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LERAY–SCHAUDER TYPE ALTERNATIVES AND THE SOLVABILITY OF COMPLEMENTARITY PROBLEMS

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Abstract. We present in this paper several existence theorems for nonlinear complementarity problems in Hilbert spaces. Our results are based on the concept of "exceptional family of elements" and on Leray–Schauder type altrenatives.

1. Introduction

The study of *complementarity problems* is a relatively new domain of applied mathematics. The complementarity theory has deep relations with several chapters of fundamental mathematics as for example, fixed-point theory, theory of variational inequalities, topological degree, functional analysis and theory of topological ordered vector spaces, among others [4], [7], [14], [15], [25].

Each complementarity problem is a mathematical model for several kinds of practical problems from economics, optimization, game theory, engineering and mechanics [4], [7], [13]–[15], [23], [25], [29].

Recently in [19] (see also [5]), we introduced a new topological method in the study of solvability of complementarity problems in Hilbert spaces. Our method is based on the concept of *exceptional family of elements* (denoted shortly by (EFE)). The notion of (EFE) is based on the topological degree and in more

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general situations on the concept of "zero-epi mappings" ([15], [16]) and because this aspect, it is different of the notion of *exceptional sequence of elements* introduced in [29], which is strongly dependent of the Euclidean structure in $\mathbb{R}^n$ and of the ordering defined by $\mathbb{R}^n_+$.

Applying the notion of (EFE) it was recently obtained the solvability of complementarity problems for several classes of mappings, [5], [6], [13]–[23], [31]–[35]. In our papers [17] and [22] it is shown that, in the method of (EFE), we can replace the topological degree by the Leray–Schauder alternative.

Now, in this paper, we will develop this idea and we will obtain new existence theorems for complementarity problems. By this way we will put also in evidence several new classes of mappings for which the complementarity problem has a solution.

2. Preliminaries

Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space and $K \subset H$ a closed pointed convex cone, i.e. K is a non-empty closed set satisfying the following properties:

(k1) $K + K \subseteq K$,

(k2) $\lambda K \subseteq K$ for all $\lambda \in \mathbb{R}_+$,

(k3) $K \cap (-K) = \{0\}$.

The *dual of* K, by definition, is $K^* = \{y \in H \mid \langle x, y\rangle \geq 0 \text{ for all } x \in K\}$. Obviously, K^* is a closed convex cone. If $K \subset H$ is a closed convex cone, then the projection onto K, denoted by P_K, is well defined for every $x \in H$, i.e. for every $x \in H$, we have that $P_K(x)$ is the unique element in K such that $\|x - P_K(x)\| = \min_{y\in K} \|x - y\|$.

A classical result says that, the projection operator P_K is characterized by the following properties.

For every $x \in H$, $P_K(x)$ is the unique element in K satisfying the following conditions:

(i) $\langle P_K(x) - x, y\rangle \geq 0$ for all $y \in K$,

(ii) $\langle P_K(x) - x, P_K(x)\rangle = 0$.

3. Complementarity problems

There exist several types of complementarity problems as the reader can see in [15]. In this paper we will consider only the general nonlinear complementarity problem in Hilbert spaces.

Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ an arbitrary mapping. The nonlinear complementarity problem defined by the mapping f and the cone K is:

(NCP(f, K)) find $x_* \in K$ such that $f(x_*) \in K^*$ and $\langle x_*, f(x_*)\rangle = 0$.

The problem NCP(f, K) has many applications and generally, it is related to equilibrium problems in physical sense and also in economical sense [4], [7], [13]–[15], [23], [25], [29]. If the mapping f is an affine mapping, i.e. $f(x) = A(x) + b$, where $A : H \to H$ is a continuous linear mapping and b is an arbitrary element in H, we have that NCP(f, K) is the *linear complementarity problem*, denoted by LCP(A, b, K).

The problem LCP(A, b, K) has been very much studied, but generally, only in the Euclidean space $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. In Complementarity Theory, the study of solvability of the problem NCP(f, K) is the first important problem, because the solvability of this problem is not evident, [4], [7], [13]–[5].

Now, in this paper we will present several existence results for the problem NCP(f, K) based on Leray–Schauder type alternatives.

4. Leray–Schauder type alternatives

One of the most important theorem of nonlinear functional analysis is the *Leray–Schauder alternative*, proved in 1934 by the topological degree [24]. Now, there exist several kinds of Leray–Schauder type alternatives proved by different methods, not based on topological degree [2], [3], [9], [26]–[28]. We note that, the classical Leray–Schauder Alternative has many applications to ordinary differential equations too.

Our applications of Leray–Schauder type alternatives to the study of complementarity problems represent a new direction of applications of this classical result. In this paper we will apply the following Leray–Schauder type alternatives.

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $X \subset H$ a non-empty subset. Let $f : H \to H$ be a mapping. We say that f is *compact* on X if $f(X)$ is *relatively compact* and we say that f is *completely continuous* if f is continuous and for any bounded set $B \subset H$, $f(B)$ is *relatively compact*. We will denote by $\overline{X}$ the *closure* of X and by ∂X the *boundary* of X. We will use also the following classical notion. We say that f is a *completely continuous field*, if f has a representation of the form $f(x) = x - T(x)$, for every $x \in H$, where $T : H \to H$ is a completely continuous mapping.

Theorem 1 (Leray–Schauder alternative). *Let $D \subseteq H$ be a convex set, U a subset open in D and such that $0 \in U$. Then each continuous compact mapping $f : \overline{U} \to D$ has at least one of the following properties:*

(1) *f has a fixed-point,*
(2) *there is $(x_*, \lambda_*) \in \partial U \times]0, 1[$ such that $x_* = \lambda_* f(x_*)$.*

Proof. A proof of this classical result is given in [9] and it is based on transversality theory. □

Let $K \subset H$ be a closed pointed convex cone. For any $r > 0$ ($r \in \mathbb{R}$) we denote by $K_r = \{x \in K \mid \|x\| \leq r\}$. Let α be the Kuratowski measure of noncompactness. For this notion the reader is referred to [1], [8], [9], [13]. We say that a mapping $f : K_r \to H$ is *α-condensing* if f is continuous bounded and $\alpha(f(B)) < \alpha(B)$, for all $B \subset K_r$ such that $\alpha(B) > 0$.

The next Leray–Schauder type alternative is based on the following fixed-point theorem.

Theorem 2 (Deimling, [8]). *Let $(E, \|\cdot\|)$ be a Banach space, $K \subset E$ a closed pointed convex cone and $f : K_r \to E$ an α-condensing mapping. If the following assumptions are satisfied:*

(1) *if $x \in \partial K$, $\|x\| \leq r$, $x^* \in K^*$ and $x^*(x) = 0$, then $x^*(f(x)) \geq 0$,*
(2) *$f(x) \neq \lambda x$ for all $\lambda > 1$ and all x with $\|x\| = r$,*

then f has a fixed-point (in K_r).

Proof. For a proof of this result the reader is referred to [8]. □

A consequence of Theorem 2 is the following Leray–Schauder type alternative.

Theorem 3. *Let $(H, \langle\cdot\,,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $h : H \to H$ a mapping such that $h(x) = x - T(x)$, for all $x \in H$, where $T : H \to H$ is an α-condensing mapping. Then, for any $r > 0$, for the mapping $f(x) = P_K[x - h(x)]$, at least one of the following two situations is satisfied:*

(1) *f has a fixed-point in K_r,*
(2) *there exist $x*$ with $\|x*\| = r$ and $\lambda* \in]0, 1[$ such that $x* = \lambda* f(x*)$.*

Proof. Since $\alpha(P_K[T(B)]) \leq \alpha(T(B)) < \alpha(B)$ for all $B \subset K_r$ with $\alpha(B) > 0$, and f is continuous and bounded we deduce that f is α-condensing. The theorem is now a consequence of Theorem 2. □

We recall that a mapping $T : H \to H$ is demi-continuous if for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$, norm-convergent to an element $x* \in H$, we have that the sequence $\{T(x_n)\}_{n\in\mathbb{N}}$ is weakly-convergent to $T(x*)$. A mapping $f : H \to H$ is said to be monotone if for any $x, y \in H$ we have $\langle x-y, f(x)-f(y)\rangle \geq 0$. We recall also that a mapping $T : H \to H$ is *pseudo-contractante* if the mapping $f(x) = x - T(x)$ is monotone.

For Complementarity Theory, it is interesting to know under what condition the mapping $\Psi(x) = P_k[x - f(x)]$ is pseudo-contractante, where $K \subseteq H$ is a closed convex cone and $f : H \to H$ is a given mapping. In this sense we have the following result.

Proposition 4. *Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ a mapping. If $f(x) = x - \varphi(x)$, where $\varphi : H \to H$ is non-expansive, then the mapping $\Phi(x) = P_K[x - f(x)]$, is pseudo-contractante.*

Proof. Indeed, $\Psi(x) = x - \Phi(x)$ is monotone if for any $x_1, x_2 \in H$, we have

$$\langle x_1 - x_2, \Psi(x_1) - \Psi(x_2)\rangle = \langle x_1 - x_2, x_1 - x_2\rangle - \langle x_1 - x_2, P_K[x_1 - f(x_1)] - P_K[x_2 - f(x_2)]\rangle \geq 0,$$

which is equivalent to

$$\langle x_1 - x_2, P_K[x_1 - f(x_1)] - P_K[x_2 - f(x_2)]\rangle \leq \|x_1 - x_2\|^2. \tag{1}$$

From our assumption, we have

$$\langle x_1 - x_2, P_K[x_1 - f(x_1)] - P_K[x_2 - f(x_2)]\rangle \leq \|x_1 - x_2\|\|\varphi(x_1) - \varphi(x_2)\| \leq \|x_1 - x_2\|^2,$$

which implies that (1) is true and the proof is complete. □

Theorem 5 (Willem). *Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space, $\Omega \subset H$ a bounded open set such that $0 \in \Omega$ and $T : H \to H$ a demi-continuous pseudo-contractante mapping. If for all $(\lambda, x) \in \,]0, 1[\, \times \partial\Omega$, we have $x \neq \lambda T(x)$, then T has a fixed point in $\overline{\Omega}$.*

Proof. For a proof of this theorem, the reader is referred to [30]. □

From Theorem 5 we deduce the following alternative.

Theorem 6. *Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space, $\Omega \subset H$ a bounded open set such that $0 \in \Omega$. If $T : H \to H$ is a demicontinuous pseudo-contractante mapping, then at least one of the following situations is true:*

(1) *T has a fixed-point in $\overline{\Omega}$,*
(2) *there exist $\lambda* \in \,]0, 1[$ and $x* \in \partial\Omega$ such that $x_* = \lambda_* T(x_*)$.*

5. On the solvability of complementarity problem for quasi-bounded fields

Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space and $T : H \to H$ a mapping. We say that T is a *quasi-bounded mapping*, if and only if

$$|T|_{qb} := \inf_{r>0} \sup_{\|x\| \geq r} \|T(x)\|/\|x\| < \infty.$$

When T is quasi-bounded, the real number $|T|_{qb}$ is called the *quasi-norm* of T. The notion of quasi-bounded mapping, was introduced by A. Granas. (See [9] and its references).

Definition 1. We say that a mapping $f : H \to H$ is a *quasi-bounded field* if f has a representation of the form $f(x) = x - T(x)$ for all $x \in H$, where $T : H \to H$ is a completely continuous quasi-bounded mapping.

Remark. If $\beta = \lim_{\|x\|\to\infty} \|T(x)\|/\|x\| < \infty$ for a mapping $T : H \to H$ then we have that $|T|_{qb} = \beta$.

Theorem 7. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ a quasi-bounded field with the representation $f(x) = x - T(x)$ for all $x \in H$. If $|T|_{qb} < 1$, then the problem* NCP(f, K) *has a solution.*

Proof. It is well known that the problem NCP(f, K) has a solution, if and only if the mapping $\Phi(x) = P_K[x - f(x)]$ has a fixed-point (see [15]). We have that

$$|\Phi|_{qb} = \inf_{r>0} \sup_{\|x\|\geq r} \frac{\|\Phi\|}{\|x\|} = \inf_{r>0} \sup_{\|x\|\geq r} \frac{\|P_K[T(x)]\|}{\|x\|} = \inf_{r>0} \sup_{\|x\|\geq r} \frac{\|T(x)\|}{\|x\|} = |T_{qb}| < 1.$$

Hence, there exists $r > 0$ such that

$$\frac{\|\Phi(x)\|}{\|x\|} < 1 \quad \text{for all } x \text{ with } \|x\| \geq r. \tag{2}$$

We take $D = H$ and $U = \{x \in H \mid \|x\| < r\}$. We obtain that there are no $x_* \in \partial U$ and $0 < \lambda_* < 1$ such that $x_* = \lambda_*\Phi(x_*)$. Indeed, if such x_* and λ_* exist then we have

$$\|x_*\| = \lambda_*\|\Phi(x_*)\| < \|\Phi(x_*)\|,$$

which is a contradiction of (2).

The assumptions of Theorem 1 are satisfied for Φ. Therefore, by Theorem 1, the mapping Φ has a fixed-point in $\overline{U}$, which implies that the problem NCP(f, K) has a solution. □

The following result is an interesting consequence of Theorem 7, since it is an existence result for nonlinear complementarity problems depending of a real parameter.

Theorem 8. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone. Consider the mapping $f(x) = x - T_1(x) - \varepsilon T_2(x)$, for all $x \in H$, where T_1 and T_2 are quasi-bounded completely continuous mappings, with $|T_1|_{qb} < 1$ and $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$. Then, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in \,]0, \varepsilon_0[$, the problem* NCP$(f_\varepsilon, K)$ *has a solution $x*(\varepsilon)$.*

Proof. The mapping $T_1 + \varepsilon T_2$ is completely continuous for every $\varepsilon > 0$. Since $|T_1|_{qb} < 1$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in \,]0, \varepsilon_0[$, we have $\varepsilon|T_2|_{qb} < 1 - |T_1|_{qb}$. Since $|T_1 + \varepsilon T_2|_{qb} \leq |T_1|_{qb} + \varepsilon|T_2|_{qb} < 1$ for all $\varepsilon \in \,]0, \varepsilon_0[$, by Theorem 7 we have that for each $\varepsilon \in \,]0, \varepsilon_0[$, the problem NCP$(f_\varepsilon, K)$ has a solution $x*(\varepsilon)$. □

Remark. The mapping f_ε, considered in Theorem 8, is a generalization of the Von Kárman operator $f(x) = x - \lambda L(x) + T(x)$, used in the study of the post critical equilibrium state of thin elastic plates. In this practical problem the mathematical model is the problem $\mathrm{NCP}(f, K)$ (see [14], [15]).

6. Exceptional family of elements and the solvability of complementarity problems

Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ a mapping. Consider the problem $\mathrm{NCP}(f, K)$ defined by f and K.

Definition 2. We say that a family of elements $\{x_r\}_{r>0}$ is an *exceptional family of elements for f, with respect to K*, if for every real number $r > 0$, there exists a real number $\mu_r > 0$ such that the vector $u_r = \mu_r x_r + f(x_r)$ satisfies the following conditions:

(1) $u_r \in K^*$,
(2) $\langle u_r, x_r \rangle = 0$,
(3) $\|x_r\| \to \infty$ as $r \to \infty$.

Similarly, as the notion of completely continuous field (see Section 4), we can introduce the notion of α-condensing field. Indeed, we say that $f : H \to H$ is an *α-condensing field* if f has a representation of the form $f(x) = x - T(x)$, where $T : H \to H$ is an α-condensing mapping. Also we say that $f : H \to H$ is a *non-expansive field* if f has a representation of the form $f(x) = x - T(x)$, where $T : H \to H$ is a non-expansive mapping. Finally, we say that $f : H \to H$ is a *projectionally pseudo-contractant field with respect to* K if the mapping $\Phi(x) = P_K[x - f(x)]$ is pseudo-contractant. By Proposition 4, we have that if f is a non-expansive field, then f is projectionally pseudo-contractant with respect to any closed convex cone K.

We have the following alternative theorem for complementarity problems.

Theorem 9. *Let $(H, \langle \cdot\,, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ a continuous mapping. If f satisfies one of the following conditions:*

(1) *f is a completely continuous field,*
(2) *f is an α-condensing field,*
(3) *f is a projectionally pseudo-contractant field,*

then there exists either a solution to the problem $\mathrm{NCP}(f, K)$ *or f has an exceptional family of elements with respect to K.*

Proof. We suppose that f satisfies one of assumptions (1)–(3) and it has a representation of the form $f(x) = x - T(x)$, for any $x \in H$, and we consider the mapping

$$\Phi(x) = P_K[x - f(x)] = P_K[T(x)] \quad \text{for any } x \in H.$$

From the complementarity theory, we know that the problem $\mathrm{NCP}(f, K)$ has a solution if and only if the mapping Φ has a fixed-point in K ([13]–[15]). Therefore, if the mapping Φ has a fixed-point, this fixed-point must be in K and the problem $\mathrm{NCP}(f, K)$ has a solution.

If the problem $\mathrm{NCP}(f, K)$ has a solution, then the proof is finished. Suppose that the problem $\mathrm{NCP}(f, K)$ is without solution. Obviously, in this case, the mapping Φ is fixed-point free. Because f satisfies one of condition (1)–(3), we observe that the assumptions of Theorem 1 are satisfied with respect to each set $B_r = \{x \in H \mid \|x\| < r\}$ with $r > 0$, or the assumptions of Theorem 3, or the assumptions of Theorem 6. Then, for any $r > 0$ there exist x_r with $|x_r| = r$ and $\lambda_r \in]0, 1[$, such that $x_r = \lambda_r P_K[T(x)]$. We have

$$\left(\frac{1}{\lambda_r}\right) x_r = P_K[x_r - f(x_r)]. \tag{3}$$

Applying the properties of operatorP_K we obtain

$$\begin{cases} \langle x_r/\lambda_r - (x_r - f(x_r)), y\rangle \geq 0 & \text{for all } y \in K, \\ \langle x_r/\lambda_r - (x_r - f(x_r)), x_r/\lambda_r\rangle = 0, \end{cases}$$

which implies

$$\begin{cases} \langle (1/\lambda_r - 1)x_r + f(x_r), y\rangle \geq 0 & \text{for all } y \in K, \\ \langle (1/\lambda_r - 1)x_r + f(x_r), x_r\rangle = 0. \end{cases} \tag{4}$$

If in (4) we put $\mu_r = 1/(\lambda_r - 1)$ it follows that $\mu_r x_r + f(x_r) \in K^*$, $\langle \mu_r x_r + f(x_r), x_r\rangle = 0$ and, since $|x_r| = r$ for any $r > 0$, we have (because $x_r \in K$) that $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to K and the proof is complete. □

Corollary 10. *Let $(H, \langle\cdot\,,\cdot\rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone. If $f : H \to H$ is a completely continuous field, or an α-condensing field, or a projectionally pseudo-contractant field without exceptional family of elements with respect to K, then the problem* $\mathrm{NCP}(f, K)$ *has a solution.*

A consequence of Corollary 10 is the fact that it is important to know what functions are without exceptional family of elements, with respect to a given closed convex cone. The reader can find in our papers [16]–[18], [20]–[22], [33]–[34] and in our book [15] several classes of functions without exceptional family of elements. It is known (see [19]) that any coercive function is without exceptional family of elements, but there exists non-coercive functions without exceptional family of elements. The property to be without exceptional family of elements can be considered, as a very general coercivity condition because this fact.

7. Condition (θ-S) and the solvability of complementarity problems

In our papers [16] and [20] we introduced condition (θ) and we proved that, if a function f satisfies condition (θ), then f is without exceptional family of elements. We recall this condition.

Definition 3 ([16], [20]). We say that a mapping $f : H \to H$ *satisfies condition* (θ) *with respect to a convex cone* $K \subset H$ if there exists a real number $\varrho > 0$ such that for each $x \in K$ with $\|x\| > \varrho$, there exists $y \in K$ with $\|y\| < \|x\|$ such that $\langle x - y, f(x)\rangle \geq 0$.

By several results proved in [15]–[18], [20] we can see that several classes of mappings considered in Complementarity Theory satisfy condition (θ). In this section we will introduce another variant of condition (θ) and we will show that this variant has also interesting consequences. Let $(H, \langle\,\cdot\,,\,\cdot\,\rangle$ be a Hilbert space, $K \subset H$ a closed pointed convex cone.

Definition 4. We say that a mapping $f : H \to H$ *satisfies condition* (θ-S) *with respect to* K, if for any family $\{x_r\}_{r>0}$, such that $\|x_r\| \to \infty$ as $r \to \infty$, there exists $y_* \in K$ such that $\langle x_r - y_*, f(x_r)\rangle \geq 0$ for some $r > 0$ such that $\|x_r\| > \|y_*\|$.

Remark. We observe that condition (θ) implies condition (θ-S). If f is positive homogeneous then condition (θ-S) implies condition (θ). (In this case we take for any $r > 0$, $x_r = rx$ if $x \in K$).

We have the following result.

Theorem 11. *Let* $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ *be a Hilbert space,* $K \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a mapping. If* f *satisfies condition* (θ-S) *with respect to* K, *then* f *is without exceptional family of elements. Moreover, if* f *is a completely continuous field, or an* α*-condensing field or a projectionally pseudo-contractant field then the problem* NCP(f, K) *has a solution.*

Proof. Suppose that f has an exceptional family $\{x_r\}_{r>0}$. Since f satisfies condition (θ-S) there exists $y_* \in K$ such that $\langle x_r - y_*, f(x_1)\rangle \geq 0$ for some $r > 0$, for which we have $\|y_*\| < \|x_r\|$. In this case we have

$$
\begin{aligned}
0 &\leq \langle x_r - y_*, f(x_r)\rangle = \langle x_r - y_*, u_r - \mu_r x_r\rangle \\
&= \langle x_r, u_r\rangle - \langle y_*, u_r\rangle - \mu_r\langle x_r, x_r\rangle + \mu_r\langle y_*, x_r\rangle \\
&\leq \mu_r[\langle y_*, x_r\rangle - \|x_r\|^2] \leq \mu_r[\|x_r\|\|y_*\| - \|x_r\|^2] = \mu_r\|x_r\|[\|y_*\| - \|x_r\|] < 0,
\end{aligned}
$$

which is a contradiction. Therefore f is without exceptional family of elements with respect to K. The last conclusion of theorem is a consequence of Corollary 10. □

Remark. Our condition (θ-S) is more general as the condition used in [32, Theorem 3.1] since in condition (θ-S) the element y_* is dependent on the family $\{x_r\}_{r>0}$, while in [32, Theorem 3.1] the element $\widehat{y}$ is independent on the family $\{x_r\}_{r>0}$.

Harker and Pang in [12] studied the solvability of variational inequalities in $\mathbb{R}^n$ and in some results they used an interesting condition, which implies the existence of solution of a general variational inequality. We consider now this condition in an arbitrary Hilbert space but for complementarity problems. We will denote this condition by (HP).

Definition 5. We say that a mapping $f : H \to H$ *satisfies condition* (HP)*, with respect to* K, if there exists an element $x_* \in K$ such that the set $K(x_*) = \{x \in K \mid \langle f(x), x - x_*\rangle < 0\}$ is bounded (or empty).

We have the following result.

Theorem 12. *Let* $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ *be a Hilbert space,* $K \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a mapping. If* f *satisfies condition* (HP) *with respect to* K*, then* f *satisfies condition* (θ-S)*. Moreover, if* f *is a completely continuous filed, or an* α*-condensing field or a projectionally pseudo-contractant field, then the problem* NCP(f, K) *has a solution.*

Proof. Let $\{x_r\}_{r>0}$ be a family of elements such that $\|x_r\| \to \infty$ as $r \to \infty$. If there exists an element $x* \in K$ such that the set $K(x*)$ is bounded (or empty) then, for $r > 0$ sufficiently large, we have that $x_r \notin K(x_*)$, which implies that $\langle f(x_r), x_r - x_*\rangle \geq 0$, for r sufficiently large such that in addition $\|x_r\| > \|x*\|$. Obviously in this case, condition (θ-S) is satisfied. □

Proposition 13. *Let* $(H, \langle\,\cdot\,,\,\cdot\,\rangle)$ *be a Hilbert space,* $K \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a mapping. If* f *has an exceptional family of elements with respect to* K*, then for any point* $x_* \in K$*, the set* $\{x \in K \mid \langle f(x), x - x_*\rangle < 0\}$ *is non-empty and unbounded.*

Proof. This result is a consequence of Theorems 11 and 12. □

In [32] was considered the notion of "p-order coercivity" in $\mathbb{R}^n$. Now we will consider this notion in any infinite dimensional Hilbert space.

Definition 6 ([32]). We say that a mapping $f : H \to H$ is $(x*, p)$*-coercive with respect to a convex cone* $K \subset H$ if there exist some $p \in\ ]-\infty, 1[$ and an element $x* \in K$ such that $\lim_{x\in K, \|x\|\to\infty} \langle f(x), x - x_*\rangle/\|x\|^p = \infty$.

Remark. The case $p = 1$ is covered by the classical notion of coercivity used by many authors in the theory of variational inequalities. Any coercive mapping is p-coercive but the converse is not true as it is shown in [32], considering $H = \mathbb{R}$, $K = \mathbb{R}_+$, $f(x) = x^\alpha/(1 + x)$ with $\alpha > 0$ and $x*$ any element such that $x* \geq 1$.

THEOREM 14. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $K \subset H$ a closed pointed convex cone and $f : H \to H$ a mapping. If f is $(x*, p)$-coercive with $-\infty < p < 1$, then f is without exceptional family of elements with respect to K.*

PROOF. It is sufficient to show that the $(x*, p)$-coercivity implies condition (θ-S). Indeed, if $0 \le p < 1$ then we have

$$\lim_{x \in K, \|x\| \to \infty} \frac{\langle f(x), x - x_* \rangle}{\|x\|^p} = \infty \tag{5}$$

with $x* \in K$ defined by the $(x*, p)$-coercivity. Relation (5) implies

$$\lim_{x \in K, \|x\| \to \infty} \langle f(x), x - x_* \rangle = \infty$$

which has as consequence the fact that condition (θ-S) is satisfied. If $-\infty < p < 0$, then for every family of elements $\{x_r\}_{r>0} \subset K$, with $\|x_r\| \to \infty$ as $r \to \infty$, we have (using formula (5)) that $\langle f(x_r), x_r - x_* \rangle > 0$ for $r > 0$ sufficiently large. Therefore, again condition (θ-S) is satisfied and the proof is complete. □

DEFINITION 7. Let $f : H \to H$ be a mapping and $K \subset H$ a closed pointed convex cone. We say that a mapping $T : H \to H$ is an $(x*, p)$*-scalar asymptotic derivative of f with respect to K*, if there exists an element $x* \in K$ and a real number $p \in]-\infty, 1[$, such that

$$\lim_{x \in K, \|x\| \to \infty} \frac{\langle f(x) - T(x), x - x_* \rangle}{\|x\|^p} = 0.$$

The importance of Definition 7 is given by the following result.

PROPOSITION 15. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, $f : H \to H$ a mapping and $K \subset H$ a closed pointed convex cone. If f has an $(x*, p)$-scalar asymptotic derivative T, and T is $(x*, p)$-coercive, then f is without exceptional family of elements with respect to K.*

PROOF. The proposition is a consequence of Theorem 14 and of the following relation

$$\lim_{x \in K, \|x\| \to \infty} \frac{\langle f(x), x - x_* \rangle}{\|x\|^p} = \lim_{x \in K, \|x\| \to \infty} \frac{\langle f(x) - T(x), x - x_* \rangle}{\|x\|^p} + \lim_{x \in K, \|x\| \to \infty} \frac{\langle T(x), x - x_* \rangle}{\|x\|^p} = \infty$$
□

We recall that a mapping $f : H \to H$ is called pseudo-monotone on K, if for any $x, y \in K$, $x \ne y$ we have that $\langle y - x, f(x) \rangle \ge 0$ implies $\langle y - x, f(y) \rangle \ge 0$. The following notion is more general as a similar notion defined in [32].

Definition 8. We say that a mapping $f : H \to H$ is *weakly proper on* K, if for each family of elements $\{x_r\}_{r>0} \subset K$, with $\|x_r\| \to \infty$ as $r \to \infty$, there exists an element $x* \in K$ such that for some $r > 0$, with $\|x*\| < \|x_r\|$ we have $\langle f(x_*), x_r - x_* \rangle \geq 0$.

We have the following result.

Theorem 16. *Let* $(H, \langle \cdot , \cdot \rangle)$ *be a Hilbert space,* $K \subset H$ *a closed pointed convex cone and* $f : H \to H$ *a pseudo-monotone mapping. If* f *is a completely continuous field, or an* α*-condensing field, or a projectionally pseudo-contractant field, then the problem* NCP(f, K) *has a solution if and only if* f *is weakly proper on* K.

Proof. We suppose that the problem NCP(f, K) has a solution x_{**}. Because the fact that the solvability of the problem NCP(f, K) is equivalent to the solvability of the following variational inequality (see [14], [15])

(VI(f, K)) find $x_0 \in K$ such that $\langle f(x_0), x - x_0 \rangle \geq 0$ for all $x \in K$,

we have $\langle f(x_{**}), x - x_{**} \rangle \geq 0$ for all $x \in K$. Obviously, if we take in Definition 8, $x_* = x_{**}$, we deduce that f is weakly proper on K.

Conversely, assume that f is weakly proper on K. In this case Definition 8 implies that for each family of elements $\{x_r\}_{r>0}$, with $\|x_r\| \to \infty$ as $r \to \infty$, there exists an element $x_* \in K$ such that $\langle f(x_*), x_r - x_* \rangle \geq 0$ for some $r > 0$ such that $\|x_*\| < \|x_r\|$. Since f is pseudo-monotone we have that $\langle f(x_r), x_r - x_* \rangle \geq 0$, which implies that f satisfies condition (θ-S), with respect to K. By Theorem 11 we have that the problem NCP(f, K) has a solution and the proof is complete.□

Comments. We presented in this paper several existence theorems for nonlinear complementarity problems considered in an arbitrary Hilbert space. Our results are based on Leray–Schauder type alternatives and on the notion of exceptional family of elements for a mapping. The research on this subject must be developed now following two directions: the first is to find new and more general Leray–Schauder type alternatives and the second to find new classes of mappings without exceptional family of elements with respect to a given convex cone.

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