Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 18, 2001, 269–281

MULTIPLICITY OF SOLUTIONS FOR NONHOMOGENEUOUS NONLINEAR ELLIPTIC EQUATIONS WITH CRITICAL EXPONENTS

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Abstract. Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. We consider a semilinear boundary value problem of the form

$$\text{(P)} \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u + f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in C(\overline{\Omega})$ and $2^* = 2N/(N-2)$. We show the effect of topology of Ω on the multiple existence of solutions.

1. Introduction

Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. In this paper we consider the existence and multiplicity of solutions of problem

$$\text{(P)} \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u + f & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $2^* = 2N/(N-2)$ and $f \in C(\overline{\Omega})$ with $f \not\equiv 0$ and $f \geq 0$ on Ω.

2000 *Mathematics Subject Classification*. 35J65.

Key words and phrases. Critical Sobolev, Dirichlet boundary value problem, semilinear elliptic problem.

We denote by (P_0) the problem (P) with $f \not\equiv 0$. Problem (P) is a simplified model of problems occur in physics and geometry, and the existence and nonexistence of solutions of problem (P) has been studied by many authors in the last decade. The difficulty to treat this problem is caused by the lack of compactness. Pohožaev ([12]) proved that problem (P_0) has no nontrivial solution when the domain Ω is star-shaped. On the other hand, the existence of a nontrivial radial solution of problem (P_0) was established by Kazdon and Warner ([10]) in the case that Ω is an annulus. In the case that domain Ω has nontrivial topology, the existence of solutions for (P_0) was established by Bahri and Coron ([2]). These results show that the shape of the domain Ω is deeply related to the existence of solutions of (P), and it comes of interest to study the effect of the topology of Ω for the multiplicity of solutions of problem (P). In [14], Rey proved that problem (P) has $\mathrm{cat}(\Omega)+1$ solutions when $\|f\|_{L^2}$ is sufficiently small. (See also Cao and Chabrowski [5]). In the present paper, we establish a multiplicity result using the homology groups of Ω.

We now state our main result:

Theorem 1.1. *There exists a residual subset $D \subset C^2(\overline{\Omega})$ and $\varepsilon_0 > 0$ such that for each $f \in D$ with $f \not\equiv 0$, $f \geq 0$ on Ω and $|f|_{C(\overline{\Omega})} < \varepsilon_0$, problem* (P) *has at least $\Sigma_{p=0}^{\infty} \dim H_p(\Omega) + 1$ solutions.*

2. Preliminaries

Throughout the rest of this paper, $c_0, c_1, \dots$, and $m_1, m_2, \dots$ stands for various constants independent of $(z,a) \in \Omega \times (1,\infty)$. For simplicity, we put $H = H_0^1(\Omega)$. For each domain $U \subset \mathbb{R}^N$, we denote by $|\cdot|_q$ the norm of $L^q(U)$, $q > 1$. We put

$$D^1(\mathbb{R}^N) = \{v \in L^{2^*}(\mathbb{R}^N) : |\nabla v|_2 \in L^2(\mathbb{R}^N)\}.$$

For each $v \in D^1(\mathbb{R}^N)$, we put $\|v\|^2 = \int_{\mathbb{R}^N} |\nabla v|^2$. The symbol $\|\cdot\|$ is also used to denote the norm of H defined by $\|v\|^2 = |\nabla v|_2^2$ for $v \in H$. $\langle \cdot, \cdot \rangle$ stands for the inner product in H. $B_r(x) \subset H$ stands for the open ball centered at $x \in H$ with radius $r > 0$. For each normed space X, a subset $A \subset X$ and $x \in X$, we put $d(A,x) = \inf\{\|x-y\| : y \in A\}$. For subspaces Y, Z of X, we denote by $D(Y,Z)$ the distance of two spaces Y and Z. That is $D(Y,Z) = \sup\{d(Y,z) : z \in Z \text{with } \|z\| \leq 1\}$. For each $d > 0$, Ω_d stands for the set $\Omega_d = \{x \in \Omega : d(\partial\Omega, x) < d\}$. For each $a \in \mathbb{R}$ and each functional $F \colon H \to \mathbb{R}$, we denote by F_a the set $F_a = \{v \in H : F(v) \leq a\}$. We call a real number d a critical level of a functional F if there exists a sequence $\{v_n\} \subset H$ such that $\lim_{n\to\infty} F(v_n) = d$ and $\lim_{n\to\infty} \|\nabla F(v_n)\| = 0$. For a pair of topological space (X,Y) with $Y \subset X$, we denote by $H_*(X,Y)$ the relative singular homology groups (cf. Spanier [15]). For two topological space X, Y, we write $X \cong Y$ when X and Y are of the same homotopy type. We define a functional I on H by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u^+|^{2^*}, \tag{2.1}$$

where $u^+(x) = \max\{0, u(x)\}$ for $x \in \Omega$. The solutions of (P_0) correspond to critical points of functional I. Let (P_∞) be the problem defined by

$$(\mathrm{P}_\infty) \qquad \begin{cases} -\Delta u = |u|^{2^*-2}u & \text{for } u \in D^1(\mathbb{R}^N), \\ u > 0 & \text{on } \mathbb{R}^N. \end{cases}$$

We denote by I^∞ the functional on $D^1(\mathbb{R}^N)$ defined by (2.1) with $\Omega = \mathbb{R}^N$. Then each critical point of functional I^∞ is a solution of problem (P_∞). For each $(z, a) \in \mathbb{R}^N \times (1, \infty)$, we put

$$u_{(z,a)}(x) = m\bigg(\frac{a}{1 + a^2|x - z|^2}\bigg)^{(N-2)/2}$$

where $m = (N(N-2))^{(N-2)/4}$. It is known that each $u_{(z,a)}$ is a critical point of I^∞. By the invariance of the norm of $D^1(\mathbb{R}^N)$ under translation and scaling

$$u \to u_R(x) = R^{-N/2^*}u(x/R) \quad \text{for } u \in D^1(\mathbb{R}^N),\ R > 0, \tag{2.2}$$

we have that each $u_{(z,a)}$ have the same critical value. We put $c = I^\infty(u_{(z,a)})$ for $(z, a) \in \mathbb{R}^N \times (0, \infty)$. We also set

$$S = \bigg\{v \in H\backslash\{0\} : \int_\Omega |\nabla v|^2 = \int_\Omega |v^+|^{2^*}\bigg\}.$$

It is easy to see that if $v \in H$ satisfies $v^+ \not\equiv 0$, there exists a unique positive number t such that $tv \in S$. It is also known that $I(v) > c$ for all $v \in S$ (cf. [2]). The following concentrate compactness lemma play an important role for our argument.

LEMMA 2.1 (cf. Bahri and Coron [2], Passarero [11]). *Let* $\{v_n\} \subset S$ *such that* $\lim_{n\to\infty} I(v_n) = c$. *Then there exist* $\{a_n\} \subset \mathbb{R}^+$ *and* $\{z_n\} \subset \Omega$ *such that* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} \|v_n - u_{(z_n,a_n)}\| = 0$.

Since $\partial\Omega$ is smooth, we can choose $0 < d_0 < 1$ such that for each $x \in \Omega$ with $d(\partial\Omega, x) < d_0$, there exists a unique point $y \in \partial\Omega$ such that $|x - y| = d(\partial\Omega, x)$. We put $d(z) = \min\{d(\partial\Omega, z), d_0\}$ for each $z \in \Omega$. For each $\rho > 0$, we put

$$\Pi(\rho) = \{(z, a) \in \Omega \times (1, \infty) : d(z) \cdot a = \rho\},$$
$$\overline{\Pi}(\rho) = \{(z, a) \in \Omega \times (1, \infty) : d(z) \cdot a \geq \rho\}.$$

Let $n \geq 2$ be an integer and $\varphi \in C^\infty([0,\infty),[0,1])$ be a function such that $\varphi(x) = 1$ for $x \in [0, 1-1/n]$, $-2n \leq \varphi'(x) \leq 0$ on $[1-1/n, 1]$ and $\varphi(x) = 0$ for $x \in [1,\infty)$. For each $(z,a) \in \Omega \times (1,\infty)$, we define a function $v_{(z,a)} \in H$ by

$$v_{(z,a)}(x) = \varphi\bigg(\frac{x-z}{d(z)}\bigg)u_{(z,a)}(x) \quad \text{for } x \in \Omega.$$

Then by the invariance of the value of I under the scaling (2.2), we have that $I(v_{(z,a)}) = I(v_{(z',a')})$ for $(z,a),(z',a') \in \Omega \times (1,\infty)$ with $d(z)a = d(z')a'$. We also have from the definition that

$$\lim_{d(z)a\to\infty} \|v_{(z,a)} - u_{(z,a)}\| = 0 \quad \text{for each } z \in \Omega. \tag{2.3}$$

For $z = (z_1,\ldots,z_n) \in \mathbb{R}^N$ and $a \in (1,\infty)$, we consider the eigenvalue problem

$$-\Delta w = \mu g(u_{(z,a)})w, \quad w \in D^1(\mathbb{R}^N) \tag{2.4}$$

where $g(t) = (2^*-1)|t^+|^{2^*-2}$ for $t \in \mathbb{R}$. Since $u_{(z,a)}$ is a solution of problem (P_∞), it is obvious that $\mu_{-1} = 1/(2^*-1)$ is an eigenvalue of (2.4) with eigenfunction $u_{(z,a)}$. It is also known that μ_{-1} is the unique eigenvalue of problem (2.4) satisfying $\mu < 1$, and μ_{-1} is simple. We put

$$T_{(z,a)} = \operatorname{span}\{u_{(0,z,a)},\ldots,v_{(N,z,a)}\}$$

where

$$u_{(0,z,a)}(x) = \frac{\partial}{\partial a}u_{(z,a)} = \frac{m(N-2)}{2}\frac{a^{(N-4)/2}(1-a^2|x-z_i|^2)}{(1+a^2|x-z_i|^2)^{(N/2)}},$$

$$u_{(i,z,a)}(x) = \frac{\partial}{\partial x_i}u_{(z,a)} = -m(N-2)\frac{a^{(N+2)/2}(x_i-z_i)}{(1+a^2|x-z_i|^2)^{(N/2)}},$$

for $1 \leq i \leq N$. Then recalling that each $u_{(z,a)}$ is a solution of problem (P_∞), we have by differentiating (P_∞) by $x_1,\ldots,x_N$ and a that each element of $T_{(z,a)}$ is an eigenfuction of problem (2.4) corresponding to the eigenvalue $\mu_0 = 1$. We denote by $E^{(-)}_{(z,a)}$ and $E^{(0)}_{(z,a)}$ the subspaces of $D^1(\mathbb{R}^N)$ spanned by eigenfunctions corresponding to the eigenvalues μ_{-1} and 1, respectively. We also put $E^{(+)}_{(z,a)} = (E^{(-)}_{(z,a)} \cup E^{(0)}_{(z,a)})^\perp$. Here $u \perp v$ implies that $\int_{\mathbb{R}^N}\langle\nabla u,\nabla v\rangle = 0$ for $u,v \in D^1(\mathbb{R}^N)$. Then one can verify easily that for each $(z,a) \in \Omega \times (1,\infty)$,

$$\langle -\Delta v - g(u_{(z,a)})v, z\rangle = 0 \tag{2.5}$$

for all $v \in E^{(+)}_{(z,a)}$ and $z \in E^{(-)}_{(z,a)} \oplus E^{(0)}_{(z,a)}$. It is known that the following lemma holds.

LEMMA 2.2 (cf. [8]).

(1) *There exists* $\mu_1 > 0$ *such that for each* $(z,a) \in \mathbb{R}^N \times (1,\infty)$,

$$\langle -\Delta v - g(u_{(z,a)})v, v\rangle \geq \mu_1 \int_{\mathbb{R}^N} g(u_{(z,a)})v^2 \quad \text{for all } v \in E^{(+)}_{(z,a)}. \tag{2.6}$$

(2) $T_{(z,a)} = E^{(0)}_{(z,a)}$ *for* $(z,a) \in \Omega \times (1,\infty)$. *That is* $v \in D^1(\mathbb{R}^N)$ *is a solution of problem*

$$-\Delta v = g(u_{(z,a)})v \tag{2.7}$$

if and only if $v \in T_{(z,a)}$.

In case that $|f|_{C(\overline{\Omega})}$ is small, the existence of a solution of problem (P) near the origin is known. That is

LEMMA 2.3 (cf. [6]). *There exists* $\varepsilon_0 > 0$ *and* $C_0 > 0$ *such that for each* $f \in C(\overline{\Omega})$ *with* $f \geq 0$ *and* $|f|_{C(\overline{\Omega})} < \varepsilon_0$, *there exists a unique solution* $u_0 \in H$ *of (P) satisfying* $|u_0|_{C^1(\overline{\Omega})} \leq C_0|f|_{C(\overline{\Omega})}$ *and*

$$c_0 = \int_\Omega \left(\frac{1}{2}|\nabla u_0|^2 - \frac{1}{2^*}|u_0|^{2^*} - fu_0\right) < \frac{c}{2}. \tag{2.8}$$

PROOF. We give a sketch of the proof. Let λ_1 be the first eigenvalue of eigenvalue problem

$$-\Delta v = \lambda v \quad \text{for } v \in H.$$

Fix $\lambda \in (0,\lambda_1)$. Let h be a truncation function of the mapping $t \to (t^+)^{2^*-1}$ defined by $h(t) = |t^+|^{2^*-1}$ for $t \in (-\infty, t_0]$ and $h(t) = \lambda t$ for $t \geq t_0$ where t_0 satisfies $t_0^{2^*-2} = \lambda$. Then since $|h(t)| \leq \lambda|t|$, we have by a standard argument that there exists a unique positive solution u_0 of problem

$$-\Delta u = h(u) + f \quad \text{for } u \in H_0^1(\Omega).$$

It is easy to see that $\|u_0\| \leq C|f|_2$ for some $C > 0$. It also follows, by the Schauder estimate, that there exists $C_0 > 0$ such that $|u_0|_{C^1(\overline{\Omega})} \leq C_0|f|_{C(\overline{\Omega})}$ for each $f \in C(\overline{\Omega})$. Then, choosing ε_0 sufficiently small, we have that $|u_0|_{C(\overline{\Omega})} < t_0$ for $f \in C(\overline{\Omega})$ with $|f|_{C(\overline{\Omega})} < \varepsilon_0$. Then since $h(t) = |t|^{2^*-1}$ for $0 \leq t < t_0$, we have that u_0 is a solution of problem (P). □

Let $f \in C(\overline{\Omega})$ with $f \not\equiv 0$, $f \geq 0$ on Ω and $|f|_{C(\overline{\Omega})} < \varepsilon_0$, and u_0 be the solution obtained in Lemma 2.3. Then it follows from the maximal principle and Lemma 2.3 that, there exists $\ell_1 > 0$ such that

$$\frac{\ell_1}{2} < -\frac{\partial u_0(x)}{\partial n} < \ell_1 \quad \text{for all } x \in \partial\Omega, \tag{2.9}$$

where $\partial/\partial n$ denotes the outer normal derivative. Then from Lemma 2.3 and the inequality above, we have that there exists $\ell_2 > 0$ and

$$\ell_2 d(x) \le u_0(x) \quad \text{for } x \in \Omega. \tag{2.10}$$

Throughout the rest of this paper, we assume that $f \in C(\overline{\Omega})$ satisfying $f \not\equiv 0$, $f \ge 0$ and $|f|_{C(\Omega)} < \varepsilon_0$, and $u_0 \in H$ is the solution obtained by Lemma 2.3. We define a functional $J\colon H \to R$ by

$$J(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 - \frac{1}{2^*}((v+u_0)^+)^{2^*} - u_0^{2^*} - 2^* u_0^{2^*-1} v]\right) \quad \text{for } v \in H.$$

It is then easy to see that for each critical point $v \in H$ of J, $v + u_0$ is a solution of problem (P). From the definition of J, we can see that the following lemma holds.

Lemma 2.4 (cf. [3]). *There exists $\varepsilon_1 > 0$ such that for each $f \in C(\overline{\Omega})$ with $f \not\equiv 0$, $f \ge 0$ on Ω, $|f|_{C(\overline{\Omega})} < \varepsilon_1$ and $v \in H$ satisfying $v^+ \not\equiv 0$, there exists a unique positive number t_v such that $J(tv)$ is increasing on an interval $[t_1, t_v)$ with $t_1 > 0$, decreasing on (t_v, ∞), and $J(t_v v) = \max\{J(tv) : t \ge 0\}$.*

We put $\mathcal{S} = \{t_v v : v \in H\backslash\{0\}\}$. Then we have by Lemma 2.4 that $J(v) > 0$ on S, and $\langle \nabla J(v), v\rangle = 0$ if and only if $v \in \mathcal{S} \cup \{0\}$. Therefore each critical point of J different from 0 is contained in $\mathcal{S}$. We also have the following Lemma as a direct consequence of concentrate compactness principle.

Lemma 2.5. *J satisfies Palais–Smale condition on $(0, c)$.*

Proof. For completeness we give a sketch of proof. Let $\{v_n\} \subset H$ be a sequence such that $\lim_{n\to\infty} \nabla J(v_n) = 0$ and $\lim J(v_n) = d \in (0, c)$. Then we find that there exists a solution $v \in H$ of (P) and a sequence $(z_n, a_n) \subset \mathbb{R}^N \times \mathbb{R}^+$ such that

$$\begin{gathered} v_n - \lambda u_{(z_n,a_n)} \to v \quad \text{weakly in } H, \\ I(v_n) \to J(v) + \lambda I(u_{(z_n,a_n)}) = d, \quad \text{as } n \to \infty, \end{gathered}$$

where $\lambda = 0$ or 1(cf. [16]). Suppose that $\lambda = 1$. Then since each solution $J(v) \ge 0$, we find that $v = u_0$. That is $d = c$. This contradicts to the assumption. Therefore we have that $\lambda = 0$. Then we find that v is a critical point with critical value d. $\square$

In the following, we fix a positive number $\rho_0 > 2$. Then by the definition of $v_{(z,a)}$, we have

Lemma 2.6. *For each $(z, a) \in \overline{\Pi}(\rho_0)$,*

$$\|v_{(z,a)}\|^2 \le Nc + O(d(z)a)^{-(N-2)}, \tag{2.11}$$

$$|v_{(z,a)}|_{2*}^{2*} \ge Nc - O(d(z)a)^{-N}, \tag{2.12}$$

$$\int_\Omega u_0 v_{(z,a)}^{2^*-1} \geq O(d(z)a^{-(N-2)/2}), \tag{2.13}$$

$$|v_{(z,a)}|_{N/(N-2)}^{N/(N-2)} \leq O(a^{-N/2}|\log a|), \tag{2.14}$$

$$\int_\Omega u_0^{N/(N-2)}(v_{(z,a)})^{N/(N-2)} \leq O(d(z)^{N/(N-2)}a^{-N/2}|\log a|). \tag{2.15}$$

Proof. We first note that $\|u_{(z,a)}\|^2 = Nc$ for each $(z,a) \in \Omega \times (1,\infty)$ (cf. [3]). Let $(z,a) \in \overline{\Pi}(\rho_0)$ and put $d = d(z)$. Then from the definition

$$\|v_{(z,a)}\|^2 = |u_{(z,a)}\nabla\varphi + \varphi\nabla u_{(z,a)}|_2^2.$$

Since $0 \leq \varphi \leq 1$ and $\varphi(t) = 0$ for $t \geq 1$, we find

$$\begin{aligned}\int_\Omega |\varphi\nabla u_{(z,a)}|^2 &\leq Nc - m^2(N-2)^2\int_d^\infty \frac{a^{N+2}r^{N+1}}{(1+a^2r^2)^N}\,dr \\ &\leq Nc - m^2(N-2)(da)^{-(N-2)},\end{aligned}$$

and

$$\int_\Omega |u\nabla\varphi|^2 \leq \int_{(1-1/n)d}^d \left(\frac{\varphi'}{d}\right)^2 \frac{a^{N-2}r^{N-1}}{(1+a^2r^2)^{N-2}}\,dr \leq Cn(da)^{-(N-2)}$$

for some $C > 0$. Then we have assertion (2.11). We also have that, for some $c_1 > 0$,

$$|\varphi u_{(z,a)}|_{2*}^{2*} \geq Nc - \int_{(1-1/n)d}^\infty \frac{a^N r^{N-1}}{(1+a^2r^2)^N}\,dr \geq Nc - c_1(da)^{-N},$$

and then (2.12) holds. On the other hand, we have by (2.10) that $u_0(x) \geq (\ell_2/2)d(z)$ for $d(z)/2 \leq d(x) \leq 2d(z)$. We also note that that $d(z) > 2/a$, because $\rho_0 > 2$. Then

$$\int_\Omega u_0 v_{(z,a)}^{2^*-1} \geq (\ell_2/2)d(z)\int_0^{1/a} \frac{a^{(N+2)/2}r^{N-1}}{(1+a^2r^2)^{(N+2)/2}}\,dr \geq \frac{\ell_2}{4N2^{(N+2)/2}}\,da^{-(N-2)/2}.$$

Then (2.13) holds. Similarly, we find that the inequality (2.14) holds (cf. [3]). From (2.9), we find that there exists $m_0 > 0$ such that $u_0(x) \leq m_0 d(z)$ for $z \in \Omega$ and $x \in \Omega$ with $|z-x| \leq d(z)$. Then, by (2.14), we find that (2.15) holds. □

Now we put

$$\begin{aligned}\mathcal{M} &= \{v_{(z,a)} : (z,a) \in \Omega \times (1,\infty)\}, \\ \mathcal{N} &= \{\lambda v_{(z,a)} : (z,a) \in \Omega \times (1,\infty), \lambda \in (1/2,2)\}.\end{aligned}$$

We denote by $\mathcal{T}_{(z,a)} \subset H_0^1(\Omega)$ the tangent space of $\mathcal{N}$ at $v_{(z,a)}$. Put $F_{(z,a)}^- = \{\lambda v_{(z,a)} : \lambda \in \mathbb{R}\}$, $F_{(z,a)}^+ = \mathcal{T}_{(z,a)}^\perp$ and $F_{(z,a)} = F_{(z,a)}^- \oplus F_{(z,a)}^+$. For each $v = v_+ + v_-$, $v_- \in F_{(z,a)}^-, v_+ \in F_{(z,a)}^+$, we put $Pv = v_+ - v_-$. From the definition, we have that $\|Pv\| = \|v\|$ for $v \in F_{(z,a)}$. Then we have

Lemma 2.7. *There exist positive numbers r_1, ρ_1 , C_1 and ε_2 such that $\rho_1 > \rho_0$, and for $f \in C(\overline{\Omega})$ with $|f|_{C(\overline{\Omega})} < \varepsilon_2$, $(z,a) \in \overline{\Pi}(\rho_1)$, and $w \in B_{r_1}(v_{(z,a)})$,*

$$\langle -\Delta v - g(w+u_0)v, Pv\rangle \geq C_1\|v\|^2 \quad \textit{for all } v \in F_{(z,a)}.$$

Proof. To prove (2.16), it is sufficient to show that (2.16) holds for $u_0 = 0$ (i.e. the case that $f \equiv 0$). In fact, if (2.16) holds for $u_0 = 0$, then by choosing $\varepsilon_2' > 0$ sufficiently small, we have that (2.16) holds with $u_0 \in C(\overline{\Omega})$ satisfying $\|u_0\|_{C(\overline{\Omega})} < \varepsilon_2'$. On the other hand, by Lemma 2.3, we can choose ε_2 sufficiently small that $\|u_0\|_{C(\overline{\Omega})} < \varepsilon_2'$ for $f \in C(\overline{\Omega})$ with $|f|_{C(\overline{\Omega})} < \varepsilon_2$. Therefore we give a proof for the case that $u_0 = 0$. Suppose that the inequality (2.16) does not hold. Then there exist sequences $\{(z_n, a_n)\} \subset \overline{\Pi}(\rho_0)$ and $(v_n, w_n) \in F_{(z_n,a_n)} \times H$ such that $\|v_n\| = 1$ for $n \geq 1$, $d(z_n)a_n = \rho_n \to \infty$ as $n \to \infty$, $\lim_{n\to\infty} \|w_n - v_{(z_n,a_n)}\| = 0$ and

$$\limsup_{n\to\infty} \langle -\Delta v_n - g(w_n)v_n, Pv_n\rangle \leq 0.$$

Here we put $v_n = v_{+,n} + v_{-,n}$, where $v_{+,n} \in F^+_{(z_n,a_n)}$ and $v_{-,n} \in F^-_{(z_n,a_n)}$ for $n \geq 1$. Since $\lim_{n\to\infty} \rho_n = \infty$, we have by (2.3) that $\lim_{n\to\infty} \|v_{(z_n,a_n)} - u_{(z_n,a_n)}\| = 0$. Then, by the definition,

$$\lim_{n\to\infty} D(F_{(z_n,a_n)}, \operatorname{span}\{T_{(z_n,a_n)}, v_{(z,a)}\}) = 0$$

holds. Then $\lim_{n\to\infty} d(E^+_{(z_n,a_n)}, v_{+,n}) = 0$. Therefore we have by (2.6) that

$$\langle -\Delta v_{+,n} - g(w_n)v_{+,n}, v_{+,n}\rangle \geq (\mu_1/2)\int_\Omega g(w_n)v_{+,n}^2.$$

for n sufficiently large. If $\limsup \int_\Omega g(w_n)v_{+,n}^2 \geq 1/2$, we have that

$$\limsup_{n\to\infty} \langle -\Delta v_{+,n} - g(w_n)v_{+,n}, v_{+,n}\rangle \geq \mu_1/4.$$

On the contrary, if $\limsup \int_\Omega g(w_n)v_{+,n}^2 < 1/2$, then we find

$$\limsup_{n\to\infty} \langle -\Delta v_{+,n} - g(w_n)v_{+,n}, v_{+,n}\rangle \geq 1/2.$$

Therefore we find that

$$\limsup_{n\to\infty} \langle -\Delta v_{+,n} - g(w_n)v_{+,n}, v_{+,n}\rangle \geq \min\{1/2, \mu_1/4\}. \tag{2.17}$$

It is obvious from the definition of $v_{-,n}$ that

$$\limsup_{n\to\infty} \langle -\Delta v_{-,n} - g(w_n)v_{-,n}, v_{-,n}\rangle \leq 1 - 1/\mu_{-1} = 2 - 2^*. \tag{2.18}$$

On the other hand, recalling that $\lim_{n\to\infty} w_n - v_{(z_n,a_n)} = \lim_{n\to\infty} w_n - u_{(z_n,a_n)} = 0$ and $\lim_{n\to\infty} d(E^+_{(z_n.a_n)}, v_{+,n}) = \lim_{n\to\infty} d(E^-_{(z_n.a_n)}, v_{-,n}) = 0$, we find by (2.5) that

$$\limsup_{n\to\infty} \langle -\Delta v_{-,n} - g(w_n)v_{-,n}, v_{+,n}\rangle = 0. \tag{2.19}$$

Then combining (2.17)–(2.19), we find that $\liminf_{n\to\infty}\langle -\Delta v_n - g(w_n)v_n, Pv_n\rangle > 0$. This is a contradiction and the proof is completed. $\square$

By Lemma 2.7, we have

LEMMA 2.8. *There exists $\rho_2 > 0$ and $C_2 > 0$ satisfying that for each $f \in C(\overline{\Omega})$ with $|f|_{C(\overline{\Omega})} < \varepsilon_2$ and each $(z,a) \in \overline{\Pi}(\rho_2)$, there exists $w_{(z,a)} \in S \cap B_{r_1/2}(v_{(z,a)})$ such that $\|w_{(z,a)} - v_{(z,a)}\| \le C_2\|\nabla J(v_{(z,a)})\|$ and*

$$
\begin{aligned}
J(w_{(z,a)}) &= \min_{v\in F^{(+)}_{(z,a)}\cap B_{r_1/2}(0)}\ \max_{w\in F^{(-)}_{(z,a)}\cap B_{r_1/2}(0)} J(v_{(z,a)}+v+w) \\
&= \max\{J(w_{(z,a)}+w) : w \in F^{(-)}_{(z,a)} \cap B_{r_1/2}(0)\} \\
&= \min\{J(w_{(z,a)}+w) : w \in F^{(+)}_{(z,a)} \cap B_{r_1/2}(0)\}.
\end{aligned}
\tag{2.20}
$$

PROOF. Let $f \in C(\overline{\Omega})$ with $|f|_{C(\overline{\Omega})} < \varepsilon_2$ and $(z,a) \in \overline{\Pi}(\rho_1)$. Recall that $J''(u)v = -\Delta v - g(u)v$ for $u, v \in H$ and that $\|\nabla J(v_{(z,a)})\|_* \to 0$, as $\rho = d(z)a \to \infty$. Then from (2.16), we find by a standard argument (cf. [1]) that if ρ_2 is sufficiently large, there exists a saddle point $w_{(z,a)} = v_{(z,a)} + z_{(z,a)}$ satisfying (2.20) for each $(z,a) \in \overline{\Pi}(\rho_2)$ with $z_{(z,a)} \in B_{r_1/2}(0) \cap F_{(z,a)}$. Since $v_{(z,a)} \in F_{(z,a)}$, we find $w_{(z,a)} \in F_{(z,a)}$. Then we have $\lambda w_{(z,a)} \in F_{(z,a)}$ for $\lambda \ge 0$. Then since $w_{(z,a)}$ is a saddle point in $F_{(z,a)}$, $\langle \nabla J(w_{(z,a)}), w_{(z,a)}\rangle = 0$. Therefore it follows that $w_{(z,a)} \in \mathcal{S}$. On the other hand, we have that

$$
\begin{aligned}
&\langle \nabla J(w_{(z,a)}), P(w_{(z,a)} - v_{(z,a)})\rangle \\
&\quad = \langle \nabla J(v_{(z,a)}) + J''(v_{(z,a)})(w_{(z,a)} - v_{(z,a)}), P(w_{(z,a)} - v_{(z,a)})\rangle \\
&\qquad + o(\|w_{(z,a)} - v_{(z,a)}\|^2).
\end{aligned}
$$

Then since $\langle \nabla J(w_{(z,a)}), P(w_{(z,a)} - v_{(z,a)})\rangle = 0$, we find by (2.16) that there exists $C_2 > 0$ satisfying $\|w_{(z,a)} - v_{(z,a)}\| \le C_2\|\nabla J(v_{(z,a)})\|$ for all $(z,a) \in \overline{\Pi}(\rho_2)$. This completes the proof. $\square$

3. Transversality theorem

In this section, we state a transversality theorem which is needed for our argument. Let X, Y be Banach spaces and $\Psi: X \to Y$ be a C^1 mapping. An element $y \in Y$ is called a regular value of Ψ if for each $x \in \Psi^{-1}(y)$, the derivative $D\Psi(x)$ is surjective. Then we have

THEOREM 3.1. *Let X, Y and Z be separable Banach spaces, $\Psi: X \times Y \to Z$ a C^1-mapping, and $z \in Z$. Assume that*

1. *For each $(x,y) \in \Psi^{-1}(z)$, $D_x\Psi(x,y): X \to Z$ is Fredholm mapping of index* 0.
2. *For each $(x,y) \in \Psi^{-1}(z)$, $D\Psi(x,y): X \times Y \to Z$ is surjective.*

Then the set of $y \in Y$ satisfying that z is a regular value of $\Psi(\cdot, y)$ is residual in Y.

The theorem above is known in more general form(cf. [13] and [4]). We apply Theorem 3.1 to our problem:

Proposition 3.2 (cf. [4]). *There exists a dense subset $D \subset C^2(\overline{\Omega})$ such that for $f \in D$, each solution u of problem* (P) *is nondegenerate.*

Proof. Proposition 3.2 is essentially the same as Theorem 3.a1 of [4]. Then we give a sketch of the proof. We put

$$X = H^2(\Omega) \cap H_0^1(\Omega), \quad Y = C^2(\overline{\Omega}), \quad Z = L^2(\Omega),$$
$$\Psi(u, f) = \Delta u + |u|^{2^*-1}u + f \quad \text{for } u \in X \text{ and } f \in Y.$$

Then for each $u \in X$, the mapping $v \to \Psi_u(u, f)v = \Delta v + g(u)v \colon X \to Z$ is a Fredholm mapping of index 0 and then satisfies condition (1). Then to apply Theorem 3.1 for $z = 0$, it is sufficient to prove (2) is satisfied with $z = 0$. Let $(u, f) \in \Psi^{-1}(0)$. That is (u, f) satisfies $-\Delta u = |u|^{2^*-1}u + f$. Then we have that $u \in C(\overline{\Omega})$. It then follows that the kernel of $\Delta + g(u)$ is a finite dimensional space contained in $C^2(\overline{\Omega})$. Now let $h \in Z$. We look for $v \in X$ and $\overline{f} \in Y$ satisfying

$$D\Psi(u, f)v = D_u\Psi(u, f)v + D_f\Psi(u, f) = \Delta v + g(u)v + \overline{f} = h.$$

Let Q be the projection from X onto the kernel $\Delta + g(u)$. Then from the observation above that $Qh \in C^2(\overline{\Omega})$. Here we put $\overline{f} = Qh$. Then it is obvious that that there exists a unique element $v \in X$ satisfying the equality above. □

4. Proof of Theorem 1.1

Lemma 4.1. *Let $f \equiv 0$. Then for each $\rho \geq \rho_2$, there exists $c_\rho > c$ such that*

$$J(w_{(z,a)}) > c_\rho \quad \textit{for all } (z, a) \in \Pi(\rho).$$

Proof. Let $f \equiv 0$. We first note that there exists $m_0 > 0$ such that $\|v_{(z,a)}\| > m_0$ for all $(z, a) \in \overline{\Pi}(\rho_2)$. From the definition, $\|\nabla J(v_{(z,a)})\| \to 0$ as $d(z)a \to \infty$. Then, since $\|w_{(z,a)} - v_{(z,a)}\| \leq C_2\|\nabla J(v_{(z,a)})\|$ for each $(z, a) \in \overline{\Pi}(\rho_2)$, we may assume, taking ρ_2 sufficiently large, that

$$\|w_{(z,a)} - v_{(z,a)}\| < m_0/4 \quad \text{for all } (z, a) \in \overline{\Pi}(\rho_2). \tag{4.1}$$

Let $\rho \geq \rho_2$. Since $f \equiv 0$ and $w_{(z,a)} \in S$, we have that $J(w_{(z,a)}) > c$ for all $(z, a) \in \Pi(\rho)$. Suppose that $\inf\{J(w_{(z,a)}) : (z, a) \in \Pi(\rho)\} = c$. Then, by Lemma 2.1, we have that there exist sequences $\{(z_n, a_n)\} \subset \Pi(\rho)$ and $\{(z'_n, a'_n)\} \subset \Omega \times (1, \infty)$ such that $\lim_{n\to\infty} d(z'_n)a'_n = \infty$ and $\lim_{n\to\infty} \|w_{(z_n,a_n)} - u_{(z'_n,a'_n)}\| = 0$. By the definition of $v_{(z,a)}$, it follows that $\lim_{n\to\infty} \|v_{(z'_n,a'_n)} - u_{(z'_n,a'_n)}\| = 0$ and then $\lim_{n\to\infty} \|w_{(z_n,a_n)} - v_{(z'_n,a'_n)}\| = 0$. Then from (4.1), $\lim_{n\to\infty} \|v_{(z_n,a_n)} -$

$v_{(a'_n,z'_n)}\| \le m_0/4$. On the other hand, recalling that $d(z_n)a_n = \rho$ for all $n \ge 1$, we have by the definition of $v_{(z,a)}$ that $\lim_{n\to\infty} \|v_{(z_n,a_n)} - v_{(a'_n,z'_n)}\| \ge 2m_0$. This is a contradiction. $\square$

As a direct consequence of Lemma 4.1 we have that

Lemma 4.2. *There exists $\varepsilon_3 > 0$ such that $\varepsilon_3 < \min\{\varepsilon_1, \varepsilon_2\}$ and that for each $f \in C(\overline{\Omega})$ with $|f|_{C(\overline{\Omega})} < \varepsilon_3$,*

$$\inf\{J(w_{(z,a)}) : (z,a) \in \Pi(\rho_2)\} > c.$$

Lemma 4.3. *Let $f \in C(\overline{\Omega})$ with $f \not\equiv 0$, $f \ge 0$ on Ω and $|f|_{C(\overline{\Omega})} < \varepsilon_3$. Then there exists $\rho_3 > 0$ and $C_3, C_4 > 0$ such that*

$$J(w_{(z,a)}) \le c + C_3(d(z)a)^{-(N-2)} - C_4 d(z)a^{-(N-2)/2} \quad \textit{for } (z,a) \in \overline{\Pi}(\rho_3).$$

Proof. We prove Lemma 4.3 by a parallel argument with that of Lemma 3 of [5]. Let $(z,a) \in \overline{\Pi}(\rho_2)$ and put $\widetilde{v}_{(z,a)} = t_v v_{(z,a)}$, where $t_v = t_{v_{z,a}}$ is the positive number defined in Lemma 2.4. Then $J(\widetilde{v}_{(z,a)}) = \max\{J(tv_{(z,a)}) : t > 0\}$. Therefore, by the definition of $w_{(z,a)}$, we find that $J(w_{(z,a)}) \le J(\widetilde{v}_{(z,a)})$. Then to prove the assertion it is sufficient to show that

$$J(\widetilde{v}_{(z,a)}) \le c + C_3(d(z)a)^{-(N-2)} - C_4 d(z)a^{-(N-2)/2} \quad \text{for } (z,a) \in \overline{\Pi}(\rho_3). \tag{4.2}$$

From the definition of $v_{(z,a)}$, we can see that there exist positive numbers t_1, t_2 such that $t_1 < t_{v_{(z,a)}} < t_2$ for all $(z,a) \in \overline{\Pi}(\rho_2)$. Then from the definition of $\widetilde{v}_{(z,a)}$, we have that

$$\begin{aligned} J(\widetilde{v}_{(z,a)}) \le{}& \max_{t\ge 0}\left\{\frac{t^2}{2}\int_\Omega |\nabla v_{(z,a)}|^2 - \frac{t^{2^*}}{2^*}\int_\Omega v_{(z,a)}^{2^*}\right\} \\ &- \min_{t_1\le t\le t_2}\left\{\frac{1}{2^*}\int_\Omega ((tv_{(z,a)}+u_0)^{2^*} - u_0^{2^*} - (tv_{(z,a)})^{2^*} - 2^* t u_0^{2^*-1} v_{(z,a)})\right\}. \end{aligned}$$

Then, by (2.11) and (2.12), it is easy to verify (cf. [5]) that

$$\max_{t\ge 0}\left\{\frac{t^2}{2}\int_\Omega |\nabla v_{(z,a)}|^2 - \frac{t^{2^*}}{2^*}\int_\Omega v_{(z,a)}^{2^*}\right\} \le c + O((d(z)a)^{-(N-2)}). \tag{4.3}$$

On the other hand, we have by [3] (cf. also [5]) that

$$\begin{aligned} &\int_\Omega ((tv_{(z,a)}+u_0)^{2^*} - u_0^{2^*} - (tv_{(z,a)})^{2^*} - 2^* t u_0^{2^*-1} v_{(z,a)}) \\ &\qquad\qquad \ge \int_\Omega u_0 (tv_{(z,a)})^{2^*-1} - C\int_\Omega u_0^{N/(N-2)} (tv_{(z,a)})^{N/(N-2)}, \end{aligned}$$

for some $C > 0$. Then, by (2.13) and (2.15),

$$
\begin{aligned}
(4.4)\quad -\min_{t_1\le t\le t_2}&\left\{\frac{1}{2^*}\int_\Omega ((tv_{(z,a)}+u_0)^{2^*}-u_0^{2*}-(tv_{(z,a)})^{2^*}-2^*tu_0^{2^*-1}v_{(z,a)})\right\}\\
&\le -\frac{1}{2^*}\left(t_1^{2^*-1}\int_\Omega u_0v_{(z,a)}^{2^*-1}-Ct_2^{N/(N-2)}\int_\Omega u_0^{N/(N-2)}v_{(z,a)}^{N/(N-2)}\right)\\
&\le -m_1d(z)a^{-(N-2)/2}+m_2d(z)^{N/(N-2)}a^{-N/2}|\log a|
\end{aligned}
$$

for some $m_1, m_2 > 0$. Since $|\log a| \le a^{1/2}$ for a sufficiently large,

$$
d(z)^{N/(N-2)}a^{-N/2}|\log a| \le \frac{d(z)^{(N+2)/2(N-2)}}{\rho^{1/2}}d(z)a^{-(N-2)/2}
$$

for large a. Then we find that by choosing $\rho_3 > \rho_2$ sufficiently large that for $(z,a) \in \overline{\Pi}(\rho_3)$,

$$
\begin{aligned}
(4.5)\quad -\min_{t_1\le t\le t_2}\left\{\frac{1}{2^*}\left(\int_\Omega (tv_{(z,a)}+u_0)^{2^*}-u_0^{2*}-(tv_{(z,a)})^{2^*}-2^*tu_0^{2^*-1}v_{(z,a)})\right)\right\}&\\
\le -m_3d(z)a^{-(N-2)/2}&
\end{aligned}
$$

for some $m_3 > 0$. Then, by (4.3) and (4.5), (4.2) holds. □

Here we put $\Gamma = \{(z,a) \in \overline{\Pi}(\rho_3) : C_4a^{(N-2)/2} > C_3d(z)^{-(N-1)}\}$. Then one can see that $\Gamma \cong \overline{\Pi}(\rho_3) \cong \Omega$. It also follows from Lemma 4.3 that

$$
(4.6)\qquad J(w_{(z,a)}) < c \quad \text{for } (z,a) \in \Gamma.
$$

Proof of Theorem 1.1. We assume that $f \in D$ with $f \not\equiv 0$, $f \ge 0$ on Ω and $|f|_{C(\overline{\Omega})} \le \varepsilon_3$. Then by Lemma 2.8, we can define a functional $\overline{J}: \overline{\Pi}(\rho_3) \to \mathbb{R}$ by $\overline{J}(z,a) = J(w_{(z,a)})$. It is easy to see that for each critical point $(z,a) \in \operatorname{int}\overline{\Pi}(\rho_3)$ of $\overline{J}$, $w_{(z,a)}$ is a critical point of J (cf. [2], [9], [14]). We also have by (2.5) that $\overline{J}$ satisfies Palais–Smale condition on $(0,c)$. Since $\Gamma \cong \overline{\Pi}(\rho_3) \cong \Omega$, we have that $H_*(\Gamma) \cong H_*(\Omega)$. For each $[\alpha] \in H_*(\Gamma)$ with $[\alpha] \ne \{0\}$, we put

$$
(4.7)\qquad c_\alpha = \min_{\alpha\in[\alpha]}\max_{(z,a)\in\alpha} J(w_{(z,a)}).
$$

By (4.6), $c_\alpha < c$. On the other hand, by Lemma 4.2, $J(w_{(z,a)}) > c$ for $(z,a) \in \partial\overline{\Pi}(\rho_3) = \Pi(\rho_3)$. This implies that each $\alpha \in [\alpha]$ with $\max\{J(w_{(z,a)}) : (z,a) \in \alpha\} < c$ is contained in the interior of $\overline{\Pi}(\rho_3)$. Therefore c_α is a critical value of $\overline{J}$ and there exists a critical point $(z,a) \in \overline{\Pi}(\rho_3)$ of $\overline{J}$ with $\overline{J}(z,a) = c_\alpha$. That is there exists a critical point $w_{(z,a)}$ of J with $J(w_{(z,a)}) = c_\alpha$. Since each critical point $w_{(z,a)}$ is nondegenerate by the assumption, we obtain that the number of critical points obtained by the formula (4.7) is $\Sigma_{p=0}^\infty \dim H_p(\Omega)$. □

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Manuscript received April 3, 2001

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