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INDEX BUNDLE, LERAY–SCHAUDER REDUCTION AND BIFURCATION OF SOLUTIONS OF NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

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Dedicated to Andrzej Granas

Abstract. We show that a family F_p; $p \in P$ of nonlinear elliptic boundary value problems of index 0 parametrized by a compact manifold admits a reduction to a family of compact vector fields parametrized by P if and only if its index bundle $\operatorname{Ind} F$ vanishes. Our second conclusion is that, in the presence of bounds for the solutions of the boundary value problem, the non vanishing of the image of the index bundle under generalized J-homomorphism produces restrictions on the possible values of the degree of F_p. The most striking manifestation of this arises when the first Stiefel–Whitney class of the index bundle is nontrivial. In this case, the degree of F_p must vanish! From this we obtain a number of corollaries about bifurcation from infinity for solutions of nonlinear elliptic boundary value problems.

Introduction

The degree theory for compact perturbations of identity, compact vector fields in Granas terminology, was introduced by Leray and Schauder in their celebrated paper [20] as a fundamental tool for the study of the existence of solutions of nonlinear elliptic boundary value problems. Their purpose was to improve the well-known continuation method of Bernstein. The use of the degree theory leads to a sharp improvement of this technique. One embeds the

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nonlinear operator F associated with a given boundary value problem into a one-parameter family F_t, $0 \le t \le 1$ with $F_1 = F$ and such that F_0 is much simpler to deal with; say a linear isomorphism. If the problem $F_t(x) = 0$ can be restated in terms of the existence of zeroes of a family of compact vector fields then assuming that the vector field associated with F_0 has nonzero degree and using the homotopy invariance, the existence of solutions of $F(x) = 0$ becomes a consequence of the existence of a priori bounds for the solutions of $F_t(u) = 0$, $0 \le t \le 1$. In this way they get rid of the condition of having invertible Frechet derivative of F_t at any point, which was necessary in the Bernstein method. Later, in [17], Granas introduced a more elementary approach to continuation based on homotopy extension property for compact vector fields.

Using Schauder estimates, Leray and Schauder were able to show that the nonlinear operators in Hölder spaces associated with the Dirichlet boundary value problem for second order quasilinear elliptic equations can be reduced, in quite an explicit way, to a compact vector field whose zeroes are in one to one correspondence with the solutions of the problem. In the same paper they considered also the Dirichlet problem for fully nonlinear second order elliptic equations and introduced a second type of reduction. This later reduction was further developed under the name of intertwined representation by Browder, Browder–Nussbaum in [12], [11] and by Krasnosel'skiĭ–Zabreĭko in [18]. For the oblique derivative problem on the two-disk, for quasilinear second order elliptic equations, the direct application of Leray–Schauder's method does not produce a compact vector field (cf. [19]). In the survey article [21] Nirenberg raised the question of whether the existence of solutions for general nonlinear elliptic boundary value problems on bounded domains can be reduced to the existence of zeroes of compact vector fields. What should be understood by a reduction was not specified. In [14] Fitzpatrick and Pejsachowicz, using results of Schnirelman [28] and Babin [8], proved that an appropriate modification of the Leray–Schauder method allows to recast to a compact vector field any fully nonlinear elliptic operator subjected to general boundary conditions of the Shapiro–Lopatinskiĭ type, provided that the data are smooth enough and the problem is Fredholm of index 0. More precisely, in [14] was shown that the map F induced by the nonlinear differential operator in Hölder and Sobolev spaces can be reduced to a compact vector field by composing on the left with a compact family of isomorphisms.

The purpose of this paper is to examine the existence of the reduction and the continuation property for solutions of nonlinear elliptic boundary value problems when a topologically nontrivial parameter space comes into play. Our results can be summarized as follows:

Not every family $\{F_p : p \in P\}$ of elliptic boundary value problems, depending smoothly on a parameter belonging to a compact connected manifold P, admits a reduction of the type described above to a parametrized family of compact vector fields. The obstruction to this is a homotopy invariant of the family of linearized equations at a given point, called analytical index or index bundle.

Our second result is that in the presence of bounds for the solutions, the non-vanishing of the image of the index bundle under generalized J-homomorphism produces restrictions on the possible values of the degree of F_p. The most striking manifestation of this arises when the first Stiefel–Whitney class of the index bundle is nontrivial. In this case, if the solutions are bounded, then the degree of F_p must vanish! From this we obtain a number of corollaries about bifurcation from infinity for solutions of elliptic boundary value problems.

In the very special case of compact perturbations of a fixed linear Fredholm map (e.g. for semilinear equations) the above results were announced in [22] with the proofs sketched only. The proof for the general elliptic boundary value problems goes through a reduction to this special case. However, for the sake of completeness, we included the full proofs here. Therefore this paper is independent of [22]. Considerable improvements of the results in [22] were obtained by Bartsch [9] for the semilinear case. Among other things he proved that the bifurcating branches are global. I don't know whether the techniques used in [9] for the semilinear case can be extended to fully nonlinear problems.

The paper is organized as follows: the main results of the paper are stated in Section 1 together with some consequences regarding bifurcation. Section 2 is devoted to proofs of the main theorems. In Section 3 we compute our invariant in a very particular case; the oblique derivative problem for second order elliptic equation on the disk with constant vector field on the boundary regarded as parameter. This result was obtained in collaboration with P. M. Fitzpatrick. Finally we give some applications to bifurcation from infinity of solutions of boundary value problems of positive index.

Section 1

1.1. Reduction and degree. Let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary. A nonlinear elliptic boundary value problem is a nonlinear partial differential equation of the form

$$(1.1)\qquad \begin{cases} f(x, u(x), \dots, D^{2k}u(x)) = 0, & x \in \Omega, \\ g_i(x, u(x), \dots, D^{m_i}u(x)) = 0, & x \in \partial\Omega,\ 1 \le i \le k,\ m_i \le 2k-1, \end{cases}$$

such that, for any $u \in C^\infty(\Omega)$ the (formal) linearization of $(f, g_1, \dots, g_k)$ at u given by

$$(1.2)\quad \begin{cases} \mathcal{L}(u)v = \displaystyle\sum_{|\alpha|\le 2k} \partial_\alpha f(x,u(x),\dots,D^{2k}u(x))D^\alpha v(x) & x\in\Omega,\\ \mathcal{B}_i(u)v = \displaystyle\sum_{|\alpha|\le m_i} \partial_\alpha g_i(x,u(x),\dots,D^{m_i}u(x))D^\alpha v(x) & x\in\partial\Omega,\ 1\le i\le k. \end{cases}$$

has the following two properties:

(i) the linear operator $\mathcal{L}(u)$ is uniformly elliptic in Ω,

(ii) the boundary operators $\mathcal{B}_i(u); 1\le i\le k$ verify the Shapiro–Lopatinskiĭ covering conditions with respect to the differential operator $\mathcal{L}(u)$

Here $f(x,\eta)$, $g_i(x,\eta)$ are assumed smooth on all arguments, $\alpha=(\alpha_1,\dots,\alpha_n)$ is a multi-index, ∂_j denotes the first partial derivative with respect to the j-th variable. $D^\alpha=\prod_{i=1}^n(\partial_i)^{\alpha_i}$ is the α-th partial derivative. Let us recall (cf. [14]) that a linear partial differential operator $\mathcal{L}$ given by

$$\mathcal{L}u(x)=\sum_{|\alpha|\le 2k} a_\alpha(x)D^\alpha u(x)$$

is called uniformly elliptic if its principal symbol

$$p^0(x,\xi)\equiv\sum_{|\alpha|=2k} a_\alpha(x)(i\xi)^\alpha$$

verifies the following property: there is some $c>0$ such that

$$|p^0(x,\xi)|\ge c|\xi|^{2k}\quad\text{for all } x\in\Omega,\ \xi\in\mathbb{R}^n.$$

Given a uniformly elliptic operator $\mathcal{L}$ of order 2k on Ω and given a set of k boundary operators

$$\mathcal{B}_i u(x)=\sum_{|\alpha|\le m_i} b^i_\alpha(x)D^\alpha u(x);\quad 0\le m_i\le 2k-1,\ 1\le i\le k.$$

For each $x\in\partial\Omega$, for each , $\eta\in\mathbb{R}^n\backslash\{0\}$ normal to $\partial\Omega$ at x and $\xi\in\mathbb{R}^n\backslash\{0\}$ with $\langle\xi,\eta\rangle=0$, consider the $(k+1)$ polynomials of a single complex variable τ

$$(1.3)\quad \begin{cases} \tau\mapsto p^0(x,\xi+\tau\eta),\\ \tau\mapsto \displaystyle\sum_{|\alpha|=m_i} b^i_\alpha(x)i^\alpha(\xi+\tau\eta)^\alpha\equiv q_i(\tau)\quad\text{for } 1\le i\le k. \end{cases}$$

Let $\tau_1,\dots,\tau_k$ be the k complex zeros of $p^0(x,\xi+\tau\eta)$ with positive imaginary part. The family $\mathcal{B}_i$, $1\le i\le k$ verifies the Shapiro–Lopatinskiĭ covering condition with respect to $\mathcal{L}$ provided that the polynomials $\{q_i\}_{i=1}^k$ are linearly independent modulo the ideal generated by $q(\tau)=\prod_{i=1}^k(\tau-\tau_i)$.

Let X' be either the Hölder space $C^{2k+2,\alpha}(\Omega)$, $0\le\alpha\le 1$ or the Sobolev space $H^{2k+2+s}(\Omega)$, $s\ge 0$ and Y' be either $C^{2,\alpha}(\Omega)\times\prod_{i=1}^k C^{2k+2-m_i,\alpha}(\partial\Omega)$

or $H^{2+s}(\Omega) \times \prod_1^k H^{2k+s-m_i+3/2}(\partial\Omega)$. Any nonlinear elliptic boundary value problem $(f, g_1, \dots, g_k)$ induces a map $F: X' \to Y'$ defined by

$$F(u) = (f(x, \dots, D^{2k}u), \gamma g_1(x, \dots, D^{m_1}u), \dots, \gamma g_k(x, \dots, D^{m_k}u)),$$

where γ is the restriction to the boundary $\partial\Omega$. It is well known that for smooth data $\{f, g_1, \dots, g_k\}$ the map F is smooth on Hölder spaces and can be made differentiable of any order on the Sobolev spaces provided one take the exponent s large enough. Its Frechet derivative $DF(u)$ at a given point u is the map induced on the same spaces by the linearized partial differential operators (1.2). By elliptic regularity (e.g. Agmon–Douglis–Nirenberg estimates), $DF(u)$ is a Fredholm operator for any $u \in X'$ and hence F is a Fredholm map. As such it has an index ind F defined as the index of the Frechet derivative $DF(u)$ at any $u \in X'$, because $\operatorname{ind} DF(u) = \dim\ker DF(u) - \dim \operatorname{coker} DF(u)$ is independent of the choice of u.

It was shown in [14] (cf. Proposition 2.8, Theorem 10.15 and 10.19) that any F as above with ind $F = 0$ is equivalent to a compact vector field in the following sense: there exist a compact map M from X' into the set $\mathrm{GL}(X', Y')$ of all isomorphisms from X' to Y' and a compact map $C: X' \to Y'$ such that

$$F(u) = M(u)(\mathrm{Id} + C)(u). \tag{1.4}$$

Notice that the spaces X', Y' are not the natural spaces on which an operator of order $2k$ acts. We require the order of differentiability of functions in our spaces raised by two. This is due to the method in proof of (1.4). Only with this extra differentiability requirement we were able to prove that the map F induced by the nonlinear boundary value problem is a quasilinear Fredholm map (see Section 2). We will keep this condition everywhere in the paper.

Since the method in [14] is rather abstract the reduction obtained there is less explicit than the one given by Leray and Schauder in the second order case. However it is clear from the form of (1.4) that bounds for the solutions of the equation $F(u) = 0$ give bounds for the zeroes of the compact vector field $\mathrm{Id} + C$. Moreover, one can show easily that if U is an open bounded subset of X' with no solutions of the equation $F(u) = 0$ on the boundary then the Leray–Schauder degrees $\deg_{LS}(\mathrm{Id} + C, U, 0)$ of two different compact vector fields equivalent to F in the sense of (1.4) will necessarily coincide up to a sign. In [14] after introducing a function $\varepsilon: \mathrm{GL}(X', Y') \to \{1, -1\}$, called orientation, it was proved that the number $\deg(F, U, 0) = \varepsilon(M) \deg_{LS}(\mathrm{Id} + C, U, 0)$ is independent of the choice of representation (1.4). In this way it is possible to assign a well defined degree directly to the map F. This degree verifies all the axioms of degree theory except for the homotopy invariance which holds in a weaker form. We shall not go any further into the details of the construction since here we will need only the absolute value of $\deg(F, U, 0)$, which is independent of the choice of orientation and is homotopy invariant.

1.2. The index bundle. Let us introduce some notation. In what follows and when confusion does not arise, a family of maps parametrized by a topological space P (or simply a parametrized family) will mean either a continuous map from P into a space of maps or a continuous map $F: P \times X \to Y$ of a product space into another topological space. In both cases, in order to distinguish parameters, we will use the parameter variable p as a subscript. Thus $F_p: X \to Y$ is the map defined by $F_p(x) = F(p, x)$. Also from now and except when explicitly stated Fredholm means Fredholm of index 0. Accordingly a family L of Fredholm operators parametrized by P is a continuous map $L: P \to \Phi_0(X, Y)$, where X, Y are Banach spaces and $\Phi_0(X, Y)$ is the space of all Fredholm operators of index 0 with the relative topology. Coherently with our notation $L_p \in \Phi_0(X, Y)$ will denote the value of L at the point p.

If the parameter space P is compact then, following Atiyah and Janich [5], to each family L as above we can assign a homotopy invariant called analytical index or index bundle. The index bundle is not a bundle but rather a stable equivalence class of vector bundles. Let us recall that two finite dimensional vector bundles are stable equivalent if they become isomorphic after an addition (direct sum) of a trivial bundle on both sides. The stable equivalence classes of vector bundles over P form a group $\widetilde{\mathrm{KO}}(P)$ called the reduced Grothendieck group of virtual bundles of rank 0.

Given a family $L: P \to \Phi_0(X, Y)$, its index bundle $\operatorname{Ind} L \in \widetilde{\mathrm{KO}}(P)$ is defined as follows: using compactness of P one can find a finite dimensional subspace V of Y such that $\operatorname{Im} L_p + V = Y$ for any p in P. It follows then that the family of finite dimensional spaces $E_p = L_p^{-1}(V)$ form a vector bundle E over P. By definition $\operatorname{Ind} L \in \widetilde{\mathrm{KO}}(P)$ is the stable equivalence class of E. It is easy to see that $\operatorname{Ind} L$ is well defined and that it depends only on the homotopy class of the family L. In particular it is invariant under perturbation by families of compact operators. Moreover, $\operatorname{Ind} L = 0$ if and only if the map L is homotopic to a map with values in $\mathrm{GL}(X, Y)$. More generally if $L: P \to \Phi(X, Y)$ is a family of Fredholm operators of any index, then the Atiyah–Janich construction assign an index $\operatorname{Ind} L$ in the full Grothendieck group $\mathrm{KO}(P)$. This later is the group completion of the abelian semigroup $\mathrm{Vect}\,(P)$ of all isomorphisms classes of vector bundles over P. Its elements are called virtual vector bundles because every element on $\mathrm{KO}(P)$ can be written as difference $[E] - [F]$ where E, F are vector bundles over P and $[E]$ denotes the equivalence class. Assigning to each stable equivalence class of a vector bundle E the element $[E] - [\Theta^n]$ where n is the rank of E and $\Theta^n = P \times \mathbb{R}^n$ is the trivial bundle of rank n one get an inclusion of $\widetilde{\mathrm{KO}}(P)$ in $\mathrm{KO}(P)$.

The index bundle in the general case is given by the same construction. Chosen an finite dimensional subspace V of Y transverse to $\operatorname{Im} L_p$ the index bundle or analytical index of the family is defined as the virtual bundle $\operatorname{Ind} L = [E] - [\Theta(V)] \in \mathrm{KO}(P)$, where E is the same as before and $\Theta(V)$ is the trivial bundle over P with fiber V. $\operatorname{Ind} L$ is invariant by Fredholm homotopies and verifies the following additivity property: given $L: P \to \Phi(X, Y)$ and $M: P \to \Phi(Y, Z)$ then $\operatorname{Ind} ML = \operatorname{Ind} M + \operatorname{Ind} L$ (cf. [10]).

1.3. The Generalized J-homomorphism. Endowing a vector bundle E with a norm, we can consider the associated unit sphere bundle $S(E)$ over P with fiber at $p \in P$ given by $S_p(E) = \{v \in E_p / \|v\|_p = 1\}$. A vector bundle is called stably homotopy trivial if there is a trivial bundle Θ such that the sphere bundle of $S(E \oplus \Theta)$ is fiber-wise homotopy equivalent to a trivial sphere bundle. Stably homotopy trivial bundles form a subgroup of $\widetilde{\mathrm{KO}}(P)$. The group $J(P)$ is defined as the quotient of $\widetilde{\mathrm{KO}}(P)$ by this subgroup. The generalized J homomorphism, $J: \widetilde{\mathrm{KO}}(P) \to J(P)$ is the projection to the quotient. The groups $J(P)$ were introduced by Atiyah in [6] who also proved that the J group of a compact CW-complex is finite by relating this group to the image of the classical J-homomorphism of Whitehead. Earlier, Thom showed that the Stiefel–Whitney characteristic classes of a vector bundle depend only on the fiber homotopy type of the associated sphere bundle. Since the first Stiefel–Whitney characteristic class $\omega_1 \in H^1(P, Z_2)$ and the total class $\omega = 1 + \omega_1 + \ldots + \omega_n \in H^*(P, Z_2)$ are also stable in the sense that they do not change after addition of the trivial bundle, it follows that both classes factor through $\widetilde{\mathrm{KO}}(P)$ and even through $J(P)$. In other words, they are well defined for elements of $J(P)$.

1.4. The main results. Let P be a compact connected manifold and let

$$f \in C^\infty(P \times \overline{\Omega} \times \mathbb{R}^{\overline{2k}}), \quad g_i \in C^\infty(P \times \overline{\Omega} \times \mathbb{R}^{\overline{m_i}}), \quad 1 \le i \le k,$$

be such that for each $p \in P$ the data $(f_p, g_{p,1}, \ldots, g_{p,k})$ define an elliptic boundary value problem in the sense of (1.1) and (1.2). If X', Y' are either the Hölder spaces or the Sobolev spaces defined at the beginning of this Section then our data define a differentiable map $F: P \times X' \to Y'$ such that for each $p \in P$ the map $F_p: X' \to Y'$ is Fredholm. Fix some $u \in X'$ and let $L_p = DF_p(u)$ be the Frechet derivative of F_p at u. Each L_p is the operator induced in the Hölder spaces by the linear partial differential operators $(1, 2)$ and by assumption these are Fredholm of index 0. Thus we have a family $L: P \to \Phi_0(X', Y')$ and this family has a well-defined index bundle $\operatorname{Ind} L$ in $\widetilde{\mathrm{KO}}(P)$. From the homotopy invariance of the index bundle it follows that $\operatorname{Ind} L$ is independent of the choice of the point $u \in X'$. We will denote this element by $\operatorname{Ind} F$ and refer to it as the index bundle of F. Let

$$\begin{cases} \widetilde{\mathcal{L}}_p(0)v = \displaystyle\sum_{|\alpha|=2k} \partial_\alpha f(p,x,0,\ldots,0)D^\alpha v(x) & x\in\Omega,\\ \widetilde{\mathcal{B}}_{p,i}(0)v = \displaystyle\sum_{|\alpha|=m_i} \partial_\alpha g_i(p,x,0,\ldots,0)D^\alpha v(x) & x\in\partial\Omega,\ 1\le i\le k. \end{cases}$$

be the principal part (i.e. top order terms) of the linearization of (1.2) at the point 0. Then $\operatorname{Ind} F$ depends only on $(\widetilde{\mathcal{L}}_p(0),\widetilde{\mathcal{B}}_{p,1}(0),\ldots,\widetilde{\mathcal{B}}_{p,k}(0))_{p\in P}$. This is due to the fact that lower order term perturbations of partial differential operators defined on bounded domains are compact and hence the family L is homotopic to the family of operators in $\Phi_0(X',Y')$ induced by $(\widetilde{\mathcal{L}}_p(0),\widetilde{\mathcal{B}}_{p,1}(0),\ldots,\widetilde{\mathcal{B}}_{p,k}(0))_{p\in P}$.

Under favorable circumstances, the determination of the index bundle of a family of elliptic boundary value problems can be transformed into a similar problem but for a family of pseudo-differential operators on the boundary by means of the Agranovich–Dynin reduction [3]. In this case there is an explicit formula for $\operatorname{Ind} F$ in terms of the principal symbols of the family of pseudo-differential operators given by the Atiyah–Singer theorem for families [7].

Theorem 1.1. *If $F\colon P\times X'\to Y'$ is as above then there exists a family of $M\colon P\times X\to \mathrm{GL}(Y',X')$ and a compact map $C\colon P\times X'\to Y'$ such that*

$$M(p,u)F(p,u)=u+C(p,u) \tag{1.5}$$

if and only if $\operatorname{Ind} F=0$ *in* $\widetilde{\mathrm{KO}}(P)$.

Remark. As we mentioned before a different type of reduction to compact vector fields based on the so-called intertwining representation was obtained by various authors. In particular the reductions of the Dirichlet problem for strongly elliptic equations obtained by [20] and [27] are far more explicit that (3.1). In the abstract setting the question is studied in [11], [18] and [26] using an approach based on reduction to compact vector fields by smooth change of coordinates on the domain of the map. This is called global right equivalence in singularity theory. Sapronov proved that if F is a proper, smooth enough Fredholm map defined on an open subset U of X then $\operatorname{Ind} F$ is essentially the only obstruction for the right equivalence of F with a compact vector field [26].

Our next result describes the restrictions on the degree of the map F_p imposed by non vanishing of the image of $\operatorname{Ind} F$ under the generalized J-homomorphism.

Theorem 1.2. *Assume that there are bounds of the form $\|u\|<R$ for the solutions of the nonlinear elliptic boundary value problem*

$$\begin{cases} f_p(p,x,u(x),\ldots,D^{2k}u(x))=0 & x\in\Omega,\\ g_{p,i}(p,x,u(x),\ldots,D^{m_i}u(x))=0 & x\in\partial\Omega,\ 1\le i\le k,\ m_i\le 2k-1. \end{cases}$$

Let $d = |\deg(F_p, B(0,R), 0)|$ (*d is independent of p by the homotopy property if the degree*). *Then:*

(i) *There exist a natural number m such that* $d^m \cdot J(\operatorname{Ind} F) = 0$ *in* $J(P)$.

(ii) *Moreover, d vanishes if* $\omega_1(\operatorname{Ind} F) \neq 0$, *i.e. whenever the index bundle* $\operatorname{Ind} F$ *is non orientable.*

As was mentioned in the introduction, the Leray–Schauder continuation method based on the homotopy invariance of degree reduces the proof of the existence of solutions of a given boundary value problem to finding a priori bounds for solutions of one-parameter families of problems.

The assertion (ii) of above theorem says that for families parametrized by non-contractible spaces the two conditions on which the method is based may not be always compatible; one cannot have at the same time a priori bounds and non trivial degree of F_p. The obstruction being given by the nonvanishing of $\omega_1(\operatorname{Ind} F)$. On the other hand, since the groups $J(P)$ are finite, by (i) the nonvanishing of $J(\operatorname{Ind} f)$ always imposes restrictions on the value of the degree d.

COROLLARY 1.1. *Under the assumptions of Theorem* 1.2, *if* $J(\operatorname{Ind} F) \neq 0$ *then d cannot be prime to the order of* $J(P)$ (*in particular* $d \neq 1$).

PROOF. Since both d^n and the order of $J(P)$ are multiples of the order of $J(\operatorname{Ind} F) \neq 0$ it follows that d and the order of $J(P)$ cannot be coprime. □

There is an alternative but closely related way to describe the restrictions on the degree arising from the nontriviality of the image of the index bundle under the generalized J homomorphism. This was explored by Bartsch [9] for the semilinear case. He described the restrictions in terms of the codegree of the index bundle. This has some advantage, since the notion of codegree easily extends to general parameter spaces. The disadvantage consist in that codegree is not easy to compute. On the other hand the order of $J(\operatorname{Ind} F)$ and the codegree of $\operatorname{Ind} F$ have the same primes in the prime decomposition arising perhaps with different powers. Here we prefer to use the order of $J(P)$ because this invariant has been computed in a number of interesting cases.

COROLLARY 1.2. *Under the assumptions of Theorem* 1.2, *if* $J(\operatorname{Ind} F) \neq 0$ *and the homology of P is free of* 2*-torsion then d cannot be prime to*

$$j = \prod_{q=0}^{\infty} \operatorname{order}\,[H_q(P,Z) \otimes J(S^q)].$$

PROOF. By [25, Theorem 2] the order of $J(P)$ divides k. □

The groups $J(S^n)$ have been determined for any n. They vanish for $n \equiv 3,5,7$ mod-8. For $n \equiv 1,2 \mod -8$, $J(S^n)$ is cyclic of order 2 and for $n = 4k$ it is cyclic of order given by the denominator of $B_k/4k$ expressed in lowest terms, where B_k is the k-th Bernoulli number. Thus the restriction on the degree provided by Corollary 1.2 is perfectly computable.

Let us recall that a bifurcation point from infinity for solutions of (1.6) is a point $p_0 \in P$ such that there exists a sequence (p_n, u_n) of solutions of the equation $F(p, u) = 0$ such that $p_n \to p_0$ and $\|u_n\| \to \infty$.

Corollary 1.3. *Assume that $J(\operatorname{Ind} F) \neq 0$ and that for some $q \in P$ there are bounds for the solutions of the equation $F_q(u) = 0$. If the degree $d = \lim_{R\to\infty} \deg(F_q, B(0,R), 0)$ is prime to the order of $J(P)$, then there exist at least one point of bifurcation from infinity for the solutions of the boundary value problem* (1.6). *In particular bifurcation from infinity arises whenever $d = \pm 1$.*

Proof. If F does not have any bifurcation point from infinity then, using the compactness of P, one can produce bounds for solutions contradicting Corollary 1.2.

The order of $J(P)$ is a power of two in the case of S^n; $n \equiv 1, 2 \mod -8$ and in the case of real projective spaces RP^n, $n \geq 1$. Moreover, in all those cases the J-homomorphism is injective. It follows from the above corollary that bifurcation from infinity arises whenever $\operatorname{Ind} F \neq 0$ and d is odd. □

Corollary 1.4. *Assume that for some $q \in P$ there are bounds for the solutions of the equation $F_q(u) = 0$.*

(i) *If the degree $d \neq 0$ and the index bundle* $\operatorname{Ind} F$ *is non orientable then the Lebesgue covering dimension of the set B of all bifurcation points from infinity for solutions of $F(p, u) = 0$ must be at least $n - 1$.*

(ii) *If the degree $d = \pm 1$ and the total Stiefel–Whitney class $\omega(\operatorname{Ind} F)$ does not vanish then the Lebesgue covering dimension of the set B o f all bifurcation points from infinity for solutions of $F(p, u) = 0$ must be at least* $\dim P - \min\{k/\omega_k(\operatorname{Ind} F) \neq 0\}$.

Proof. Since the first Stiefel–Whitney class of the index bundle does not vanish one can find a closed path γ in P passing through q and such that $\omega_1(\operatorname{Ind} f)$ evaluated on $\alpha = \gamma_*(\text{generator of } H_1(S^1, Z_2))$ is nonzero. From this it follows easily that, in the exact sequence of a pair of spaces, α is mapped into a nontrivial class in $H_1(P, P - B, Z_2)$ and hence determines by duality a nonzero class in the $(n-1)$-Cěch cohomology group of Bif(F). By the cohomological characterization of dimension it follows that dim $B \geq n - 1$.

The proof of (ii) is similar using the the fact that any homology class in $H_k(P, Z_2)$ is the image of the fundamental class of a submanifold. □

The assertion (i) in the above Corollary was proved by other means in [15]. Related results were also obtained by Alexander and Antman in [4], for compact vector-fields parametrized by $\mathbb{R}^n$, but under different type of assumptions, and for semilinear Fredholm maps by Bartsch in [9].

Remark. A standard approach to bifurcation from infinity, see for example [23], is to look at the spectrum of the asymptotic derivative. Notice the absence in the above corollaries of any assumption about even the existence of asymptotic derivative at infinity. Here the bifurcation is forced by the nontriviality of a topological invariant associated to the linearization of the problem at a given point of X'.

If $F(p,0) \equiv 0$ and for some $q \in P$ we have that 0 is an isolated solution of $F_q(u) = 0$, then taking $d = \lim_{R\to 0} \deg(F_q, B(0,R), 0)$ we obtain parallel results for bifurcation of solutions of the nonlinear equation $F(p,u) = 0$ from the trivial branch. As in the case of bifurcation from infinity, in presence of a topologicaly nontrivial parameter space, the appearance of bifurcation points here is caused by the non vanishing of invariants that depends only on the top-order coefficients of the linearized operator. This type of results are of a different nature from the ones that can be obtained using the classical approcach.

Section 2

2.1. Proof of Theorem 1.1. In the following definition we shall assume that the Banach space X is compactly embedded in a Banach space X_1. For spaces defined in Section 1 one can take as X_1 $C^{2k+1,\alpha}(\Omega)$ and $H^{2k+1+s}(\Omega)$. Given $D \subseteq X$, when we refer to its topological properties we will be referring to the topology induced by X, unless explicitly stated.

Definition. Let the parameter space P be as before. A mapping $F\colon P\times X \to Y$ is called a *family of quasilinear Fredholm maps parametrized* by P provided that F has a representation of the form

$$F(p,x) = L_{(p,x)}x + C(p,x) \tag{2.1}$$

where $L\colon P \times X \to \Phi(X,Y)$ is given by the restriction to X of a continuous map $\overline{L}\colon P \times X_1 \to \Phi(X,Y)$ and $C\colon P \times X \to Y$ is compact.

We will refer to formula (2.1) as a representation of the family F. The family of Fredholm operators $L\colon P \times X \to \Phi(X,Y)$ will be called the principal part of F in the the corresponding representation. While the representation is not unique the principal parts corresponding to two different representations of the same family of quasilinear Fredholm maps differ by a family of compact operators. In what follows, except when explicitly stated, we shall consider quasilinear maps of index 0 only. In this case the principal part L take values in $\Phi_0(X,Y)$. A family of quasilinear Fredholm maps having a representation whose principal part is independent of $x \in X$ will be called semilinear. Thus, a semilinear Fredholm family of index 0 can be represented in the form $F(p,x) = L_p(x) + C(p,x)$ with $L: P \to \Phi_0(X,Y)$ continuous and $C: P \times X \to Y$ compact.

Theorem 1.1 will be deduced from Lemma 2.1, Proposition 2.2 bellow and the following result proved in [14, Theorem 10.15, 10.19].

Theorem 2.1. *Let $f \in C^\infty(P \times \overline{\Omega} \times R^{\overline{2k}})$, $g_i \in C^\infty(P \times \overline{\Omega} \times R^{\overline{m_i}})$; $1 \leq i \leq k$ be such that for each $p \in P$ the above data define an elliptic boundary value problem on Ω. If X', Y' are the function spaces introduced in Section 1, then the family $F: P \times X' \to Y'$ induced by $(f_p, g_{p,1}, \dots, g_{p,k})$ is a family of quasilinear Fredholm maps.*

Our first step will consists on a reduction of quasilinear Fredholm families to semilinear families.

Definition. Let T be a metric space and let $F: T \to Y$ and $G: T \to Z$ be continuous maps from T into Banach spaces Y, Z. We will say that F and G are *compactly equivalent* if there exist a compact family of linear isomorphism $M: T \to \mathrm{GL}(Z,Y)$ such that $F(t) = M_t(G(t))$. (Recall that a map is compact if it sends bounded sets of the domain into relatively compact sets of the range.)

Lemma 2.1. *Any quasilinear family of Fredholm maps $F: P \times X \to Y$ is compactly equivalent to a semilinear family $G: P \times X \to Y$. Moreover, if F has a representation $F(p,x) = L_{(p,x)}(x) + \widetilde{C}(p,x)$ then G can be represented as $G(p,x) = L_p(x) + C(p,x)$ with $L_p = L_{(p,0)}$ and $M_{p,0} = \mathrm{Id}_Y$.*

Remark. It follows easily from the above that if each F_p is differentiable and such that $F_p(0) = 0$ then also $G_p(0) = 0$ and $DF_p(0) = DG_p(0)$.

In order to prove Lemma 2.1 we will need some properties of continuous families of linear Fredholm operators parametrized by compact spaces.

A (two sided) parametrix of a Fredholm operator $S: X \to Y$ is a Fredholm operator $R: Y \to X$ such that $RS - \mathrm{Id}_X$ and $SR - \mathrm{Id}_Y$ are compact operators. The existence of parametrix characterizes Fredholm operators (of any index) among all linear bounded operators. Since the set of all invertible elements of a topological algebra is open, using partition of unity arguments one can easily prove the following proposition (see [29, Theorem 2.8]).

Proposition 2.1. *Given any continuous family $L: T \to \Phi(X,Y)$ parametrized by a paracompact space T there exist a continuous family $R: T \to \Phi(Y,X)$ such that R_t is a parametrix of L_t for each $t \in T$.*

Such a family is called a parametrix of L. A strong parametrix of a family of Fredholm operators of index 0 is a parametrix R such that R_t is an isomorhism for each $t \in T$.

PROPOSITION 2.2. *Let T be a compact space. Then a family $L: T \to \Phi_0(X, Y)$ admits a strong parametrix if and only if* $\operatorname{Ind} L = 0$ *in* $\widetilde{\mathrm{KO}}(T)$.

PROOF. This is well-known (cf. [5], [29]). We will give the proof here for completeness since in [5] and [29], is used a different construction of the index bundle.

Assume $\operatorname{Ind} L = 0$. Let V be a finite dimensional subspace of Y such that $\operatorname{Im} L_t + V = Y$. Being the bundle $E = \bigcup_{t \in T} L_t^{-1}(V)$ stably trivial by assumption, there exists some trivial bundle Θ^k of rank k, such that $E \oplus \Theta^k$ is trivial. Let $\widetilde{V}$ be any finite dimensional subspace of Y containing V and such that $\dim \widetilde{V} = \dim V + k$. It follows from the transversality assumption that L_t induces an isomorphism between $E \oplus \Theta^k$ and the vector bundle $\widetilde{E} = \bigcup_{t \in T} L_t^{-1}(\widetilde{V})$. Hence $\widetilde{E}$ is trivial and therefore there exists an isomorphism say A between $\widetilde{E}$ and the trivial bundle $\Theta(\widetilde{V})$ with fiber $\widetilde{V}$. Let $P: T \to \mathcal{L}(X)$ be a continuous family such that P_t is a projector with $\operatorname{Im} P_t = \widetilde{E}_t$. (The existence of such a family follows easily from partition of unity arguments and the convexity of the set of all bounded projectors with a given image.) Let Q be a projector on Y with $\operatorname{Im} Q = \widetilde{V}$. Then the family M defined by $M_t = (\mathrm{Id}_{\,Y} - Q)L_t + A_t P_t$ is a compact perturbation of the family L (in the sense that $L_t - M_t$ are compact operators). Moreover, for each t, M_t is an isomorphism. From this it follows that the family $R = M^{-1}$ is the desired strong parametrix of L. The only if part follows easily from the additivity property of the index bundle (cf. [10]). □

PROPOSITION 2.3. *Let T be compact and let $L, N: T \to \Phi_0(X, Y)$ be two continuous families parametrized by T then* $\operatorname{Ind} L = \operatorname{Ind} N$ *if and only if there exists a family of isomorphisms $M: T \to \mathrm{GL}(Y)$ such that $L_t - M_t N_t$ is compact.*

PROOF. Let R be any parametrix of L. Then, by the additivity of the index bundle, $\operatorname{Ind} R = -\operatorname{Ind} L$ and hence $\operatorname{Ind} NR = 0$. By the above proposition NR has a strong parametrix $M: T \to \mathrm{GL}(Y)$. Since $\mathrm{Id}_Y - M_t N_t R_t$ is compact by composing on the right with L it follows that $L_t - M_t N_t$ is compact as well. The converse is clear. □

PROPOSITION 2.4. *Let $S \subset T$ be a pair of compact spaces such that S is a deformation retract of T (i.e. there exists a map $r: T \to S$ such that if $i: S \to T$ is the inclusion then $ri = \mathrm{Id}_S$ and ir is homotopic to Id_T). Let $L, N: T \to \Phi_0(X, Y)$ be two continuous families parametrized by T such that the restriction of $L - N$ to S is a family of compact operators. Then there exists a continuous family of isomorphisms $M: T \to \mathrm{GL}(Y)$ such that $L_t - M_t N_t \in K(X, Y)$, for any $t \in T$ and such that $M_t = \mathrm{Id}_Y$ for each $t \in S$.*

PROOF. In order to shorten notation let us introduce the equivalence relation $L \sim N$ if $L_t - N_t$ is compact for any $t \in T$. Notice that $\sim$ is additive and preserved by composition on both sides. Also, in accordance with the previous notation, we shall denote by $L_i: S \to \Phi_0(X,Y)$ the family obtained by composing $i: S \to T$ with $L: T \to \Phi_0(X,Y)$.

Since ir is homotopic to Id_T, by the homotopy invariance of the index bundle, $\operatorname{Ind} L_{ir} = \operatorname{Ind} L$ and $\operatorname{Ind} N_{ir} = \operatorname{Ind} N$. By Proposition 2.3 we can choose families $P, Q: T \to \mathrm{GL}(Y)$ such that $PL_{ir} \sim L$ and $QN_{ir} \sim N$. Composing on the right with $ir: T \to T$ we obtain also $P_{ir}L_{ir} \sim L_{ir} \sim Q_{ir}N_{ir}$, since $L_i \sim N_i$. Consider $M: T \to \mathrm{GL}(Y)$ defined by $M = PP_{ir}^{-1}Q_{ir}Q^{-1}$. Clearly, $M_i = \mathrm{Id}_Y$ and $MN \sim L$, by the previous discussion. This proves the proposition. □

Proof of the Lemma 2.1. Let $i: P \times X \to P \times X_1$ be the embedding induced by the inclusion of X into X_1. Let $N: P \times X_1 \to \Phi_0(X,Y)$ be defined by $N_{(p,x)} = L_{(p,0)}$. In order to prove the lemma it is enough to produce a compact family $M: P \times X \to \mathrm{GL}(Y)$ such that

$$ML_i \sim N_i \quad \text{and} \quad M_{(p,0)} = \mathrm{Id}_Y. \tag{2.2}$$

Indeed, if (2.2) holds then $K = ML_i - N_i$ is a compact family of compact operators and $M_{(p,x)}F(p,x)) = L_{(p,0)}(x) + C(p,x)$ with $C(p,x) = K_{p,x} + M_{(p,x)}\widetilde{C}(P,X)$ a compact map.

We build the family M inductively using Proposition 2.4. For each $n \geq 0$ define C_n as the closure in X_1 of the ball $B(0,n)$ of X and let $T_n = P \times C_n$. Since for each n, C_n is compact and convex it follows that each T_n is a deformation retract of T_{n+1}. Let L^n and N^n be the restrictions to T_n of L and N, respectively. Using Proposition 2.4 one gets a family $\overline{M}^1: T_1 \to \mathrm{GL}(Y)$ such that $\overline{M}^1 L^1 \sim N^1$ and $\overline{M}^1$ restricted to T^0 is the constant family Id_Y. Assuming we have constructed $\overline{M}^n: T_n \to \mathrm{GL}(Y)$ such that $\overline{M}^n L^n \sim N^n$ and $\overline{M}^n | T_{n-1} = \overline{M}^{n-1}$, we will produce $\overline{M}^{n+1}: T_{n+1} \to \mathrm{GL}(Y)$ with the same properties.

Let $r: T_{n+1} \to T_n$ be a retraction. Then $\overline{M}^n_r L^{n+1}|_{T_n} = \overline{M}^n L^n \sim N^n = N^{n+1}|_{T_n}$ and hence by Proposition 2.4 there exists $\widetilde{M}: T^{n+1} \to \mathrm{GL}(Y)$ with $\widetilde{M}|_{T_n} = \mathrm{Id}_Y$ and such that $\widetilde{M}\overline{M}^n_r L^{n+1} \sim N^{n+1}$. Now $\overline{M}^{n+1} = \widetilde{M}\overline{M}^n_r$ has the desired properties.

Let $T = \bigcup_{n \geq 0} T_n$ and let $\overline{M}: T \to \mathrm{GL}(Y)$ be defined by $\overline{M}_{(p,x)} = \overline{M}^n_{(p,x)}$ if $(p,x) \in T^n$. Then $\overline{M}$ is a well-defined continuous family and since $i(P \times X)$ is contained in T the family $M = \overline{M}_i$ is a compact family of isomorphisms which clearly verifies (2.2). This proves the Lemma and gives al that we need to conclude the proof of Theorem 1.1. Indeed, by Lemma 2.1, any quasilinear Fredholm family F is compactly equivalent to a semilinear family G with the same index bundle. By Proposition 2.2 the map G and hence also F have a representation of the form (1.5) if and only if $\operatorname{Ind} G = \operatorname{Ind} F = 0$. □

2.2. Proof of Theorem 1.2. Theorem 1.2 is a consequence of Theorem 2.1 and the following

Theorem 2.2. *Let* $F\colon P\times X\to Y$ *be a family of quasilinear Fredholm maps of index* 0*. If* $\{x/F(p,x)=0\}$ *is bounded, then* (i) *and* (ii) *of Theorem* 1.2 *hold with* $d=\lim_{R\to\infty}|\deg(F_p,B(0,R),0)|$.

Proof. By Lemma 2.1, the family $F\colon P\times X\to Y$ is compactly equivalent to a semilinear Fredholm family $G\colon P\times X\to Y$. Thus

$$M_{(p,x)}F(p,x)=G(p,x)=L_p(x)+C(p,x)$$

with $L_p=DG_p(0)=\widetilde{L}_{p,0}=DF_p(0)$ and $M_{(p,0)}=\mathrm{Id}_Y$. It follows that $\operatorname{Ind}F=\operatorname{Ind}L=\operatorname{Ind}G$. Clearly, there are bounds for the solutions of $F(p,x)=0$ if and only if the same holds for solutions of $G(p,x)=0$. Moreover, for any fixed $p\in P$, the homotopy $H\colon[0,1]\times X\to Y$ defined by $H(t,x)=M_{(p,tx)}F_p(x)$ is an admissible homotopy between F_p and G_p on a ball of sufficiently large radius in X. Therefore $d=|\deg(F,B(0,R),0)|=|\deg(G,B(0,R),0)|$ for R large enough. In conclusion, we can assume, without loss of generality, that our family F has a semilinear representation of the form $F(p,x)=L_p(x)+C(p,x)$ with L and C as before. Since the map F_p is a compact perturbation of a Fredholm operator we have that each F_p is proper on closed bounded sets of X. It follows from this and compactness of P that if the restriction of F_p to $B(0,R)$ has no zeroes on the boundary $\partial B(0,R)$ for any p then the same will be true for any map close enough to F in norm. Straight segment homotopy will ensure that the degree on $B(0,R)$ of the map F_p and any close enough compact perturbation of F_p will be the same. Since the map C can be arbitrarely approximated by a map with finite dimensional range we can further assume that $C\,(P\times B(0,R))$ is contained in a finite dimensional subspace V of Y. Moreover, V can be chosen big enough for the transversality condition $\operatorname{Im}L_p+V=Y$ to hold for any $p\in P$. □

Proposition 2.5. *Let* $f\colon X\to Y$ *be a semilinear map of the form* $f(x)=L(x)+C(x)$*. Assume that for some ball* $B=B(0,r)$, $0\notin f(\partial B)$ *and that* $C(B)$ *is contained in a finite dimensional subspace* V *such that* $\operatorname{Im}L+V=Y$*. Let* $W=L^{-1}(V)$ *and let* g *be the restriction of* f *to* $W\cap B=D$ *viewed as a map from* D *into* V*. Then* $\deg(f,B,0)=\pm\deg_B(g,D,0)$ *where the right hand side is the Brouwer degree of a map between finite dimensional vector spaces of the same dimension.*

Proof. Let P, Q be projectors with $\operatorname{Im}P=W$ and $\operatorname{Im}Q=V$ and let $A\colon W\to V$ be any isomorphism ($\dim V=\dim W$ by the transversality assumption). Consider the isomorphism $M\colon X\to Y$ defined by $M=(I-Q)L+AP$. Clearly $M\sim L$. Moreover, $M|_W=A$ and hence $M^{-1}|_V=A^{-1}$. By definition of the degree in Section 1, $\deg(f,B,0)=\pm\deg_{L.S}(M^{-1}f,B,0)$, where $\deg_{LS}$ stands for the Leray–Schauder degree. But $L=M+QL-AP$ and hence

$$M^{-1}f=\mathrm{Id}_X+M^{-1}QL-M^{-1}AP+M^{-1}C=\mathrm{Id}_C+A^{-1}QL-P+A^{-1}C.$$

As $[A^{-1}QL - P + A^{-1}C](B)$ is contained in W, by the reduction property of the Leray–Schauder degree we have that

$$\deg_{L\cdot S}(M^{-1}f, B, 0) = \deg_B(M^{-1}f|_{B\cap W}, B\cap W, 0).$$

On the other hand we have that

$$M^{-1}f|_{B\cap W} = A^{-1}(L+C)|_{B\cap W} = A^{-1}g$$

and therefore

$$\deg(f, B, 0) = \pm\deg_B(A^{-1}g, D, 0) = \pm\deg_B(g, D, 0).$$

The last equality holds because A is an isomorphism. This finishes the proof of the proposition. □

We will need also the following well-known fact:

PROPOSITION 2.6. *Let $B = B(0,r)$ be a ball in $\mathbb{R}^n$ and let $g\colon (B, \partial B) \to (\mathbb{R}^n, \mathbb{R}^n - \{0\})$ be a continuous map. Let $\overline{g}\colon \partial B \to S^{n-1}$ be defined by $\overline{g}(x) = \|g(x)\|^{-1}g(x)$. Then the homomorphism $\overline{g}^*\colon H^{n-1}(S^{n-1}, Z_k) \to H^{n-1}(\partial B, Z_k)$, induced by $\overline{g}$ in singular cohomology groups with coefficients in Z_k, coincides with the multiplication by $\pm d = \deg_B(g, B, 0)$.*

PROOF. Using the homotopy $h(t,x) = tg(x) + (1-t)\|g(x)\|^{-1}g(x)$ one easily shows that the following diagram is commutative.

$$\begin{array}{ccc} H_{n-1}(\partial B, Z) & \xrightarrow{\overline{g}_*} & H_{n-1}(S^{n-1}, Z) \\ \downarrow & & \downarrow \\ H_n(B, B-K, Z) & \xrightarrow[g_*]{} & H_n(R^n, R^n - \{0\}, Z) \end{array} \tag{2.7}$$

Here the vertical arrows are the natural inclusions composed with the inverse of the connecting homomorphism in the exact sequence of a pair and $K = g^{-1}(0)$.

Let $o_K \in H_n(B, B-K, Z)$ and $o_0 \in H_n(\mathbb{R}^n, \mathbb{R}^n - \{0\}, Z)$ be the fundamental classes of K and $\{0\}$ respectively (cf. [13, Chapter 8]). By naturality, one easily shows that the vertical maps send generators of $H_{n-1}(\partial B, Z)$ and $H_{n-1}(S^{n-1}, Z)$ into o_K and o_0, respectively. On the other hand, by the homological characterization of the degree (cf. [13, Chapter 8, Proposition 4.3]) one has $g_*(o_K) = \pm d o_0$. By commutativity of the diagram (2.7) the homomorphism $\overline{g}_*\colon H_{n-1}(S^{n-1}, Z) \to H_{n-1}(\partial B, Z)$ coincides with multiplication by $\pm d$. Now the Proposition follows from the universal coefficients theorem for cohomology. □

PROOF OF THEOREM 2.2, CONTINUED. By the transversality assumption that $E = \{(p,x)/L_p(x) \in V\}$ is the total space of a vector bundle over P and $\operatorname{Ind} F$ is the class of this bundle in $\widetilde{\mathrm{KO}}(P)$. Let us consider

$$B(E) = \{(p,x) \in E/\|x\|_p \le r\} \quad \text{and} \quad S(E) = \{(p,x) \in E/\|x\|_p = r\}.$$

Let us denote by g the restriction of F to $B(E)$ viewed as a map into the finite dimensional space V. Let $\dim V = n$. Since, for a fixed p the map F_p verifies all the hypothesis of Proposition 2.5 it follows from this proposition that the degree, $\deg_B(g_p, B(E_p), 0)$, of the restriction g_p of g to the fiber $B(E_p)$ coincides with $\pm d$, for each $p \in P$. On the other hand g induces a map $\overline{g}$ from the sphere bundle $S(E)$ to the unit sphere S^{n-1} of V defined by $\overline{g}(p,x) = \|g(p,x)\|^{-1} g(p,x)$. Assume that $d \neq 0$. Let $k \neq 2$ any number prime to d. If $c \in H^{n-1}(S^{n-1}, Z_k) \approx Z_k$ is any generator, then the cohomology class $u = \overline{g}^*(c) \in H^{n-1}(S(E), Z_k)$ has the property that its restriction to each fiber $S(E_p)$ is a generator of $H^{n-1}(S(E_p), Z_k)$. This follows from the Proposition 2.6 since, if i_p is the inclusion of the fiber at p then $i_p^*(u) = \overline{g}_p^*(c)$ in $H^{n-1}(S(E_p), Z_k)$ and k is prime to d. This shows that u is an orientation class for $E = \operatorname{Ind} F$ over Z_k. But any vector bundle orientable over Z_k; $k \neq 2$ is orientable over Z as well. This can easily be seen as follows: the existence of an orientation class is equivalent to the following fact: the natural action of the first homotopy group $\pi_1(P)$ on the cohomology of the fiber $H^{n-1}(S(E_p), Z_k)$ is trivial (cf. [24]). But the only automorphisms of Z are multiplications by ± 1. Hence if the action is trivial on $H^{n-1}(S(E_p), Z_k) \sim Z_k$ then it must be trivial on $H^{n-1}(S(E_p), Z) \sim Z$. In conclusion, if d does not vanish then the index bundle $\operatorname{Ind} F$ must be orientable or, what is the same, $\omega_1(\operatorname{Ind} F) = 0$ in $H^1(P, Z_2)$. This proves the second assertion of Theorem 2.2.

The first assertion follows from Proposition 2.5 and the mod$-k$ Dold's theorem of Adams [1]. Indeed [1, Theorem 1.1] states that if E is an orientable vector bundle and $\overline{g}\colon S(E) \to S(\Theta^n)$ is a map over P from the sphere bundle $S(E)$ into the sphere bundle of a trivial n-dimensional bundle Θ^n such that $\overline{g}_p\colon S(E_p) \to S(\Theta^n_p)$ is of degree $\pm d$, then for some m, $S(d^m E)$ is fiber-wise homotopy equivalent to $S(d^m \Theta^n)$. In other words

$$d^m \cdot J([E]) = d^m \cdot J(\operatorname{Ind}(F) = 0 \quad \text{in } J(P). \qquad \square$$

Section 3

3.1. Examples and applications. Let us consider a particular case of the oblique derivative problem for nonlinear second-order elliptic equation on the plane of the form:

$$\begin{cases} f\left(x, u(x), Du(x), D^2u(x)\right) = 0, & x \in \Omega, \\ D_v(u) = 0, & x \in \partial\Omega. \end{cases} \tag{3.1}$$

Here Ω is the open disk $|x| < 1$ in $\mathbb{R}^2$, u is a real valued function, v is a vector field on $\partial\Omega$ of norm $|v(x)| = 1$ and $D_v(u)(x) = \langle v(x), \nabla u(x)\rangle$ is the directional derivative of u in direction v.

We identify the plane R^2 with the complex plane $\mathbb{C}$ and consequently we will use the complex notation $z = x_1 + ix_2$ and $v(z) = v_1(z) + iv_2(z)$ as well. Introducing the conjugate gradient $\overline{\nabla} u = \partial_1 u - i\partial_2 u$ the boundary condition in (3.1) can be written in the following complex form $\Re(v\overline{\nabla}u(z)) = 0$ for $|z| = 1$, where $\Re$ denotes the real part.

The linearization of (3.1) is given by

$$\mathcal{L}(u)h(x) = \sum_{|\alpha|\leq 2} \partial_\alpha f(x, \dots , D^2u(x))D^\alpha h(x)$$

with boundary operator $\mathcal{B} = \gamma D_v(h)$. $\mathcal{L}$ is a real, elliptic, second order operator. In this case the polynomial $p^0(x, \xi+\tau\eta)$ of (1.3) has two complex conjugate roots with nonzero imaginary part while the corresponding equation for the symbol of the boundary condition has a real root. It follows then that the boundary condition $D_v u = g$ verifies the Shapiro–Lopatinkiĭ condition with respect to any second order elliptic operator and therefore (3.1) is a nonlinear elliptic boundary value problem in the sense of Section 1. Notice that v can be tangent $\partial\Omega$ at some points. That the BVP (3.1) is elliptic in spite of this happens only in dimension two.

The map $G: H^{4+s}(\Omega) \to H^{2+s}(\Omega) \times H^{5/2+s}(\partial\Omega)$ defined by

$$G(u) = (f(x, u, Du, D^2u), D_v u)$$

is Fredholm and, by the results in [14], it is a quasilinear Fredholm map. We will study the map G when the boundary condition is given a constant vector field $v(z) \equiv v_1 + iv_2$. In this case $DG(u)$ has index 2 as the following discussion shows:

$DG(u)$ is the map induced by the linear boundary value problem $\mathcal{L}(u)h = f$; $D_v h = g$. Since the boundary condition $D_v h = g$ is compatible with any elliptic operator of second order, the path $\{s\mathcal{L}(u) + (1 - s)\Delta, D_v\}_{0\leq s\leq 1}$ induces a homotopy of linear Fredholm operators in the corresponding spaces. Thus in order to find the index of $DG(u)$ it is enough to compute the index of the operator $\Delta^v: H^{4+s}(\Omega) \to H^{2+s}(\Omega) \times H^{5/2+s}(\partial\Omega)$ induced by the linear BVP $[\Delta h = f;\ D_v h = g]$. By elementary means one can show that Δ^v is surjective with two dimensional kernel (cf. [10]). We review the proof here since we will need a part of the proof later.

Let us compute first of all $\ker \Delta^v$. If $\Delta h = 0$, h is harmonic. Since the skew-gradient of an harmonic function is analytic we have that also

$$\phi = v\overline{\nabla} h = (v_1 + iv_2)(\partial_1 h - i\partial_2 h)$$

is analytic on the disk and verifies $\Re\phi = 0$ on the boundary. By the maximum principle $\Re\phi(z) = 0$ everywhere on the disk and therefore $\phi(z) = ic$, with c a real constant. Thus $\partial_1 h - i\partial_2 h = ic(v_1 - iv_2)$ and from this we get $\nabla h = c(v_2, -v_1) = cv^\perp$. Therefore

$$\begin{aligned}(3.2)\qquad \Delta^v &= \{h/\nabla h(x) = cv^\perp \text{ for all } x \in \partial\Omega\}\\ &= \{h/h(x_1,x_2) = c(v_2x_1 - v_1x_2) + d\}.\end{aligned}$$

In order to prove the surjectivity of Δ^v we first prove that the restriction of D_v to $\ker\Delta$ is surjective. For this, given $g \in H^{5/2+s}(\partial\Omega)$ let us choose an analytic function ϕ whose real part coincide with g on the boundary of the disc (take any harmonic u function which restricts to g on the boundary and then take $\phi = u + iu^*$ where u^* is any harmonic conjugate to u). Then $\psi = \overline{v}\phi$ is analytic and verifies $\Re(v\psi) = 0$ on the boundary. But the Cauchy–Riemann equations imply that an analytic function is always equal to $\overline{\nabla}(u)$ for some harmonic function u and this prove the surjectivity of D_v restricted to $\ker\Delta$. The surjectivity of Δ^v now follows from the following Lemma whose proof is left to the reader.

Lemma 3.1. *Let $L: X \to Y$ be a bounded operator with closed range and let V be a closed subspace of Y such that $\operatorname{Im} L + V = Y$ Let $E = L^{-1}(V)$ and let $L': E \to V$ be given by the restriction of L to E. Then*

(i) $\ker L = \ker L'$, $\operatorname{Im} L' = \operatorname{Im} L \cap F$, $\operatorname{Coker} L' = F/\operatorname{Im} L\cap F \equiv \operatorname{Coker} L \cong Y/\operatorname{Im} L$,
(ii) *L is Fredholm, surjective or injective if and only if L' is. Moreover,* $\operatorname{ind} L = \operatorname{ind} L'$.

We apply Lemma 3.1 to

$$\Delta^v = (\Delta, D_v): H^{4+s}(\Omega) \to H^{2+s}(\Omega) \times H^{5/2+s}(\partial\Omega)$$

with $V = \{0\} \times H^{5/2+s}(\partial\Omega)$ which verifies the transversality condition in the hypothesis because $\Delta: H^{4+s}(\Omega) \to H^{2+s}(\Omega)$ is surjective. Then L' is the restriction of γD_v to $\ker\Delta$ which is surjective by the previous discussion and hence Δ^v is surjective as well.

Therefore the map G is a quasilinear Fredholm map (qlf-map) of index 2. In order to obtain a problem of index 0 it is enough to restrict G to a subspace of codimension 2 by imposing two linearly independent conditions on u. We will further impose $u(\pm 1/2, 0) = 0$. Another reasonable condition could be $\nabla u(0,0) = 0$. Clearly the restriction of a qlf-map to a finite codimensional subspace is a qlf-map with index decreased by the codimension of the subspace. We then have that the restriction of G to the subspace $\widetilde{H}(\Omega) = \{u \in H^{4+s}(\Omega)/u(\pm 1/2, 0) = 0\}$ is a qlf-map of index 0. If we consider the constant vectorfield v as a parameter we have a family of qlf-maps of index zero parametrized by S^1. Since our boundary conditions are linear homogenous the problem can be simplified further by restricting the map to the subspace of functions verifying the boundary conditions. Namely for each v we consider the subspace X_v of $H^{4+s}(\Omega)$ given by

$$X_v = \{u \in H^{4+s}(\Omega)/u(\pm 1/2, 0) = 0 \text{ and } D_v u = 0 \text{ on } \partial\Omega\}$$

and consider the map $F_v\colon X_v \to H^{2+s}(\Omega)$ defined by

$$F_v(u) = f(x, u, Du, D^2 u).$$

Since also D_v is surjective, an easy application of Lemma 3.1 (this time with $V = H^{5/2+s}(\partial\Omega)$ shows that each F_v is also a qlf-map of index 0.

Our final observation is that the parametrization by v counts the solutions of the equation twice, since $F_v(u) = 0$ and $F_{-v}(u) = 0$ have exactly the same solutions. The description of the $\ker \Delta^v$ in (3.2) suggest that one can get rid of the extra count of parameters by associating to each vector v the line through the origin perpendicular to v. Namely, if $l = \{w/w = cv^{\perp},\ c \in R\}$ we introduce a new space

$$X_l = \{u \in H^{4+s}(\Omega)/u(\pm 1/2, 0) = 0,\ \nabla u(x) \in l \text{ for all } x \in \partial\Omega\}$$

and define $F_l\colon X_l \to H^{2+s}(\Omega)$ as the restriction of $f(x, u, Du, D^2u)$ to X_l. Notice that $X_l = X_v$ and $F_l = F_v$ but now v and $-v$ count once.

The set of all lines in the plane through the origin is the real projective space RP^1, a compact manifold diffeomorphic to S^1. While each map F_l is quasilinear, the family $\{F_l\colon X_l \to H^{2+s}(\Omega);\ l \in RP^1\}$ is not quite a family of qlf-maps as defined in Section 2. However it can be easily recast to that case. This is possible because the family $\{X_l;\ l \in RP^1\}$ form a Hilbert bundle over the space RP^1 (we postpone the proof of this for a moment) whose total space is the subspace $\widetilde{X}$ of $RP^1 \times H^{4+s}(\Omega)$ given by $\widetilde{X} = \{(l, u)/l \in RP^1,\ u \in X_l)\}$. Moreover, the family F_l defines a differentiable map $F\colon \widetilde{X} \to H^{2+s}(\Omega)$ (equivalently a bundle map $\widetilde{F}$ from the bundle $\widetilde{X}$ to the trivial bundle $RP^1 \times H^{2+s}(\Omega)$). By Kuiper's theorem $\mathrm{GL}(H)$ is contractible if H is separable and from this it follows that every Hilbert bundle with separable fiber has a trivialization, i.e. a vector bundle isomorphism with a trivial bundle $RP^1 \times X_{l_0}$. Then composing F with the trivialization T we obtain a family of qlf-maps as defined in Section 2. Hence we can apply the results obtained there to $F \circ T$ which is essentially the same as F. First of all let us compute the index bundle of this qlf-family. Since the index bundle is invariant under composition with isomorphisms it is well defined for Fredholm morphisms between Hilbert bundles. Thus we can work directly directly with the map F. But one does not need bundles really here, we just drop the trivialization T from the notation. Let $L_l = DF_l(0)$, and let $\widetilde{D_l}$ be the restriction of $\Delta\colon H^{4+s}(\Omega) \to H^{2+s}(\Omega)$ to X_l. Then as before

$H_{l,s} = sL_l + (1-s)\widetilde{D_l}$ gives a homotopy between families of Fredholm operators parametrized by RP^1. Hence $\operatorname{Ind} F = \operatorname{Ind} L = \operatorname{Ind} \widetilde{D}$.

We compute $\operatorname{Ind} \widetilde{D}$ as follows: we write each $\widetilde{D_l}$ as composition of two Fredholm operators. For this let

$$H_l^{4+s}(\Omega) = \{u \in H^{4+s}(\Omega)/\nabla u(x) \in l \text{ for all } x \in \partial\Omega\}$$

and let Δ_l be the restriction of Δ to $H_l^{4+s}(\Omega)$. Then $\widetilde{D_l} = \Delta_l \circ I_l$ where I_l is the inclusion of of X_l into $H_l^{4+s}(\Omega)$. The index of the family I, $\operatorname{Ind}(I) \in \mathrm{KO}(RP^1)$ is easy to find since each I_l is injective and $\operatorname{coker} I = \{\operatorname{coker} I_l\}_{l\in RP^1}$ is isomorphic to the trivial bundle $\Theta^2 = RP^1 \times R^2$, the isomorphism being induced by the map $u \to (u(-1/2, 0), u(1/2, 0)) \in R^2$. Thus $\operatorname{Ind} I = -[\Theta^2]$. On the other hand taking any v perpendicular to l by our previous calculation and Lemma 3.1 we get:

$$\operatorname{coker} \Delta_l = \operatorname{coker} \Delta^v = 0$$

and

$$\ker \Delta_l = \ker \Delta^v = \{h/\nabla h = cv^{\perp}\} = \{h/\nabla h \in l\}.$$

Hence the kernel bundle of $\widetilde{\Delta} = \{\Delta_l\}_{l\in RP^1}$ is isomorphic to the direct sum of the tautological line bundle ν over RP^1, whose total space is the Möebius band $\{(l, w)/l \in RP^1,\ w \in l\}$, with the trivial line bundle Θ^1. The isomorphism $\varphi\colon \ker \widetilde{\Delta} \to \nu \oplus \Theta^1$ is defined by $\varphi_l(h) = (\nabla h, h(0))$. By additivity property of the index bundle $\operatorname{Ind} \widetilde{D} = \operatorname{Ind} \widetilde{\Delta} + \operatorname{Ind} I = [\nu \oplus \Theta^1] - [\Theta^2] = [\nu] - [\Theta^1]$. This element is a generator of $\widetilde{\mathrm{KO}}(RP^1) \cong Z_2$ with $\omega_1([\nu] - [\Theta^1]) = 1$ in $H^1(RP^1, Z_2)$. Therefore we can apply our results obtained in Section 2. Moreover, since our previous discussion is not affected by introducing the parameter l also in the nonlinearity f we can consider general families of boundary value problems as bellow.

THEOREM 3.1. *Let $f(l, x, u, Du, D^2u)$ be a family of second order nonlinear elliptic operators parametrized by $l \in RP^1$*

(i) *The family of oblique derivative problems*

$$\begin{cases} f(l, x, u, Du, D^2u) = 0 & \text{for } x \in \Omega, \\ \nabla u(x) \in l & \text{for } x \in \partial\Omega, \\ u(1/2, 0) = 0 = u(-1/2, 0), \end{cases} \tag{3.3}$$

cannot be reduced to a parametrized family of compact vector fields.

(ii) *If there are a-priori bounds for solutions of* (3.3) *for all $l \in RP^1$ then for large R, $\deg(F_l, B(0, R), 0)$ must vanish. Equivalently if for some fixed line l_0 there are bounds and $\deg(F_{l_0}, B(0, R), 0) \neq 0$ then bifurcation from infinity arise.*

For example if l_0 is the y-axis and $f(l_0, x, u, Du, D^2u) = \Delta u$. Then

$$\deg(F_{l_0}, B(0,R), 0) = \pm 1$$

because Δ with this boundary conditions is an isomorphism. Hence there must some $l \in RP^1$ that is bifurcation point from infinity for solutions of (3.3).

Proof. We proved everything in the previous discussion except the fact that the family X_l is a Hilbert bundle. Since local triviality is a local question we check this on a small neighborhood of a given line l_0. But lines l close to l_0 can be described in a continuous manner in terms of a chosen normal vector $v = v(l)$ and hence in this neighbourhood $X_l = X_{v(l)}$. Now the local triviality follows from the fact that $X_{v(l)}$ are kernels of a family of surjective maps

$$M_v : H^{4+s}_{v(l)}(\Omega) \to H^{5/2+s}_{v(l)}(\partial\Omega) \times R^2$$

defined by

$$M_v(u) = (\langle v, \nabla u\rangle, u(1/2,0), u(-1/2,0)).$$

Families of kernels of surjective operators are always locally trivial in Hilbert spaces. That each M_v is surjective follows from the surjectivity of Δ_v proved above. □

One can think that the failure of having a priori bounds and nontrivial degree is caused by the presence of the two point conditions for a problem naturally of index two, but it is not so as one can see considering the parametrized family the oblique derivative problem for the same type of equation but with the boundary condition given at each point $x \in \partial\Omega$ by the derivative in direction of the normal rotated by an angle θ. In other words the boundary condition is $D_v u = 0$ but this time with the vector field given by $v(x) = x_1 \cos\theta + x_2 \sin\theta$. Here we take $w = (\cos\theta, \sin\theta)$ as parameter.

A calculation similar to the case of constant vectorfield shows that the index of BVP with this boundary condition is 0 and hence no extra conditions are needed. Passing to the line perpendicular to w as before we get a family of BVP parametrized by RP^1. For each line $l \in RP^1$ and $z \in \partial\Omega$ define the line $\lambda(l,z) = \overline{z} \cdot l$ i.e. the line obtained from l by multiplying all vectors in l by $\overline{z}$. Then we get a family of boundary value problems

$$\begin{cases} f(l,z,u,Du,D^2u) = 0 & \text{for } z \in \Omega, \\ \nabla u \in \lambda(l,z) & \text{for } z \in \partial\Omega. \end{cases} \tag{3.4}$$

Now our nonlinear map F_l is defined by $f(l,z,u,Du,D^2u)$ on the space of maps verifying the boundary conditions

$$H^{4+s}_\lambda(\Omega) = \{u \in H^{4+s}(\Omega) / \nabla u(z) \in \lambda(l,z) \text{ for } z \in \partial\Omega\}.$$

A slightly more elaborate argument which we won't reproduce here shows that for this F again $\operatorname{Ind} F = [\nu] - [\Theta^1]$ a generator of $\widetilde{\mathrm{KO}}(RP^1)$ and hence our conclusions in Theorem 3.1 hold in this case as well.

Considering general boundary conditions of the form $\nabla u \in \lambda(l, z)$ for $z \in \partial\Omega$, where $\lambda\colon RP^1 \times \partial\Omega \to S^1$ is any map, we get a problem of index $2(1-k)$ where k is the winding number of $\lambda_l\colon \partial\Omega \to S^1$ (c.f. [10]). It is easy to show that for $k \leq 1$ by adding $2(1-k)$point conditions we obtain a problem of index 0 for which the same conclusion hold. Using conformal representation everything extends to simply-connected domains with smooth boundary and boundary conditions of the general type described above.

Our next applications deals with a general nonlinear elliptic BVP of positive index $m > 0$ on a bounded domain of in R^n with smooth boundary.

$$(3.5) \quad \begin{cases} f(x, u(x), \ldots, D^{2k}u(x)) = 0 & x \in \Omega, \\ g_i(x, u(x), \ldots, D^{m_i}u(x)) = 0 & x \in \partial\Omega,\ 1 \leq i \leq k,\ m_i \leq 2k-1. \end{cases}$$

Let, as before,

$$X = H^{2k+2+s}(\Omega), \quad s \geq 0, \quad \text{and} \quad Y = H^{2+s}(\Omega) \times \prod_1^k H^{2k+s-m_i+3/2}(\partial\Omega),$$

and let $F\colon X \to Y$ be qlf-map induced by (3.5).

For each m-dimensional subspace l of X let $N_l = \{v/\langle v, w\rangle = 0 \text{ for all } w \in l\}$ be its normal in X. The restriction F_l of F to N_l is a qlf-map of index 0. It follows easily from the homotopy invariance of the absolute value of the degree that if for some l_0 we have that $F_{l_0}^{-1}(0)$ is bounded and $\deg(F_{l_0}, B(0,R) \cap N_l, 0) \neq 0$ then the set $F^{-1}(0)$ of solutions of (3.5) cannot be bounded. Here we will say something more about directions in which the solution of (3.5) escape to infinity. Let us consider the Grassmannian $G_m(X)$ of all m-dimensional subspaces of X with the distance given by $d(l, l') = \|\pi_l - \pi_{l'}\|$, where π_l is the orthogonal projector onto l. $G_m(X)$ is an infinite dimensional Banach manifold modeled by $L(R^m; X)$.

Definition. An m-dimensional subspace l of X is a *point of bifurcation from infinity in the normal direction to l* if there is a sequence (l_i) of elements of $G_m(X)$ converging to l and a sequence $u_i \in N_{l_i}$ of solutions of (3.5) such that $\|u_i\| \to \infty$.

There are several non equivalent notions of codimension of a closed subset of an infinite dimensional manifold. We will use the following one: We will say that the codimension of closed subset B of an manifold M is at most k if there exist an increasing sequence M_j of of finite dimensional submanifolds whose union is dense in M such that the Lebesgue covering dimension of $B \cap M_j$ is not less than dim $M_j - k$. With this we can state our result.

Theorem 3.2. *Given the boundary value problem* (3.5) *if, for some* l_0, $F_{l_0}^{-1}(0)$ *is bounded and for* R *big enough* $\deg(F_{l_0}, B(0,R)\cap N_l, 0) \neq 0$ *then the set* $B \subset G_m(X)$ *of bifurcation points of* (3.5) *in the normal direction is of codimension at most one in* $G_m(X)$.

Proof. Take an increasing sequence (X_j) of finite dimensional subspaces of X such that l_0 is contained in X_1 and $\bigcup_{j=1}^{\infty} X_j$ is dense in X. Let $M_j = G_m(X_j)$. There is a natural inclusion of M_j into M_{j+1} and of M_j into $G_m(X)$. Moreover, it is easy to see that $\bigcup_{j=1}^{\infty} M_j$ is dense in $G_m(X)$. The normal spaces N_l form a Hilbert bundle N over $G_m(X)$ which is trivial and hence we can consider F_l as a family of qlf-maps of index zero parametrized by the (non-compact) manifold $M = G_m(X)$.

Let L_l be the linearization at 0 of F_l. The family $\{L_l : l \in M\}$ can be written as composition of a constant family given by the linearization $DF(0)$ of the full map F at the point 0 with the family of inclusions $i_l : N_l \to X$, $l \in M$. Clearly the same happens with the restriction of the family $\{L_l\}$ to each M_j. Since M_j are compact $\operatorname{Ind}\{L_l : l \in M_j\}$ is well defined. Since $\operatorname{Ind} DF(0) = [\Theta^m]$ we have that

$$\operatorname{Ind}\{L_l : l \in M_j\} = [\Theta^m] + \operatorname{Ind}\{i_l : l \in M_j\}.$$

The later is easy to compute since $\ker i_l = 0$ and coker $i_l = X/N_l \equiv \nu_j$ where ν_j is the tautological m-plane bundle over $M_j = G_m(X_j)$. Thus $\operatorname{Ind}\{L_l : l \in M_j\} = [\Theta^m] - \nu_j$. Since $\omega_1(\Theta^m - \nu_j) \neq 0$ in $H^1(G_m(X_j), Z_2)$ by Corollary 1.4 (which holds for any qlf map of index 0) we get $\dim B \cap M_j \geq \dim M_j - 1$. This proves the theorem. Related results were obtained in [16]. □

References

[1] J. F. Adams, *On the groups* $J(X)$, I. Topology **2** (1963), 181–195.

[2] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Comm. Pure Appl. Math. **12** (1959), 623–727.

[3] M. S. Agranovich and A. S. Dynin, *General boundary value problems for elliptic systems on n-dimensional regions*, Dokl. Akad. Nauk **143** (1962), 511–514.

[4] J. C. Alexander and S. S. Antman, *Global and local behavior of bifurcating multidimensional continua of solutions for multiparameter nonlinear eigenvalue problems*, Arch. Rational. Mech. Anal. **76** (1981), 339–355.

[5] M. F. Atiyah, *K-Theory*, Benjamin, New York, 1967.

[6] ———, *Thom complexes*, Proc. London Math. Soc. **11** (1961), 291–310.

[7] M. F. Atiyah and I. M. Singer, *The index of elliptic operators* IV, V, Ann. of Math. **93** (1971), 119–149.

[8] A. V. Babin, *Finite dimensionality of the kernel and cokernel of quasilinear elliptic mappings*, Math. USSR Sbornik. **22** (1974), 427–454.

[9] T. Bartsch, *The global structure of the zero set of a family of semilinear Fredholm maps*, Nonlinear Anal. **17** (1991), 313–331.

[10] B. BOOSS AND D. D. BLEECKER, *Topology and Analysis*, Springer–Verlag, 1985.

[11] F. E. BROWDER, *Nonlinear operators and nonlinear equations of evolution in Banach spaces*, Proc. Sympos. Pure Math. **18** (1976).

[12] F. E. BROWDER AND R. D. NUSSBAUM, *The topological degree for non compact nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 671–676.

[13] A. DOLD, *Lectures on Algebraic Topology*, Grundlehren Math. Wiss. **200** (1980).

[14] P. M. FITZPATRICK AND J. PEJSACHOWICZ, *Orientation and the Leray–Schauder theory for fully nonlinear elliptic boundary value problems*, Mem. Amer. Math. Soc. **483** (1993).

[15] ______, *Nonorientability of the index bundle and several-parameter bifurcation*, J. Functional Anal. **98** (1991), 42–58.

[16] P. M. FITZPATRICK, I. MASSABÒ AND J. PEJSACHOWICZ, *Global several parameter bifurcation and continuation theorems*, Math. Ann. **263** (1983), 61–73.

[17] A. GRANAS, *Theorème sur le prolongement de l'homotopie dans les espaces de Banach et ses applications dans la théorie des équations non-linéaires*, Bull. Acad. Polon. Sci. **VII (9)** (1959).

[18] M. A. KRASNOSEL'SKIĬ AND P. P. ZABREĬKO, *Geometrical Methods in Nonlinear Analysis*, Grundlehren Math. Wiss. **263** (1984).

[19] O. A. LADYZHENSKAJA AND N. N. URAL'TSEVA, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[20] J. LERAY AND J. SCHAUDER, *Topologie et équations fonctionnelles*, Ann. Sci. Ecóle Norm. Sup. (4) **51** (1934), 45–78.

[21] L. NIRENBERG, *Variational and topological methods in nonlinear problems*, Bull. Amer. Math. Soc. (N. S.) **4** (1981), 267–302.

[22] J. PEJSACHOWICZ, *K-theoretic methods in bifurcation theory*, Contemp. Math. **72** (1988), 193–205.

[23] P. H. RABINOWITZ, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 487-513.

[24] E. SPANIER, *Algebraic Topology*, McGraw–Hill, New York, 1966.

[25] F. SIGRIST, U. SUTER, *On the exponent and order of the groups $J(X)$*, Lecture Notes in Math. **673**, Springer–Verlag, 116–122.

[26] YU. I. SAPRONOV, *The local invertibility of non-linear Fredholm maps*, Funktsional Analiz i Prilozhen. **5** (1971), 296–300.

[27] I. V. SKRYPNIK, *Nonlinear Elliptic Boundary Value Problems*, vol. 91, Teubner-Texte Math., 1986.

[28] A. I. ŠNIREL'MANN, *The degree of a quasi-ruled mapping and a non-linear Hilbert problem*, Math. USSR Sbornik **18** (1971), 373–396.

[29] M. G. ZAIDENBERG, S. G. KREIN, P. A. KUCHMENT AND A. A. PANKOV, *Banach bundles and linear operators*, Russian Math. Surveys **5** (1975), 115–175.

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