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CONTINUOUS SELECTIONS VIA GEODESICS

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Abstract. Some continuous selection results for a class of nonconvex-valued maps are obtained. One of them contains Michael's theorem, in the case of a Hilbert codomain. Methods of nonsmooth analysis and Γ-convergence are used.

1. Introduction

The classical problem of finding continuous selections from lower semicontinuous set-valued maps from a paracompact space X into a Banach space Y has been studied by many authors (see, e.g. the survey book by Repovš and Semenov [18]). If the values of the map F are convex and closed, then a continuous selection exists (Michael's theorem). In the case where the values of F are not necessarily convex, one has first to mention Filippov's counterexample (see [3, Example 1, p. 68]), showing that continuous selections may fail to exist. However, in the particular case where $Y = L^p(T, E)$, T being a nonatomic measure space and E a Banach space, and the values of F are nonconvex but decomposable, continuous selections do exist (see [2], [13], [5], [1]). In order to obtain continuous selections, a key tool is connecting points with continuity, while remaining in a prescribed set. If the values of F are convex, then

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one can use convex combinations, while if the values are decomposable, suitable "decomposable combinations" (see (3.2)) perform the task.

This paper also deals with the problem of finding continuous selections in a possibly nonconvex setting. The main idea is the simple observation that segments in convex sets are exactly the geodesics, and therefore geodesic-like curves may be good substitutes of segments in some classes of nonconvex sets (see also [15], where the idea of geodesic combinations appears in an abstract framework). More precisely, if we assume that F is locally selectionable and for any $x \in X$ the value $F(x)$ enjoy the property that any pair of points $y_1, y_2 \in F(x)$ can be connected by a curve in $F(x)$ depending continuously on y_1, y_2 and on x, then a continuous selection exists (see Proposition 3.1 below).

The main point of the present paper consists in providing a class of nonconvex multifunctions enjoying the above property of "continuous connection among points", and to obtain the corresponding selection result. The basic ingredients are already present in the literature. They are the so-called φ-convex sets, that is sets which satisfy an external sphere condition with locally uniform radius (see e.g. [12], [6], [7], [9], [17], where such property is studied under different points of view). Since a convex set satisfies an external sphere condition with constant infinite radius, it is natural to imagine that some properties which hold globally for convex sets still hold for φ-convex sets, but locally. This is the case, for example, for the existence and uniqueness properties of the metric projection (see [6]), which are at the basis of local existence and continuous dependence of geodesics (see [7]). We prove a selection theorem (Theorem 3.1) assuming the lower semicontinuity of F together with the local compactness of its graph, the locally uniform φ-convexity of the values $F(x)$, and the *global* uniqueness of geodesics in $F(x)$ (called hyperbolicity). We remark that the continuous dependence of geodesics must be established also with respect to the set in which they are taken, differently from [15], where the geodesic structure is independent of x. This is obtained with the help of a Γ-convergence argument. As a corollary, we obtain selections for continuous maps having as values embedded $\mathcal{C}^2$-manifolds with locally uniform negative curvature. A second result (Theorem 3.2), without the strong compactness requirement, comes by a strengthening of the hyperbolicity assumption: we suppose the diameter of the values of F to be small with respect to the radius of the external sphere. Since this hypothesis is automatically satisfied in the convex case, Theorem 3.2 contains Michael's selection theorem (in the particular case of a Hilbert codomain). Under a similar assumption, it is also easy to obtain the existence of fixed points of a compact continuous map of a closed φ-convex set into itself, thus generalizing Schauder's fixed point theorem. Some known examples (see the final remarks) illustrate the sharpness of our assumptions.

2. φ–convex sets

In the following, H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and X is a paracompact topological space. We denote by w-H the space H endowed with the weak topology. We say that a set $G \subset X \times H$ is *locally weakly closed* if each point $(x, y) \in G$ admits a neighbourhood $W(x, y)$ in $X \times H$ such that $G \cap W(x, y)$ is closed in $X \times w$-H. We say also that a set $G \subset X \times H$ is *locally σ-compact* if for each $x \in X$ there exists a neighbourhood $U(x)$ in X such that $G \cap (U(x) \times \overline{B}(0, k))$ is compact in $X \times H$ for all $k = 1, 2, \ldots$ Let $K \subset H$ be closed. The vector $v \in H$ is said to be a *proximal normal* to K at $x \in K$ if there exists $\sigma = \sigma(x, v)$ such that $\langle v, y - x\rangle \le \sigma\|y - x\|^2$ for all $y \in K$. The set of all proximal normals to K at $x \in K$ (which is a convex cone, see [8, §1]) is denoted by $N_K(x)$. The metric *projection* of a point $x \in H$ into K, i.e. the set of all $y \in K$ such that $\|y - x\| = \inf\{\|y' - x\| : y' \in K\} := d(x, K)$ is denoted by $\pi_K(x)$. We now introduce the class of sets we consider.

Definition 2.1. We say that a closed set $K \subset H$ is φ-convex if there exists a continuous function $\varphi : K \to [0, \infty)$ such that for all $x, y \in K$, $v \in N_K(x)$

$$\langle v, y - x\rangle \le \varphi(x)\, \|v\|\|y - x\|^2. \tag{2.1}$$

In other words, φ-convexity means that for each $x \in K$ and for each $v \in N_K(x)$ there exists a closed ball, with radius continuously depending on x and center placed in the half line $x + v\mathbb{R}^+$, which touches K only at x. The name "φ-convex" is taken from [14], [6], [7]. This class of sets was studied in infinite dimensional Hilbert spaces first by A. Canino ([6], [7]), where several properties, including local existence and uniqueness of the metric projection and of geodesics, were established. Later, φ-convex sets were characterized by means of the local smoothness of the distance function in [17], [9]; a finite dimensional version of this result is contained in [12, §4]. Observe that convex sets as well as sets with $\mathcal{C}^{1,1}$-boundary are φ-convex, while other examples can be found in [6]. The following properties will be used in the sequel.

Lemma 2.1. *Let $K \subset H$ be a nonempty closed and φ-convex set. Then there exists an open set $\mathcal{U} \supset K$ such that the metric projection π_K is well defined, single-valued and locally Lipschitzean in $\mathcal{U}$. More precisely:*

(a) *given $x \in K$ and $\eta > \varphi(x)$ the projection is a singleton for all points in the ball $x + \delta B$, where $\delta > 0$ is such that $4\eta\delta < 1$ and $\varphi(y) \le \eta$ for all $y \in K \cap (x + 3\delta B)$,*

(b) *given $r > 0$ and $\eta \ge 0$ such that $2r\eta < 1$, for all $z_1, z_2 \in \mathcal{U} \cap (K + r\overline{B})$, $\varphi(\pi_K(z_1)) \le \eta$, and $\varphi(\pi_K(z_2)) \le \eta$, such that the following Lipschitz estimate holds*

$$\|\pi_K(z_1) - \pi_K(z_2)\| \le (1 - 2r\eta)^{-1}\|z_1 - z_2\|.$$

Furthermore, for all $z \in \overline{\mathrm{co}}\, K \cap \mathcal{U}$ *it holds*

$$d(z, K) = \|z - \pi_K(z)\| \le \varphi(\pi_K(z))(\operatorname{diam} K)^2 \tag{2.2}$$

where $\overline{\mathrm{co}}$ *means the closed convex hull and* $\operatorname{diam} K$ *is the diameter of the set* K.

Proof. For the proof of (a) and (b) we refer to [7, Proposition 1.12], [9, Theorem 6.1] and to [6, Proposition 2.9]. To prove the last statement, let us take $z \in \mathrm{co}\, K \cap \mathcal{U}$ and let $z = \sum_{i=1}^n \lambda_i x_i$, $\lambda_i \ge 0$, $\sum_{i=1}^n \lambda_i = 1$, $x_i \in K$. Since $z - \pi_K(z) \in N_K(\pi_K(z))$ we have by the definition of φ-convexity:

$$\begin{aligned} \langle z - \pi_K(z), x_i - \pi_K(z)\rangle &\le \varphi(\pi_K(z))\|z - \pi_K(z)\|\|x_i - \pi_K(z)\|^2 \\ &\le \varphi(\pi_K(z))\|z - \pi_K(z)\|(\operatorname{diam} K)^2, \end{aligned}$$

$i = 1, \dots, n$. Multiplying by λ_i and summing we clearly obtain (2.2). □

In what follows, we consider the space $L^2(0,1;H)$ of all H-valued strongly measurable functions $u(\,\cdot\,)$ with $\int_0^1 \|u(t)\|^2 < \infty$, and we denote by $W^{1,2}(0,1;H)$ the space of all absolutely continuous functions whose derivative (which exists for a.e. $t \in [0,1]$ by Theorem 2.1 [4, p. 16]) belongs to $L^2(0,1;H)$. The space $W^{1,2}(0,1;H)$ with the scalar product

$$\langle u(\,\cdot\,), v(\,\cdot\,)\rangle = \langle u(0), v(0)\rangle + \int_0^1 \langle \dot u(t), \dot v(t)\rangle\, dt$$

is a Hilbert space.

We use the symbol w-$W^{1,2}(0,1;H)$ to denote the space $W^{1,2}(0,1;H)$ supplied with the weak topology. Moreover, the symbol $\widetilde{w}$-$W^{1,2}(0,1;H)$ indicates the (finer) topology generated by the union of the weak topology with the topology of uniform convergence.

Fix $K \subset H$ and $y_1, y_2 \in K$. Let us consider the set

$$H^K_{y_1,y_2} := \{\gamma \in W^{1,2}(0,1;H) : \gamma(t) \in K \text{ for all } t,\ \gamma(0) = y_1,\ \gamma(1) = y_2\} \tag{2.3}$$

which is nonempty by Proposition 2.16 ([6]), if K is assumed to be closed, φ-convex and connected. Define the energy functional

$$J^K_{y_1,y_2}(\gamma) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 \|\dot\gamma(t)\|^2\, dt & \text{if } \gamma \in H^K_{y_1,y_2}, \\ \infty & \text{if } \gamma \in W^{1,2}(0,1;H) \setminus H^K_{y_1,y_2}. \end{cases} \tag{2.4}$$

Proposition 2.1. *Let* $K \subset H$ *be closed,* φ*-convex and connected, and let* $y_1, y_2 \in K$. *Then the functional* $J^K_{y_1,y_2}$ *admits a minimizer, provided either* K *is weakly closed, or* $\|y_1 - y_2\|$ *is small enough.*

Proof. Observe that $J^K_{y_1,y_2}$ is coercive (see Definition 1.12 [10, p. 12]) and lower semicontinuous in w-$W^{1,2}(0,1;H)$. If K is weakly closed, then it is easy to show that $H^K_{y_1,y_2}$ is weakly closed in $W^{1,2}(0,1;H)$, so that $J^K_{y_1,y_2}$ admits a minimizer. The remainder of the statement is contained in [7, Theorem 3.3]. □

Definition 2.2. We say that a closed connected φ-convex set $K \subset H$ is hyperbolic if for all $y_1, y_2 \in K$ the above defined functional $J^K_{y_1,y_2}$ admits exactly one minimizer in $W^{1,2}(0,1;H)$.

The term "hyperbolic" is taken from Riemannian geometry. Although the minimizers of $J^K_{y_1,y_2}$ are proved in [7] to be *locally* unique and continuously depending on the extreme points, global uniqueness can be violated, even in $\mathbb{R}^2$.

In what follows, the concept of Γ-convergence will be used. The relevant definitions and results can be found, for example, in [10, Chapters 4 and 7].

3. Geodesic curves and selection results

We begin with a general selection method.

Proposition 3.1. *Let F be a multivalued map from X into a normed space E with the following two properties*

(1) *For all $x \in X$ there exists a neighborhood V_x, and a continuous function $f_x : V_x \to E$ such that $f_x(y) \in F(y)$ whenever $y \in V_x$,*
(2) *For all $x \in X$ and all pairs $y_1, y_2 \in F(x)$ there exists a continuous function $\gamma^x(y_1, y_2; \,\cdot\,) : [0,1] \to F(x)$ such that*
   (a) *$\gamma^x(y_1, y_1; \alpha) \equiv y_1$ and $\gamma^x(y_1, y_2; 0) = y_1$, $\gamma^x(y_1, y_2; 1) = y_2$ for all $y_1, y_2 \in F(x)$, $x \in X$,*
   (b) *$\gamma^x(y_1, y_2; \alpha) = \gamma^x(y_2, y_1; 1-\alpha)$ for all $\alpha \in [0,1]$, $y_1, y_2 \in F(x)$, $x \in X$,*
   (c) *given $x \in X$, $y_i \in F(x)$, $i = 1,2$, and $\varepsilon > 0$ there exist $\delta > 0$ and a neighbourhood $U(x)$ such that*
$$\|\gamma^{x'}(y_1', y_2'; \alpha) - \gamma^x(y_1, y_2; \alpha)\| \le \varepsilon \quad \textit{for all } \alpha \in [0,1],$$
   whenever $x' \in U(x)$ and $y_i' \in F(x)$, $i = 1,2$, satisfy $\|y_i - y_i'\| \le \delta$.

Then F admits a continuous selection.

Proof. For $n = 1,2,\ldots$ denote by Δ_n the simplex $\{(\alpha_1, \ldots, \alpha_n) : \alpha_i \ge 0,\ \sum_{i=1}^n \alpha_i = 1\}$ and observe that for $(\alpha_1, \ldots, \alpha_n) \in \Delta_n$ and $\alpha_n < 1$ we have $(\alpha_1/(1-\alpha_n), \ldots, \alpha_{n-1}/(1-\alpha_n)) \in \Delta_{n-1}$. Fix now $x \in X$, $y_1, \ldots, y_n \in F(x)$, and define the functions $\gamma^x_n(y_1, \ldots, y_n; \cdot) : \Delta_n \to F(x)$ by recursion as follows:

$$\gamma^x_n(y_1; 1) = y_1,$$
$$\gamma^x_n(y_1, \ldots, y_n; \alpha_1, \ldots, \alpha_n) = \begin{cases} \gamma^x\left(\gamma^x_{n-1}\left(y_1, \ldots, y_{n-1}; \dfrac{\alpha_1}{1-\alpha_n}, \ldots, \dfrac{\alpha_{n-1}}{1-\alpha_n}\right), y_n; \alpha_n\right) & \text{if } \alpha_n < 1, \\ y_n & \text{if } \alpha_n = 1, \end{cases}$$

where γ^x is taken from the condition (2). From (c) and an induction argument it follows easily that given $x \in X$ and $y_i \in F(x)$, $i = 1, \dots, n$, for all $\varepsilon > 0$ there exist $\delta_n > 0$ and a neighbourhood $U_n(x)$ such that

$$\|\gamma_n^{x'}(y_1', \dots, y_n'; \alpha_1, \dots, \alpha_n) - \gamma_n^x(y_1, \dots . y_n; \alpha_1, \dots, \alpha_n)\| \le \varepsilon \tag{3.1}$$

for all $(\alpha_1, \dots, \alpha_n) \in \Delta_n$, $x' \in U_n(x)$ and $y_i' \in F(x)$, $i = 1, \dots, n$, with $\|y_i - y_i'\| \le \delta_n$. Observe that by induction it follows also that the functions $\gamma_n^x(y_1, \dots, y_n; \cdot)$ are continuous in Δ_n. To see this it is enough to take into account that $\gamma^x(a, y; \alpha) \to y$ as $\alpha \to 1$ for each point $a \in F(x)$, and to apply a compactness argument on the same line as in [15, p. 567].

Condition (a) implies inductively that if $\alpha_i = 0$ for some $i = 1, \dots, n$ then

$$\begin{aligned}\gamma_n^x&(y_1, \dots, y_i, \dots, y_n; \alpha_1, \dots, \alpha_i, \dots, \alpha_n) \\ &= \gamma_{n-1}^x(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n; \alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n),\end{aligned}$$

while from (b) we see that $\gamma_n^x(y_1, \dots, y_n; \alpha_1, \dots, \alpha_n)$ does not depend on the ordering of the pairs (y_i, α_i). In other words, the value $\gamma_n^x(y_1, \dots, y_n; \alpha_1, \dots, \alpha_n)$ remains the same if we change the places of arbitrary points y_i, y_j together with the corresponding numbers α_i and α_j.

Let us consider now a locally finite refinement $(W_\iota)_{\iota \in I}$ of the covering $(V_x)_{x \in X}$ (which appears in the property (1)), together with a continuous partition of unity $(\alpha_\iota)_{\iota \in I}$ subordinate to it. For all $\iota \in I$ let $x_\iota \in X$ be such that $W_\iota \subset V_{x_\iota}$, and set $f_\iota(x) = f_{x_\iota}(x)$, $x \in W_\iota$. Define, for $x \in X$, $I(x) = \{\iota \in I : \alpha_\iota(x) > 0\} = \{\iota_1, \dots, \iota_m\}$, and

$$f(x) = \gamma_m^x(f_{\iota_1}(x), \dots, f_{\iota_m}(x); \alpha_{\iota_1}(x), \dots, \alpha_{\iota_m}(x)).$$

By the above, the function $f : X \to E$ is well defined and $f(x) \in F(x)$ for all $x \in X$. Continuity of f now follows from the property (3.1), the continuity of all the functions $f_\iota(x), \alpha_\iota(x)$, and the local finiteness of the covering $(W_\iota)_{\iota \in I}$. The proof is complete. □

Remark. If $F(x)$ is convex, then it is natural to choose as $\gamma^x(y_1, y_2; \cdot)$ the convex combination $(1 - \alpha)y_1 + \alpha y_2$. If $E = L^p(T, Y)$, with (T, μ) a finite nonatomic measure space and Y a Banach space, then being $(T_\alpha)_{\alpha \in [0,1]}$ an increasing chain of measurable subsets of T such that $\mu(T_\alpha) = \alpha\mu(T)$ and $F : X \to E$ a multifunction with decomposable values, one can set

$$\gamma^x(y_1, y_2; \alpha)(\cdot) = \chi_{T_{1-\alpha}}(\cdot)y_1(\cdot) + \chi_{T_\alpha}(\cdot)y_2(\cdot), \tag{3.2}$$

where $\chi_S(t) = 1$ if $t \in S$, and $= 0$ if $t \notin S$. However, the above result does not improve the selection technique for decomposable-valued maps, since it requires local selections to be given.

The next two results make Proposition 3.1 applicable to a class of maps with φ-convex values. We prove first the local selectionability.

From now on, X is supposed to be a *metric* space.

Proposition 3.2. *Let $F : X \to H$ be a lower semicontinuous multivalued map, with locally weakly closed graph, and assume that there exists an upper semicontinuous function $\varphi : \operatorname{graph} F \to \mathbb{R}^+$ such that*

(i) *$y \mapsto \varphi(x,y)$ is continuous on $F(x)$ for all x,*
(ii) *$F(x)$ is $\varphi(x,\cdot)$-convex for all x.*

Then F satisfies the (1) *of Proposition* 3.1. *More precisely, for all $x_0 \in X$, $y_0 \in F(x_0)$ there exists a neighbourhood $U(x_0)$ such that the map*

$$x \mapsto \pi_{F(x)}(y_0)$$

is well defined, single-valued and continuous in $U(x_0)$.

Proof. Fix $x_0 \in X$ and $y_0 \in F(x_0)$. Choose $\eta, \delta > 0$ such that $4\eta\delta < 1$, and a neighbourhood $U(x_0)$ with the property that $\varphi(x,y) < \eta$ whenever $x \in U(x_0)$ and $y \in F(x)$ with $\|y - y_0\| < 3\delta$. By the hypotheses on F we can assume that

$$d(y_0, F(x)) < \delta \quad \text{for all } x \in U(x_0),$$

and that $\operatorname{graph} F \cap (U(x_0) \times \overline{B}(y_0,\delta))$ is closed in $X \times w\text{-}H$. By (a) in Lemma 2.1, for all $x \in U(x_0)$ the projection of y_0 into $F(x)$ is a singleton, that we call $f(x)$. We show that f is continuous in $U(x_0)$. To this aim, take $x \in U(x_0)$ and a sequence $\{x_n\} \subset U(x_0)$ converging to x, and set $y_n = f(x_n)$. Observe that, by lower semicontinuity,

$$\limsup_{n\to\infty} d(y_0, F(x_n)) \le d(y_0, F(x)), \tag{3.3}$$

so that, in particular, the sequence $\{y_n\}$ is relatively weakly compact. Thus any subsequence of $\{y_n\}$ admits a subsequence, still denoted by $\{y_n\}$, weakly converging to a point y which belongs to $F(x)$ by graph closedness. By combining the weak lower semicontinuity of the norm with (3.3), we obtain that $\|y - y_0\| \le d(y_0, F(x))$. Then from the uniqueness of the projection both $y = \pi_{F(x)}(y_0) = f(x)$ and $\limsup_{n\to\infty} \|y_n - y_0\| \le \|y - y_0\|$ follow. The above inequalities yield that $f(x_n)$ converges strongly to $f(x)$; thus the proof is concluded. $\square$

The following result concerns the continuous dependence of geodesics.

Proposition 3.3. *Let a multivalued map $F : X \to H$ and an u.s.c. function $\varphi : \operatorname{graph} F \to \mathbb{R}^+$ be such as in Proposition* 3.2. *Assume, moreover, that the sets $F(x)$ are connected and hyperbolic for all x, and that* graph F *is locally*

σ-compact. Then for all $x \in X$ and all pairs $y_1, y_2 \in F(x)$ there exists a continuous curve $\gamma^x(y_1, y_2; \cdot) : [0,1] \to F(x)$ such that the properties (a)–(c) *of Proposition* 3.1 *hold.*

Proof. Fix $x \in X$ and, for all $y_1, y_2 \in F(x)$, consider the set $H^x_{y_1,y_2} := H^{F(x)}_{y_1,y_2}$ and the energy functional $J^x_{y_1,y_2} := J^{F(x)}_{y_1,y_2}$ as defined, respectively, in (2.3) and (2.4). Define $\gamma^x(y_1, y_2; \cdot)$ to be the unique minimizer of $J^x_{y_1,y_2}$. Properties (a) and (b) follow immediately from the definition. To prove (c), let $\{x_n\} \subset X$, $x_n \to x$, and $y_i^n \in F(x_n)$, $y_i^n \to y_i$, $i = 1,2$. It is our purpose to show that $\gamma^{x_n}(y_1^n, y_2^n; \cdot)$ converges to $\gamma^x(y_1, y_2; \cdot)$ in $\widetilde{w}$-$W^{1,2}(0,1;H)$. Set $H_n := H^{x_n}_{y_1^n,y_2^n}$, $J_n := J^{x_n}_{y_1^n,y_2^n}$, $J := J^x_{y_1,y_2}$, and observe that the family of the functionals $\{J_n : n \geq 1\}$ is *equi-coercive* in the space $\widetilde{w}$-$W^{1,2}(0,1;H)$ (see Definition 7.6 in [10, p. 70]). Then the result will follow from Corollary 7.24 in [10, p. 84] if we establish that the sequence J_n, $n = 1,2,\ldots$, Γ-converges to J in $\widetilde{w}$-$W^{1,2}(0,1;H)$. To do this, it is enough to prove that given $u \in W^{1,2}(0,1;H)$

(α) for all sequences $\{u_n\}$ converging to u in $\widetilde{w}$-$W^{1,2}(0,1;H)$ it holds

$$\liminf_{n\to\infty} J_n(u_n) \geq J(u), \tag{3.4}$$

(β) there exists a sequence $\{v_n\}$, $v_n \in H_n$, $n = 1,2,\ldots$, converging to u in $\widetilde{w}$-$W^{1,2}(0,1;H)$ such that

$$\limsup_{n\to\infty} J_n(v_n) \leq J(u) \tag{3.5}$$

(see Proposition 8.1 [10, p. 87]).

To show (α), observe that $u_n(t) \to u(t)$ in H for all $t \in [0,1]$, and if $u_n \in H_n$ for all n, then $J_n(u_n) = J(u_n)$ and, by local closedness of graph F, $u \in H^x_{y_1,y_2}$. Since J is lower semicontinuous in w-$W^{1,2}(0,1;H)$, (3.4) now follows. Next, we construct a sequence $v_n \in H_n$ converging to u in $\widetilde{w}$-$W^{1,2}(0,1;H)$ and such that (3.5) holds. Taking a sequence $\lambda_n \to 0+$ such that $\|y_i^n - y_i\|^2 \leq \lambda_n^2$, $n \geq 1$, $i = 1,2$, define the functions $u_n : [0,1] \to H$ to be equal to $ut((t-\lambda_n)/(1-2\lambda_n))$ for $t \in [\lambda_n, 1-\lambda_n]$, $u_n(0) = y_1^n$, $u_n(1) = y_2^n$, and to be affine in the remainder of the interval $[0,1]$. We have that the u_n are absolutely continuous, $\|u_n(t) - u(t)\| \to 0$ as $n \to \infty$ uniformly on $[0,1]$, and

$$\int_0^1 \|\dot u_n(t)\|^2\,dt \leq \frac{1}{1-2\lambda_n}\int_0^1 \|\dot u(t)\|^2\,dt + 2\lambda_n, \quad n \geq 1. \tag{3.6}$$

Let $\eta, \delta > 0$ with $4\eta\delta < 1$, and a neighbourhood $U(x)$ be such that $\varphi(x', y) \leq \eta$ for all $x' \in U(x)$ and $y \in F(x')$ with $\|y - u(t)\| \leq 3\delta$ for some $t \in [0,1]$. Assume also that $d(u(t), F(x')) \leq \delta/2$ whenever $x' \in U(x)$ and $t \in [0,1]$. We use here the upper semicontinuity of the function φ on graph F and the lower semicontinuity of F. Since for n large enough $d(u_n(t), F(x_n)) \le \delta$, by Lemma 2.1 the projection $v_n(t) = \pi_{F(x_n)}(u_n(t))$ is well defined, is absolutely continuous, and

$$\|\dot{v}_n(t)\| \le 2\|\dot{u}_n(t)\| \tag{3.7}$$

for a.e. $t \in [0,1]$. Thus $v_n \in H_n$ and by (3.6) the sequence $\{v_n\}_{n\ge 1}$ is relatively compact in the weak topology of $W^{1,2}(0,1;H)$. Since, moreover, $v_n(t) \to u(t)$ as $n \to \infty$ uniformly in t, we obtain that $\{v_n\}$ converges to u in $\widetilde{w}$-$W^{1,2}(0,1;H)$. Set $r_n = \sup_{t\in T} d(u_n(t), F(x_n))$, and observe that $r_n \to 0$ as $n \to \infty$ by the lower semicontinuity of F. Applying the local Lipschitzianity of the projection (see Lemma 2.1(b)), we can improve the estimate (3.7): $\|\dot{v}_n(t)\| \le (1 - 2r_n\eta)^{-1}\|\dot{u}_n(t)\|$ for a.e. $t \in [0,1]$. This together with (3.6) already imply (3.5). The proof is complete. □

We are now ready to prove our main result.

Theorem 3.1. *Let a multivalued map $F : X \to H$ together with an u.s.c. function $\varphi : \operatorname{graph} F \to \mathbb{R}^+$ satisfy the same hypotheses of Proposition* 3.3. *Then F admits a continuous selection $f(x) \in F(x)$, $x \in X$. Moreover, given $x_0 \in X$ and $y_0 \in F(x_0)$, the selection $f(x)$ can be chosen such that $f(x_0) = y_0$.*

Proof. Let U_0 be an open neighborhood of x_0 and $f_0 : U_0 \to H$ be a continuous selection of $F(x)$ such that $f_0(x_0) = y_0$, given by Proposition 3.2. Choose an open neighbourhood V_0 of x_0 such that $\overline{V}_0 \subset U_0$, and define a continuous function $\alpha : X \to [0,1]$ equal to 1 on $\overline{V}_0$ and to 0 outside U_0. If g is an arbitrary continuous selection from F obtained by applying Propositions 3.1–3.3, then the function

$$f(x) = \begin{cases} f_0(x) & \text{if } x \in V_0, \\ \gamma^x(g(x), f_0(x); \alpha(x)) & \text{if } x \in U_0 \setminus V_0, \\ g(x) & \text{if } x \notin U_0, \end{cases}$$

where $\gamma^x(y_1, y_2; \cdot\,)$ is the curve appearing in the statement of Proposition 3.3, is a continuous selection with $f(x_0) = y_0$. □

Corollary 3.1. *Let $F : X \to \mathbb{R}^n$ be a continuous multivalued map admiting as values closed simply connected $\mathcal{C}^2$-manifolds with negative sectional curvature uniformly bounded from below. Then the statement of Theorem* 3.1 *holds.*

Proof. Let $\kappa < 0$ be the lower bound for the curvature of $F(x)$. Then $F(x)$ is -2κ-convex. Moreover, by [11, Theorem 3, p. 248], there exists a unique geodesic curve connecting any two points in $F(x)$. Therefore, Theorem 3.1 can be applied. □

In conclusion we exploit the simple fact that if the diameter of a φ-convex set K is small enough, then π_K is a retraction of K in the convex hull $\overline{\operatorname{co}}\, K$ (see Lemma 2.1). Though being actual corollaries, the statements below contain

Michael's selection and Schauder fixed point theorems for the particular case of Hilbert space-valued functions.

Theorem 3.2. *Let a multivalued map $F : X \to H$ be lower semicontinuous, and let an upper semicontinuous function $\varphi_0 : X \to \mathbb{R}^+$ and $0 < \eta(x) < 1/2$, $x \in X$, be given. Assume furthermore that*

(i) *$F(x)$ is $\varphi_0(x)$-convex for all $x \in X$,*
(ii) *$\varphi_0(x) \operatorname{diam} F(x) < \eta(x)$ for all $x \in X$,*
(iii) *for all $(x,y) \in \operatorname{graph} F$ the set*

$$\operatorname{graph} F \cap (\overline{B}(x, \varphi_0(x)) \times \overline{B}(y, 2\eta(x) \operatorname{diam} F(x))$$

is closed in $X \times w$-H.

Then F admits a continuous selection which can be chosen to pass through an arbitrary point of graph F.

Proof. It is easy to see that the multifunction $\overline{\operatorname{co}}\, F(x)$ is lower semicontinuous, so that it admits a continuous selection (passing through a given point of graph F) which we call $f(x)$. By Lemma 2.1(a), the hypothesis (ii) implies that $f(x)$ admits a unique projection into $F(x)$ for all x. We claim now that the map $x \mapsto \pi_{F(x)}(f(x))$ is continuous, so it is a selection we look for.

Fix $x \in X$ and assume that $f(x) \notin F(x)$, which may occur only when $\varphi_0(x) > 0$. Let a sequence $\{x_n\} \subset X$, $x_n \to x$ as $n \to \infty$, be given. In accordance with Lemma 2.1 (see the estimate (2.2)), by the lower semicontinuity of F, the upper semicontinuity of $\varphi_0(\,\cdot\,)$ and the hypotheses (i), (ii), we can assume that

$$d(f(x), F(x_n)) < \eta(x) \operatorname{diam} F(x) < \frac{1}{4\varphi_0(x_n)} \quad \text{for all } n = 1, 2, \ldots \tag{3.8}$$

Hence, by Lemma 2.1 ((a) and (b)) it holds that the projections $\pi_{F(x_n)}(f(x))$ are single-valued, and

$$\|\pi_{F(x_n)}(f(x_n)) - \pi_{F(x_n)}(f(x))\| \le 2\|f(x_n) - f(x)\| \to 0, \quad n \to \infty.$$

Moreover, by (3.8), the points $(x_n, \pi_{F(x_n)}(f(x)))$ belong to the set

$$\operatorname{graph} F \cap (\overline{B}(x, \varphi_0(x)) \times \overline{B}(\pi_{F(x)}(f(x)), 2\eta(x) \operatorname{diam} F(x))$$

for all n large enough. This permits (see hypothesis (iii)) to apply the same argument of the proof of Proposition 3.2, which yields $\|\pi_{F(x_n)}(f(x)) - \pi_{F(x)}(f(x))\| \to 0$ as $n \to \infty$. Thus the continuity at the point x follows. If $f(x) \in F(x)$, instead, we immediately obtain

$$\|\pi_{F(x_n)}(f(x_n)) - f(x)\| \le 2\|f(x_n) - f(x)\| + d(f(x), F(x_n)) \to 0, \quad n \to \infty,$$

which concludes the proof. □

Proposition 3.4. *Let $\varphi_0 \geq 0$ be fixed, and let $K \subset H$ be φ_0-convex and such that $4\varphi_0 \operatorname{diam} K < 1$. Then each continuous and compact mapping $f : K \to K$ admits a fixed point.*

Remarks. (1) Filippov's counterexample (see [3, p. 68]) uses a continuous multifunction from $(-1, 1)$ into $\mathbb{R}^2$ whose values $F(t)$ are, for $t \neq 0$, an arc of ellipse with arbitrarily decreasing smaller axis and arbitrary increasing speed of rotation as t approaches 0, while $F(0) = [-1, 1]$. It is clear that all the values do satisfy the φ-convexity assumption, but φ is not u.s.c. with respect to t at $t = 0$. The upper semicontinuity assumption prevents sudden "changes of shape" of $F(x)$.

(2) Example 2 in [3, p. 69] shows that the hyperbolicity assumption cannot – in general – be dropped. In that example a continuous multivalued map F from the unit ball of $\mathbb{R}^2$ into the (closed) subsets of the sphere $\{x \in \mathbb{R}^2 : \|x\| = 1\}$ is constructed, with no continuous selections nor fixed points. All the values $F(x)$ are 1/2-convex, but are allowed to be the whole circumference, which is not hyperbolic.

(3) The idea of introducing a parameter estimating the nonconvexity of a set, and to generalize Michael's selection theorem to maps the nonconvexity of whose values is controlled by this parameter, was already present in [16] (see also the last generalization in [19]). However, the notion of paraconvexity introduced there is not related directly to φ-convexity. For example, in $\mathbb{R}^2$, a set whose shape is the symbol $\vee$ is paraconvex, but not φ-convex, while a set whose shape is the symbol $\cup$ is φ-convex and hyperbolic, but not paraconvex.

(4) With a little more of effort, an alternative proof of Theorem 3.2 can be given following the same scheme of Theorem 3.1. Indeed, it can be shown that the family of curves $\gamma^x(y_1, y_2; \alpha) = \pi_{F(x)}((1-\alpha)y_1 + \alpha y_2)$, with $y_1, y_2 \in F(x)$, satisfies the requirements (a)–(c) of Proposition 3.1. Thus our technique provides an actual generalization of Michael's selection theorem.

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