$), which is essential in applying Theorem 2 (or Theorem 2').

Lemma 1. *M_f is a closed subset* (*in the topology of the uniform convergence on compact subintervals of* $\mathbb{R}$) *of* $C(\mathbb{R},B)$.

Proof. Let $M_f \ni g_l \xrightarrow{\mathrm{loc}} g$ hold on $\mathbb{R}$, by which $g \in C(\mathbb{R},B)$. Assume that $(\varepsilon,k,a,\tau) \in \Omega_f$, for all $t \in \mathbb{R}$. Then, for each $\delta > 0$, then exists l_0 such that, for all $l > l_0$, we have

$$|g(t+\tau)-g_l(t+\tau)| < \delta/2 \ \text{ and } \ |g(t)-g_l(t)| < \delta/2.$$

It follows from the inequality

$$|g(t+\tau)-g(t)| \le |g(t+\tau)-g_l(t+\tau)| + |g_l(t+\tau)-g_l(t)| + |g_l(t)-g(t)|$$

that

$$l > l_0 : |g(t+\tau)-g(t)| < \delta + \varepsilon.$$

Since δ can be chosen arbitrarily small, we arrive at $|g(t+\tau)-g(t)| < \varepsilon$. $\square$

Remark 1. Using the Bochner transform, one can prove quite analogously that M'_f is a closed subset of $L_{\mathrm{loc}}(\mathbb{R},B)$, where

$$\begin{aligned}\Omega'_f &:= \{(\varepsilon,k,a,\tau) \in \mathbb{R}^4 \mid \text{for all } t \in \mathbb{R} \ \tau \in [a,a+k]\\ &\qquad \text{and } |f^b(t+\tau)-f^b(t)| < \varepsilon\},\\ M'_f &:= \{g \in L_{\mathrm{loc}}(\mathbb{R},B) \mid (\varepsilon,k,a,\tau) \in \Omega'_f\\ &\qquad \Rightarrow \text{for all } t \in \mathbb{R} \ |g^b(t+\tau)-g^b(t)| < \varepsilon\}.\end{aligned}$$

Moreover, M_f as well as M'_f can be proved to be convex (cf. [4]).

Observe that if $\alpha \in [-1,1]$, $\beta \in \mathbb{R}$, then $\alpha M_f + \beta \subset M_f$, and subsequently $M_f \cap Q_C$ is convex, closed subset of $C(\mathbb{R},B)$, where $Q_C := \{g \in C(\mathbb{R},B) \mid \sup_{t\in\mathbb{R}} |g(t)| \le C\}$.

The following Bohr–Neugebauer-type statement is true.

Lemma 2. *Consider the linear evolution equation in the Banach space B:*

$$X' + AX = P(t), \tag{1}$$

where $A: B \to B$ is a linear, bounded operator whose spectrum does not intersect the imaginary axis and $P \in S$ is an essentially bounded S-a.p. *function. Then* (1) *possesses a unique uniformly* a.p. *solution $X(t) \in AC_{\mathrm{loc}}(\mathbb{R}, B)$ of the form*

$$X(t) = \int_{-\infty}^{\infty} G(t-s)P(s)\,ds, \tag{2}$$

where $G(t-s)$ is the principal Green function for (1)*, which takes the form*

$$G(t-s) = \begin{cases} e^{A(t-s)}P_- & \text{for } t > s, \\ -e^{A(t-s)}P_+ & \text{for } t < s, \end{cases}$$

and P_-, P_+ stand for the corresponding spectral projections to the invariant subspaces of A (*for more details, see e.g.* [16, pp. 79–81]).

Proof. It is well-known (see e.g. [16], [25]) that, under the above assumptions, equation (1) has exactly one solution $X(t)$ of the form (2).

In order to prove that $X(t)$ is uniformly a.p., we will equivalently show (see e.g. [8], [4]) that the set of functions $X_\tau(t) := X(t+\tau)$, $\tau \in \mathbb{R}$, is precompact in the topology $\|X\|_S = \|X^b\|_{BC(\mathbb{R}, L([0,1]))}$.

Since $P(t)$ is S-a.p., we can choose from the sequence $\{P_{-\tau_k}(t)\}$ a Cauchy subsequence $\{P_{-\tau_{k_j}}(t)\}$. Having apparently

$$X_{\tau_{k_j}}(t) = X(t+\tau_{k_j}) = \int_{-\infty}^{\infty} G(t-s)P(s-\tau_{k_j})ds = \int_{-\infty}^{\infty} G(t-s)P_{-\tau_{k_j}}(s)ds,$$

it follows that $X_{\tau_{k_j}}(t)$ is a Cauchy sequence (in the BC-topology) as well. In fact (cf. [16, p. 88]),

$$\begin{aligned} \|X^b_{\tau_{k_j}}(t) - {} & X^b_{\tau_{k_i}}(t)\|_{BC} \\ &= \left\| \int_{-\infty}^{\infty} G^b(t-s)[P_{-\tau_{k_j}}(s) - P_{-\tau_{k_i}}(s)]\,ds \right\|_{BC} \\ &\le \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{\infty} |G(t-s)| \|P^b_{-\tau_{k_j}}(s) - P^b_{-\tau_{k_i}}(s)\|_{BC}\,ds \right| \\ &\le \|P^b_{-\tau_{k_j}}(t) - P^b_{-\tau_{k_i}}(t)\|_{BC} \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{\infty} |G(t-s)|\,ds \right| \\ &\le C(A)\|P^b_{-\tau_{k_j}}(t) - P^b_{-\tau_{k_i}}(t)\|_{BC}, \end{aligned}$$

where $C(A)$ is a finite constant depending only on A.

This already means that the set of functions $X_\tau(t)$ is precompact, which completes the proof. $\square$

Remark 2. Another Bohr–Neugebauer-type theorem has been proved in [12]. Although this theorem even deals with (1), where $A = A(t, X)$ can be time-dependent and nonlinear, it only applies to our situation in particular cases (see e.g. [6]).

Remark 3. For $B = \mathbb{R}^n$, the $(n \times n)$-matrix A can be arbitrary in order every entirely bounded solution of (1) to be (uniformly) a.p. (see e.g. [19, p. 86]). In $\mathbb{R}^2$, A can be, more generally, a maximal monotone operator for the same goal (see [22]).

3. Banach-like approach

In this chapter, Theorem 1 will be applied to the differential inclusion in a real separable Banach space with the norm $|\cdot|$, namely

$$X' + AX \in F(X) + \Sigma(t), \tag{3}$$

where $A: B \to B$ is again a (single-valued) bounded, linear operator whose spectrum does not intersect the imaginary axis, $F: B \to 2^B \setminus \{\emptyset\}$ is a Lipschitz-continuous multifunction with bounded, closed, convex values and $\Sigma: \mathbb{R} \to 2^B \setminus \{\emptyset\}$ is an essentially bounded S-a.p. multifunction with closed, convex values. By a *solution* $X(t)$ of (3) we mean everywhere the function belonging to the class $AC_{\mathrm{loc}}(\mathbb{R}, B)$ and satisfying (3) almost everywhere.

Let us recall that by the *Lipschitz-continuity* of F we mean:

$$\exists L \in [0, \infty) : d_H(F(X), F(Y)) \leq L|X - Y| \quad \text{for all } X, Y \in B,$$

where $d_H(\,\cdot\,,\,\cdot\,)$ stands for the Hausdorff metric, and by an *S-a.p. multifunction* Σ the measurable one (i.e. $\{t \in \mathbb{R} \mid \Sigma(t) \subset U\}$ is a measurable set, for each open $U \in B$) satisfying that, for every $\varepsilon > 0$, there exists a positive number $k = k(\varepsilon)$ such that, in each interval of the length k, there is at least one number τ with

$$\sup_{a \in \mathbb{R}} \int_a^{a+1} d_H(\Sigma(t), \Sigma(t+\tau))\, dt < \varepsilon.$$

Let us note that F admits, under our assumptions, a Lipschitz-continuous selection $(F \supset) f: B \to B$ if and only if B is finite dimensional (see [23, p. 101]). However, even for $B = \mathbb{R}^n$, the Lipschitz constant need not be the same (for the related estimates and more details, see [23, pp. 101–103]). On the other hand, although a uniformly a.p. multifunction need not admit a uniformly a.p. selection (see [10]), $S(t)$ possesses (see [15], [17]) an S-a.p. selection $\sigma \subset \Sigma$.

Hence, consider still a one-parameter family of linear inclusions

$$X' + AX \in F(q(t)) + \sigma(t), \quad q \in Q, \tag{4}$$

where $\sigma \subset \Sigma$ is an (existing) S-a.p. selection and Q is the Banach space of uniformly a.p. functions $q \in C(\mathbb{R}, B)$.

Since one can easily check the composition $F(q)$ to be S-a.p. (see e.g. [6], [15]), it can be Castaing-like represented in the form (see [15], [17])

$$F(q(t)) = \overline{\bigcup_{n\in\mathbb{N}} f_n(q(t))},$$

where $f_n(q)$, $n \in \mathbb{N}$, are related S-a.p. selections. Therefore, denoting (cf. (2) and (4))

$$T(q) := \int_{-\infty}^{\infty} G(t-s)\bigg[\overline{\bigcup_{n\in\mathbb{N}} f_n(q(s))} + \sigma(s)\bigg]\,ds, \quad q \in Q,$$

where the integral is understood in the sense of R. J. Aumann (cf. [23]), one can already discuss the possibility of applying Theorem 1. It is required that

(i) Q is complete,
(ii) $T\colon Q \to 2^Q \setminus \{\emptyset\}$ is a Lipschitz-continuous multifunction with nonempty, closed values, having a Lipschitz constant $L_0 \in [0,1)$.

Since Q is (by the hypothesis) Banach, only (ii) remains to be verified.

Taking into account the well-known elementary properties of S-a.p. functions (the S-limit of a sequence of S-a.p. functions is an S-a.p. function and the sum of two S-a.p. functions is an S-a.p. function as well) and applying Lemma 2 to (4) (when taking separately the indicated S-a.p. selections on the right-hand side of (4)), we get that $T(Q) \subset Q$. Moreover, the set of values of T can be verified quite analogously as in e.g. [3] or [2] to be nonempty, closed and convex, for every $q \in Q$. Thus, we only need to show that T is a contraction.

If F is Lipschitzean with a sufficiently small Lipschitz constant $L \in [0,1)$, then we obtain (see [23, p. 199])

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} d_H(T(q_1),T(q_2)) \\
&\qquad = \sup_{t\in\mathbb{R}} d_H\bigg(\int_{-\infty}^{\infty} G(t-s)F(q_1(s))\,ds, \int_{-\infty}^{\infty} G(t-s)F(q_2(s))\,ds\bigg) \\
&\qquad \le \sup_{t\in\mathbb{R}} \bigg|\int_{-\infty}^{\infty} |G(t-s)| d_H(F(q_1(s)),F(q_2(s)))\,ds\bigg| \\
&\qquad \le L \sup_{t\in\mathbb{R}} \bigg|\int_{-\infty}^{\infty} |G(t-s)| \sup_{t\in\mathbb{R}} |q_1(t)-q_2(t)|\,ds\bigg| \\
&\qquad \le LC(A) \sup_{t\in\mathbb{R}} |q_1(t)-q_2(t)| = LC(A)d(q_1,q_2),
\end{aligned}$$

where $C(A)$ is a constant depending only on A (cf. [16]).

So, the desired contraction takes place, when $L_0 := LC(A) < 1$.

We are in position to give the first main result.

Theorem 3. *Let the above assumptions be satisfied. Then inclusion* (3) *admits a uniformly a.p. solution, provided the Lipschitz constant L satisfies the inequality $L < 1/C(A)$, where $C(A)$ is a constant depending only on A such that*

$$\sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{\infty} |G(t-s)| ds \right| \leq C(A),$$

(G *denotes the principal Green function for* (1)).

Remark 4. For $B = \mathbb{R}^n$, the explicit estimate of $C(A)$ can be found, under some additional restrictions in [6] (cf. also [7]).

4. Schauder–Tikhonov-like approach

Consider again inclusion (3), but this time assume, for a moment, that F is no longer Lipschitz-continuous, but upper-semi-continuous (i.e. for any open subset $U \subset B$, the set $\{X \in B \mid F(X) \subset U\}$ is open) and such that

$$|F(X)| \leq L|X| + M, \tag{5}$$

where $0 \leq L < 1/C(A)$ and $M \geq 0$ is an arbitrary constant. Let all the other assumptions be satisfied.

Applying Theorem 2 to (3), one can establish quite analogously as in [3] (cf. Theorem 17 in [3]) the following statement.

Proposition 1. *Let all the above assumptions be satisfied* (*jointly with* (5), *where $L < \delta/C(A)$ and $\delta \leq 1$ is a given constant related to the fact that the moduls of frequencies of S-a.p. multifunctions involve those of their S-a.p. selections* (*see* [15], [17]). *Assume, furthermore, that*

$$\gamma(F(\Omega)) < \frac{1}{C(A)}\gamma(\Omega), \quad \textit{for every bounded } \Omega \subset B, \tag{6}$$

where γ stands for the Hausdorff measure of noncompactness (*and $C(A)$ has the same meaning as above*). *Then inclusion* (3) *admits a uniformly a.p. solution, provided:*

(H1) $F(q) \in \left\{ G\colon \mathbb{R} \to 2^B \setminus \{\emptyset\} \textit{ is measurable } \middle| \, (\varepsilon, k, a, \tau) \in \widetilde{\Omega}'_0 \right.$
$$\left. \Rightarrow \sup_{b\in\mathbb{R}} \int_b^{b+1} d_H(G(t), G(t+\tau))\, dt < \delta L_1 \varepsilon \right\},$$
for every $q \in \widetilde{M}_\sigma := \{g \in C(\mathbb{R}, B) \mid (\varepsilon, k, a, \tau) \in \widetilde{\Omega}'_\sigma \Rightarrow \|g(t+\tau) - g(t)\|_S < \varepsilon\}$, where $\widetilde{\Omega}'_\sigma := \{(\varepsilon, k, a, \tau) \in \mathbb{R}^4 \mid \tau \in [a, a+k]$ and $\|\sigma(t+\tau) - \sigma(t)\|_S < \varepsilon/\Delta\}$, $L_1 < 1/C(A)$ and $\Delta \gg 1$ is sufficiently big,

(H2) $\delta = \delta(\varepsilon)$ *in* (H1) *is independent of $\varepsilon > 0$,*

(H3) $T(Q) \subset \widetilde{M}_\sigma$ ($\Rightarrow T(Q) \subset Q$), *where*

$$T(q) := \int_{-\infty}^{\infty} G(t-s)\left[\overline{\bigcup_{n\in\mathbb{N}} \widetilde{f}_n(q(s))} + \sigma(s)\right] ds,$$

$\widetilde{f}_n(q) \in \widetilde{M}'_\sigma := \{g \in L_{\mathrm{loc}}(\mathbb{R}, B) \mid (\varepsilon, k, a, \tau) \in \widetilde{\Omega}'_\sigma \Rightarrow \|g(t+\tau) - g(t)\|_S < L\varepsilon/\delta\}$,

and $\widetilde{f}_n(q) \subset F(q)$, *for every* $n \in \mathbb{N}$,

$$Q := \widetilde{M}_\sigma \cap Q_C$$

(*observe that* Q *is again a closed, convex subset of* $C(\mathbb{R}, B)$),

$$Q_C := \{g \in C(\mathbb{R}, B) \mid \sup_{t\in\mathbb{R}} |g(t)| \le C\},$$

and $C > 0$ *is a constant such that*

$$C \ge \frac{C(A)}{1 - C(A)L_1}(M + \sup \mathrm{ess}_{t\in\mathbb{R}}|\Sigma(t)|).$$

Remark 5. Conditions (H1), (H2) imply (see [15], [17]) that $F(q)$ can be Castaing-like represented in the form

$$F(q(t)) = \overline{\bigcup_{n\in\mathbb{N}} \widetilde{f}_n(q(t))} \quad \text{for every } n \in \mathbb{N},$$

where $\widetilde{f}_n(q) \in \widetilde{M}'_\sigma$ and $\widetilde{f}_n(q) \subset F(q)$, for every $n \in \mathbb{N}$.

Since satisfying conditions (H1)–(H3) without a Lipschitz-continuity of F seems to be, even in particular single-valued cases, a difficult task, we still give

Proposition 2. *Assume* (H2) *and let the assumptions of Theorem* 3 *hold with* $L < \delta/C(A)$, *where* $\delta \le 1$ *is a given constant. Then all conditions in Proposition* 1 *are satisfied.*

Proof. It is well-known (see e.g. [24, p. 85]) that, under the above assumptions, the Lipschitz-continuity of F with the constant $L < \delta/C(A)$, $\delta \le 1$, implies (6). As (5) follows immediately, we restrict ourselves to checking only (H1) and (H3).

Since the Lipschitz-continuity of F implies

$$\sup_{b\in\mathbb{R}} \int_b^{b+1} d_H\,(F(q(t)), F(q(t+\tau)))\,dt \le L \sup_{b\in\mathbb{R}} \int_b^{b+1} |q(t+\tau) - q(t)|\,dt < \delta L_1 \varepsilon,$$

for every $q \in \widetilde{M}_\sigma$, hypothesis (H1) is satisfied and so, in view of Remark 5, $\widetilde{f}_n(q) \in \widetilde{M}'_\sigma$, for every $n \in \mathbb{N}$.

As concerns (H3), consider a uniformly a.p. solution $X(t)$ of the equation

$$X' + AX = \widetilde{f}(q(t)) + \sigma(t), \quad q \in Q \ (= \widetilde{M}_\sigma \cap Q_C),$$

where $\widetilde{f}(q) \in \widetilde{M}'_\sigma$ and $\widetilde{f}(q) \subset F(q)$. We have that

$$
\begin{aligned}
&\|X^b(t+\tau) - X^b(t)\|_{BC} \\
&\quad = \left\| \int_{-\infty}^{\infty} G^b(t-s)[\widetilde{f}(g(s+\tau)) - \widetilde{f}(g(s)) + \sigma(s+t) - \sigma(s)]\,ds \right\|_{BC} \\
&\quad \le \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{\infty} |G(t-s)| \|\widetilde{f}^b(q(s+\tau)) - \widetilde{f}^b(q(s)) + \sigma^b(s+\tau) - \sigma^b(s)\|_{BC}\,ds \right| \\
&\quad \le \|\widetilde{f}^b(q(t+\tau)) - \widetilde{f}^b(q(t)) + \sigma^b(t+\tau) - \sigma^b(t)\|_{BC} \sup_{t\in\mathbb{R}} \left| \int_{-\infty}^{\infty} |G(t-s)|\,ds \right| \\
&\quad \le C(A)\|\widetilde{f}^b(q(t+\tau)) - \widetilde{f}^b(q(t)) + \sigma^b(t+\tau) - \sigma^b(t)\|_{BC} \\
&\quad < C(A)\left(\frac{L\varepsilon}{\delta} + \frac{\varepsilon}{\Delta}\right) = \varepsilon C(A)\left(\frac{L}{\delta} + \frac{1}{\Delta}\right) < \varepsilon,
\end{aligned}
$$

by the hypothesis $L < \delta/C(A)$ and $\Delta >> 1$. Thus, $X(t) = T(q) \in \widetilde{M}_\sigma$, for every $q \in Q$, which completes the proof. $\square$

Remark 6. If F is single-valued, then one can obviously take $\delta = 1$. If $B = \mathbb{R}^n$, then a lower estimate for δ can be obtained explicitly for a Lipschitz-continuous F, namely $\delta \le 1/n(12\sqrt{3}/5 + 1)$ (see [23, pp. 101–103]). In the both cases, (H2) holds automatically.

Hence, we can conclude this section by the second principal result.

Theorem 4. *Let the assumptions of Theorem* 3 *be satisfied, where F is single-valued or $B = \mathbb{R}^n$. Then inclusion* (3) *admits* (*on the basis of Theorem* 2) *a uniformly a.p. solution belonging to the set Q, provided $L < 1/C(A)$ or $L < 1/C(A)n(12\sqrt{3}/5+1)$, respectively.*

5. Concluding remarks (comparison of approaches)

We could see that if F is a contraction with $L < \delta/C(A)$, $\delta \le 1$, then we arrived, under (H2), at the same results, when applying Theorem 1 or Theorem 2. In fact, due to (H3), Theorem 2 gives us a bit more, namely that an a.p. solution belongs to Q. On the other hand, it is a question, whether or not conditions (H1)–(H3) can be satisfied in particular (especially, single-valued) cases without the Lipschitz-continuity of F. If so, then (in spite of the fact that the application of Theorem 1 is apparently more straightforward) Theorem 2 might be more efficient in this field.

In the single-valued case, the problem reduces to verifying only (H1), because (H2) holds trivially (see Remark 6) and (H3) is then satisfied, whenever $C(A) < 1$. As a simplest example for $C(A) < 1$, we can take $B = \mathbb{R}^2$ and $A = \operatorname{diag}(a_{11}, a_{22})$, where $a_{11} > 0 > a_{22}$ and $(1/a_{11} - 1/a_{22}) < 1$ (see [7], where more examples can be found). Moreover, according to Corollary to Lemma 3 in [15], $F(q) = \widetilde{f}(q)$ is an S-a.p. function whose modul of frequencies is involved in the one of any $q \in \widetilde{M_\sigma}$. This, however, does not yet mean (H1).

Nevertheless, for differential inclusions in Banach spaces, Theorem 3 seems to be a new result. On the other hand, the conclusion of Theorem 4 can be also obtained by different techniques (see e.g. [2], [27]).

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