STRUCTURE OF LARGE POSITIVE SOLUTIONS OF SOME SEMILINEAR ELLIPTIC PROBLEMS WHERE THE NONLINEARITY CHANGES SIGN

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ABSTRACT. Existence and uniqueness of large positive solutions are obtained for some semilinear elliptic Dirichlet problems in bounded smooth domains \( \Omega \) with a large parameter \( \lambda \). It is shown that the large positive solution has flat core. The distance of its flat core to the boundary \( \partial \Omega \) is exactly measured as \( \lambda \to \infty \).

1. Introduction

In this paper we study the following eigenvalue problem

\[
-\Delta u = \lambda f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \( (N \geq 2) \) with smooth boundary \( \partial \Omega \), \( \lambda > 0 \). We are interested in the structure of positive solutions of (1.1) for large positive \( \lambda \) in the case that \( f(0) = 0, f'(0) = 0, f(a) = f(b) = 0, 0 < a < b, f \) changes sign on \( [0, \infty) \). More precisely, we assume that \( f \in C^1((0, \infty) \setminus \{b\}) \cap C^0([0, \infty)) \) satisfies the following conditions:

\( f_1 \) \( f(0) = 0, f'(0) = 0, f \) has two positive zeros \( a \) and \( b \) such that \( a < b; f < 0 \) in \( (0, a) \), \( f > 0 \) in \( (a, b) \); there exists \( 0 < \delta < b - a \) such that \( f'(s) < 0 \) for \( s \in (b - \delta, b) \) and \( f \) has no other positive zeros.

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(f_2) \lim_{s \to \beta} f(s)/(b - s)^\omega = C_1, \lim_{s \to b^-} f'(s)/(b - s)^{\omega - 1} = -C_2 \text{ for some } 0 < C_1, C_2 < \infty \text{ and } 0 < \omega < 1,
(f_3) \int_0^b f(s) \, ds > 0 \text{ and } \beta \text{ is the unique number in } (a, b) \text{ such that } \int_0^\beta f(s) \, ds = 0.

Note that (f_2) implies \lim_{s \to \beta} f'(s) = -\infty.

Problem (1.1) has appeared in various models in applied mathematics, including population genetics and chemical reactor theory (see e.g. [16] and the references therein) and has been studied by many authors (see for example [1], [7]–[9], [19], [20], [5], [16]). Notice that if we set \( \varepsilon = 1/\lambda \), (1.1) can be viewed as a singularly perturbed problem. The case that \( f'(0) = 0 \) can be viewed as a border line case of singular perturbation problems (see [10]). Benci and Cerami [2] raised the question what happens for the structure of positive solutions in this borderline case, also called the zero mass case.

In paper [4], Clement and Sweers obtained that (1.1) has a unique positive solution \( u_\lambda \) with \( \max u_\lambda \to b \) as \( \lambda \to \infty \) and \( u_\lambda \to b \) in compact sets of \( \Omega \) as \( \lambda \to \infty \) if \( f \) satisfies (f_1) and (f_3) with \( f'(0) < 0 \) and \( -\infty < f'(b) < 0 \). Notice that under such conditions on \( f \), the fact that max \( u_\lambda < b \) can be obtained by the maximum principle. In a recent paper [6], Dancer studied (1.1) in a domain \( D \) of type \( R_N \) with \( f'(0) = 0 \) and \( -\infty < f'(b) < 0 \). He showed that when \( f \) satisfies (f_1), (f_3) and some extra conditions, (1.1) has exactly 2 positive solutions \( \pi_\lambda \), \( u_\lambda \) with \( 0 < \|u\|_\infty < b \) for all large positive \( \lambda \): \( \pi_\lambda \) is a large solution, i.e. \( \pi_\lambda \to b \) uniformly on compact subsets of \( D \) as \( \lambda \to \infty \); \( u_\lambda \) is a small solution, i.e. \( \|u_\lambda\|_\infty < b \) and \( v_\lambda(y) := \frac{u_\lambda}{\lambda} \lambda^{-1/2} \to V \) as \( \lambda \to \infty \) in \( C^2_{\text{loc}}(\mathbb{R}^N) \), where \( V = V(y) \) is the unique positive (radial) solution of
\[
\Delta V + f(V) = 0 \quad \text{in } \mathbb{R}^N, \quad V'(|y|) < 0, \quad V(y) \to 0 \quad \text{as } |y| \to \infty.
\]

In this paper we shall show that when \( f \) satisfies (f_1)–(f_3), (1.1) has a unique large positive solution \( u_\lambda \) for \( \lambda \) sufficiently large. By a large solution \( u_\lambda \) of (1.1), we mean that \( u_\lambda \in C^2(\Omega) \) and that there exists an open set \( \Omega_0 \subset \Omega \) independent of \( \lambda \) with meas \( (\Omega_0) > 0 \) such that
\[
\lim_{\lambda \to \infty} \inf_{x \in \Omega_0} u_\lambda(x) > a.
\]

Since \( \lim_{s \to b^-} f'(s) = -\infty \) (see (f_2)), the large positive solution \( u_\lambda \) of (1.1) may have flat core, i.e.
\[\Gamma_\lambda = \{x \in \Omega \mid u_\lambda(x) = b\} \neq \emptyset\]
(see [22]). We shall prove that under the assumptions (f_1)–(f_3), there exists flat core for the large positive solution \( u_\lambda \) of (1.1) when \( \lambda \) is sufficiently large. We also give the exact estimate of the flat core of \( u_\lambda \).

The flat core properties of the positive solutions of elliptic equations similar to (1.1) have also been discussed by several authors, see for example [17], [18], [21]. In [21], Sweers obtained a positive solution of (1.1) which has flat core
for \( \lambda \) sufficiently large. In a recent paper [18], Melian and Lis studied the flat core properties of the positive solutions of some elliptic problems involving \( p \)-Laplacian, but with simpler nonlinearity. It was known from [18] that under the conditions: \( f \in C^2(0, b) \) with \( f(s) > 0 \) in \((0, b)\); \( \lim_{s \to 0^+} f(s)/s = m > 0 \); \( f(b) = 0 \); \( f(s)/s \) is decreasing in \((0, b)\) and \( \lim_{s \to b^-} f(s)/(b - s)^\omega = C > 0 \) with \( 0 < \omega < 1 \), (1.1) has a unique positive solution for \( \lambda \) sufficiently large and flat core of this solution exists.

Since the large positive solution \( u_\lambda \) of (1.1) has flat core, one will see that the problem studied in this paper becomes more difficult. For example, it is known from [11] that if \( f \) is a Lipschitz continuous function, the positive solutions of (1.1) with \( \Omega \) being an \( N \)-ball is radially symmetric. We shall see in Section 5 below that such result is also true for the large positive solutions of (1.1), but \( f(s) \) in our case is not Lipschitz for \( s \) near \( s = b \). Moreover, we shall see later that it is difficult to establish the sweeping out results when we use sub- and supersolution argument because of the flat core of \( u_\lambda \).

2. Existence of large positive solutions

In this Section we study the existence of large positive solutions of (1.1). The results in this section are strongly related to [4], but we need to overcome a difficulty arising from the singularity of \( f'(s) \) at \( s = b \). To deal with the case that \( f'(b) = -\infty \), we modify \( f \) in the following way. For any \( \varepsilon > 0 \) sufficiently small, define \( f_\varepsilon(s) = f(s) - \varepsilon \). Then there exists \( a(\varepsilon) > a \) and \( b(\varepsilon) < b \) such that \( f_\varepsilon(a_\varepsilon) = 0 \), \( f_\varepsilon(b_\varepsilon) = 0 \) and \( f_\varepsilon(0) = -\varepsilon \). (It is easy to see that \( a(\varepsilon) \to a \) and \( b(\varepsilon) \to b \) as \( \varepsilon \to 0 \) and \( f_\varepsilon \in C^1([0, b(\varepsilon)]) \).) We make an extension \( F_\varepsilon \) of \( f_\varepsilon \):

\[
\begin{align*}
F_\varepsilon(s) & \text{ is bounded,} \\
F_\varepsilon(s) & \equiv 0 \quad \text{for } s \in (-\infty, -1], \\
F_\varepsilon & \in C^1(-\infty, b(\varepsilon)) \text{ and } F_\varepsilon < 0 \quad \text{for } s \in (-1, 0), \\
F_\varepsilon & \to 0 \quad \text{uniformly for } s \in [-1, 0] \text{ as } \varepsilon \to 0, \\
\lim_{s \to 0^-} F_\varepsilon'(s) & = 0, \lim_{s \to -1^+} F_\varepsilon'(s) = 0, \\
F_\varepsilon & \equiv f_\varepsilon \quad \text{for } s \in [0, b(\varepsilon)], \\
F_\varepsilon(s) & < 0 \quad \text{for } s \in [b(\varepsilon), \infty), \\
\int_{-1}^{b(\varepsilon)} F_\varepsilon(s) \, ds & > 0.
\end{align*}
\]

**Lemma 2.1.** Let \( F_\varepsilon \) be defined as above. Then there exists \( \mu_0 > 0 \) independent of \( \varepsilon \) such that for \( \mu > \mu_0 \), there exists \( v_{\varepsilon, \mu} \in C^1(\mathbb{R}^N) \), radially symmetric, which satisfies

\[
(2.1) \quad -\Delta v = \mu F_\varepsilon(v) \quad \text{in } \mathbb{R}^N, \quad v(1) = -1.
\]
Moreover, $\max v_{\varepsilon, \mu} < b(\varepsilon)$ and $\max v_{\varepsilon, \mu} \to b(\varepsilon)$ as $\mu \to \infty$.

**Proof.** Define $\tilde{f}_\varepsilon(s) = F_\varepsilon(s - 1)$. We have that $\tilde{f}_\varepsilon$ satisfies $\tilde{f}_\varepsilon(0) = 0$ and $\tilde{f}_\varepsilon'(0) = 0$. Moreover, $\tilde{f}_\varepsilon$ is bounded in $[0, b(\varepsilon) + 1]$. Since $\tilde{f}_\varepsilon(b(\varepsilon) + 1) = F_\varepsilon(b(\varepsilon)) = 0$, without loss of generality, we assume $\tilde{f}_\varepsilon(s) \equiv 0$ for $s \in [b(\varepsilon) + 1, \infty)$. Now we consider the problem

$$
\begin{align*}
-\Delta u &= \mu \tilde{f}_\varepsilon(u) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B,
\end{align*}
$$

where $B$ is the unit ball in $\mathbb{R}^N$. By the arguments similar to that in [4], we can obtain a global minimizer $y_{\varepsilon, \mu} \in H^1_0(B)$ to the functional

$$
I_\mu(u) = \frac{1}{2} \int_B |\nabla u|^2 - \mu \int_B \tilde{F}_\varepsilon(u),
$$

where $\tilde{F}_\varepsilon(s) = \int_0^s \tilde{f}_\varepsilon(\xi) \, d\xi$. It is known from the regularity of $-\Delta$ and the maximum principle that $y_{\varepsilon, \mu} \in C^2_0(B)$ which is a positive solution of (2.2). By [11], we know that $y_{\varepsilon, \mu}$ is radially symmetric and $y'_{\varepsilon, \mu} < 0$ for $r \in (0, 1]$. Moreover, the fact that $\max y_{\varepsilon, \mu} \to b(\varepsilon) + 1$ can also be obtained from the argument similar to that in [4].

Set $v_{\varepsilon, \mu}(r) = y_{\varepsilon, \mu}(r) - 1$ for $r \in [0, 1]$ and

$$
v_{\varepsilon, \mu}(r) = \begin{cases}
-1 + \frac{1}{2 - N}(r^2 - N - 1)y'_{\varepsilon, \mu}(1) & \text{for } r \in (1, \infty) \text{ if } N > 2, \\
-1 + y'_{\varepsilon, \mu}(1) \log r & \text{for } r \in (1, \infty) \text{ if } N = 2.
\end{cases}
$$

Since $F_\varepsilon = 0$ on $(-\infty, -1]$, one verifies that $v_{\varepsilon, \mu}$ is the required function. This completes the proof. \hfill \Box

**Remark.** By the well-known result of [11], we know that all the positive solutions of (2.2) are radially symmetric for $\varepsilon > 0$. But we do not know whether such conclusion is true or not when $\varepsilon = 0$ since $\tilde{f}_0$ is not Lipschitz continuous near $s = b + 1$.

**Corollary 2.2.** Let $(\mu, v_{\varepsilon, \mu})$ be as in Lemma 2.1, and let $\alpha_{\varepsilon, \mu} \in (0, 1)$ be the unique zero of $v_{\varepsilon, \mu}$. Then for $y \in \Omega$ and $\lambda > \mu \cdot \alpha^{2}_{\varepsilon, \mu} \cdot d(y, \partial \Omega)^{-2}$,

$$
w_{\mu, \varepsilon}(\lambda, y; x) := v_{\varepsilon, \mu}(\lambda/\mu^{1/2}, (x - y)), \quad x \in \Omega
$$

is a subsolution of the problem

$$
-\Delta u = \lambda F_\varepsilon(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
$$

**Remark.** We can show that for any $\varepsilon > 0$ sufficiently small, $\alpha_{\varepsilon, \mu} \to 1$ as $\mu \to \infty$. In fact, for any sequence $\{\mu_n\}$ with $\mu_n \to \infty$ as $n \to \infty$, by the arguments similar to that in the proof of Lemma 2.1, we have that $\tau_n :=
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\[ v_{\varepsilon,\mu_n}(0) = \max_B v_{\varepsilon,\mu_n} \to b(\varepsilon) \text{ as } n \to \infty. \]

Defining \( y = \mu_n^{1/2} r \) and \( \tilde{v}_{\varepsilon,\mu_n}(y) = v_{\varepsilon,\mu_n}(r) \), we have that \( \tilde{v}_{\varepsilon,\mu_n} \) satisfies

\[
\tilde{v}_{\varepsilon,\mu_n}'' + \frac{N - 1}{y} \tilde{v}_{\varepsilon,\mu_n}' + F_{\varepsilon}(\tilde{v}_{\varepsilon,\mu_n}) = 0, \quad \tilde{v}_{\varepsilon,\mu_n}(0) = 0, \quad \tilde{v}_{\varepsilon,\mu_n}(b) = \tau_n.
\]

Since \( \tau_n \to b(\varepsilon) \) as \( n \to \infty \) and \( b(\varepsilon) \) is the unique solution of the problem

\[
u'' + \frac{N - 1}{y} u' + F_{\varepsilon}(u) = 0 \quad \text{in } (0, \infty), \quad u'(0) = 0, \quad u(0) = b(\varepsilon),
\]

one obtains from the theory of ordinary differential equations that

\[
\tilde{v}_{\varepsilon,\mu_n} \to b(\varepsilon) \text{ in } C^1_{\text{loc}}(0, \infty) \text{ as } n \to \infty.
\]

(We can choose subsequences if necessary.) This implies that \( \alpha_{\varepsilon,\mu_n} \to 1 \text{ as } n \to \infty. \)

Let \( x^* \in \Omega \). We define \( \lambda^* := \mu d(x^*, \partial \Omega)^{-2} > \mu \alpha_{\mu,\varepsilon} d(x^*, \partial \Omega)^{-2} \) and \( z_\lambda = w(\lambda, x^*) \), where \( \mu, \alpha \) are as defined in Corollary 2.2. Note that \( \lambda^* \) is independent of \( \varepsilon. \)

**Theorem 2.3.** Let \( f \) satisfy \((f_1)\)–\((f_3)\). Then there exists \( \lambda_0 > 0 \) such that

for \( \lambda > \lambda_0 \), \((1.1)\) has at least one large positive solution \( u_\lambda \) such that

\[
\max u_\lambda \to b \quad \text{as } \lambda \to \infty.
\]

Moreover, \( u_\lambda \to b \) on compact sets of \( \Omega \) as \( \lambda \to \infty. \)

The proof of this theorem is similar to that in [4], but we need to overcome a difficulty arising from the singularity of \( f'(s) \) at \( s = b \). We first present the following lemmas.

**Lemma 2.4.** Let \( F_{\varepsilon} \) be as above. Then

(i) for \( \lambda > \lambda^* \) \((2.4)\) has a solution \( u_{\lambda}^{(\varepsilon)} \in [z_\lambda, b(\varepsilon)] \),

(ii) there exist \( \lambda^{**} > \lambda^* \), \( c > 0 \) and \( \tau \in (a, b(\varepsilon)) \), such that for \( \lambda > \lambda^{**} \) every solution \( u_{\lambda}^{(\varepsilon)} \in [z_\lambda, b(\varepsilon)] \) of \((2.4)\) satisfies

\[
u_{\lambda}^{(\varepsilon)}(x) > \min\{cz^{1/2}d(x, \partial \Omega), \tau\} \quad \text{for all } x \in \Omega.
\]

**Proof.** By Corollary 2.2, for \( \lambda > \lambda^* \) we have that \( z_\lambda \) is a subsolution of \((2.4)\) and \( z_\lambda < b(\varepsilon). \) Since \( b(\varepsilon) \) is a supersolution of \((2.4)\) and there exists \( M_\varepsilon > 0 \) such that \( f_{\varepsilon}(s) + M_\varepsilon s \) is strictly increasing in \((\min \Omega z_\lambda, b(\varepsilon))\), by a monotone method, there exists a minimal solution \( u_{\lambda}^{(\varepsilon)} \in [z_\lambda, b(\varepsilon)] \) of \((2.4)\) for \( \lambda > \lambda^*. \) This completes the proof of the first assertion.
Since $\Omega$ satisfies a uniform interior sphere condition, there exists $\eta_0 > 0$ such that $\Omega = \bigcup\{B(x, \eta) \mid x \in \Omega_\eta\}$ for $\eta \in (0, \eta_0]$, where $\Omega_\eta = \{x \in \Omega \mid d(x, \partial \Omega) > \eta\}$. Set

$$
\lambda^{**} = \max(\lambda^*, \mu \eta_0^{-2}),
$$

(2.7)

$$
c = \mu^{-1/2} \inf \{(r-\tau)^{-1} v(r) \mid r \in [0, \alpha)\},
$$

$$
\tau = v(0),
$$

with $\mu$, $v$ and $\alpha$ as in Corollary 2.2. (Note that $\lambda^{**}$ is independent of $\varepsilon$.)

Let $(\lambda, u_{\varepsilon, \lambda})$ be any solution of (2.4), $\lambda > \lambda^{**}$ and $u_{\varepsilon, \lambda} \in [z_\lambda, b(\varepsilon))$. Since for $\lambda > \lambda^{**}$, $\Omega_{\alpha(\mu/\lambda)^{1/2}}$ is arcwise connected and since $w(\lambda, \cdot)$ is a subsolution for $y \in \Omega_{\alpha(\mu/\lambda)^{1/2}}$ with $w(\lambda, y) < 0$ on $\partial \Omega$, by the sweeping out result (see [5]) we obtain

$$
u_{\varepsilon, \lambda} > w(\lambda, y) \quad \text{in } \Omega \text{ for all } y \in \Omega_{\alpha(\mu/\lambda)^{1/2}}.
$$

Hence, a similar argument to that in [4] implies

$$
u_{\varepsilon, \lambda} > c \lambda^{1/2} d(x, \partial \Omega) \quad \text{for all } x \in \Omega \setminus \Omega_{\alpha(\mu/\lambda)^{1/2}},
$$

(2.8)

$$
u_{\varepsilon, \lambda}(x) > \tau \quad \text{for all } x \in \Omega_{\alpha(\mu/\lambda)^{1/2}},
$$

(2.9)

which completes the proof. □

REMARKS. (1) The sweeping out result as in [5] holds here since there exists $M_\varepsilon > 0$ such that $|F^\varepsilon(s)| \leq M_\varepsilon$ for $s \in [0, b(\varepsilon)]$.

(2) It follows from (2.8)–(2.9) that the minimal solution $u_{\lambda}^{(\varepsilon)} > 0$ for $\lambda > \lambda^{**}$, and max $u_{\lambda}^{(\varepsilon)} \in (a, b(\varepsilon))$ for $\mu$ and $\lambda$ sufficiently large. This implies that $u_{\lambda}^{(\varepsilon)}$ is a positive solution of

$$
-\Delta u = \lambda(f(u) - \varepsilon) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.
$$

(2.10)

(3) We know that the constant $c$ in (2.7) depends upon $\varepsilon$. But we can show that $c \geq c_0/2 > 0$ for any $\varepsilon$ sufficiently small, where $c_0$ is independent of $\varepsilon$. In fact, we know that for any fixed $\mu$ sufficiently large, and any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$, there exists a subsequence (still denoted by $\{\varepsilon_n\}$) such that $v_{\varepsilon_n, \mu} \to v_{0, \mu}$ in $C^1(B)$ and $v_{0, \mu}$ is a positive radial solution of the problem

$$
-\Delta u = \mu F_0(u) \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B,
$$

where $F_0(s) = f(s)$ in $[0, b]$ and $F_0(s) \equiv 0$ in $(-\infty, 0]$. We can choose

$$
c_0 = \mu^{-1/2} \inf \{(1 - r)^{-1} v_{0, \mu}(r) \mid r \in [0, 1]\}.
$$

Note that for $\mu$ sufficiently large, max $v_{0, \mu} = b$ may hold.

By the conditions on $f$, we can choose a fixed $\tilde{b} \in (b - \delta, b)$ and $M > 0$ such that $f(s) + Ms$ is increasing for $s \in [0, \tilde{b}]$. Let $\gamma > 1$ be a fixed number. Now,
setting
\[ g(s) = \begin{cases} 
Ms & \text{for } s \in (0, \hat{b}], \\
M\hat{b} + \gamma f(\hat{b}) - \gamma f(s) & \text{for } s \in (\hat{b}, b].
\end{cases} \]
we can easily show that \( g \) is continuous in \([0, b]\) and \( g \) is increasing in \([0, b]. \) Moreover, we also know that \( f(s) + g(s) \) is increasing in \([0, b]. \) (Note that \( f'(s) < 0 \) for \( s \in (\hat{b}, b). \))

**Lemma 2.5 (Maximum Principle).** If \( u_1, u_2 \in H^1_0(\Omega) \cap C^0(\overline{\Omega}) \) such that, for any \( \phi \in H^1_0(\Omega), \phi \geq 0, \)
\begin{equation}
\int_\Omega \nabla u_2 \cdot \nabla \phi + \int_\Omega g(u_2)\phi \geq \int_\Omega \nabla u_1 \cdot \nabla \phi + \int_\Omega g(u_1)\phi
\end{equation}
and
\[ u_2 \geq u_1 \quad \text{on } \partial \Omega, \]
then \( u_2 \geq u_1 \) in \( \Omega. \)

**Proof.** Let us choose \( \phi = (u_1 - u_2)^+ \in H^1_0(\Omega) \cap C^0(\overline{\Omega}). \) Then it follows from (2.11) that
\[ 0 \geq \int_\Omega |\nabla (u_1 - u_2)^+|^2 + \int_\Omega [g(u_1) - g(u_2)](u_1 - u_2)^+. \]
This implies that \((u_1 - u_2)^+ = 0\) in \( \Omega \) and thus, \( u_1 \leq u_2 \) in \( \Omega. \)

**Proof of Theorem 2.3.** Since \( u^{(c)}_\lambda \) is a positive subsolution of (1.1) with \( \max u^{(c)}_\lambda < b, \) \( b \) is a supersolution of (1.1) and \( f(s) + g(s) \) is increasing in \((0, b), \)
by a monotone argument as in Theorem 2.4 in [3] that there exist a minimal positive solution \( u_\lambda \in C^2(\Omega) \) of the problem (1.1) such that
\begin{equation}
\tag{2.12}
 u^{(c)}_\lambda \leq u_\lambda(x) \leq b \quad \text{in } \Omega.
\end{equation}
Here we use Lemma 2.5. It follows from Lemma 2.4 that for any \( \varepsilon > 0 \) and \( \lambda > \lambda^*, \)
\[ u_\lambda > \min\{c\lambda^{1/2}d(x, \partial \Omega), \tau\} \quad \text{for all } x \in \Omega. \]
Since \( \tau \to b \) as \( \varepsilon \to 0 \) and \( \mu \to \infty, \) we have from (2.9) (we may choose \( \lambda = \mu^2 \)) that for \( \lambda \) sufficiently large that
\begin{equation}
\tag{2.13}
 u_\lambda > \tau \quad \text{for all } x \in \Omega_{\alpha(\mu/\lambda)^{1/2}}.
\end{equation}
Since \( \lim_{\lambda \to \infty} \mu/\lambda = 0 \) (here we use \( \lambda = \mu^2 \)) and \( \tau \to b \) as \( \lambda \to \infty \) and \( \varepsilon \to 0 \)
(noticing that \( u_\lambda \) is independent of \( \varepsilon \) and \( \alpha > \tilde{\alpha} > 0), \) we have from (2.13) that
\[ u_\lambda \to b \quad \text{on compact sets of } \Omega \text{ as } \lambda \to \infty. \]
We easily know that \( u_\lambda \) is a large positive solution of (1.1) according to the definition of large positive solutions. This completes the proof. \( \square \)
Remarks. (1) Note that the monotone argument was used to the weak solutions in [3], but the arguments can be applied for our case to obtain the solution in $C^2(\Omega)$ since $g$ is bounded.

(2) We can obtain that if $u_\lambda$ is a positive solution of (1.1) with $u_\lambda \in [z_\lambda, b]$, then $u_\lambda \geq u_\lambda^{(e)}$ in $\Omega$. In fact, $u_\lambda^{(e)}$ can be obtained by the similar monotone argument to that in the proof of Theorem 2.3 with $f(s)$ replaced by $F_\epsilon$. Note that by modifying $g(s) = Ms$ for $s \leq 0$ we can show that $F_\epsilon(s) + g(s)$ is increasing in $(-\infty, b(\epsilon))$. Thus, the monotone argument can be used for the problem (2.4). On the other hand, by modifying $f(s) \equiv 0$ for $s < 0$ and $g$ as above for $s < 0$, we also know that $f(s) + g(s)$ is increasing for $s \in (-\infty, b)$. Since $z_\lambda$ is a subsolution for both (2.4) and (1.1), we can use the monotone argument starting from $z_\lambda$, i.e.

\[-\Delta \zeta_n^{(1)} + \lambda g(\zeta_n^{(1)}) = \lambda(F + g)(\zeta_n^{(1)}) \quad \text{in } \Omega, \quad \zeta_n^{(1)} = 0 \quad \text{on } \Omega\]

with $\zeta_0^{(1)} = z_\lambda$ and

\[-\Delta \zeta_n^{(2)} + \lambda g(\zeta_n^{(2)}) = \lambda(f + g)(\zeta_n^{(2)}) \quad \text{in } \Omega, \quad \zeta_n^{(2)} = 0 \quad \text{on } \Omega\]

with $\zeta_0^{(2)} = z_\lambda$. Since $f(s) > F_\epsilon(s)$ on $(0, b]$, then it follows from the maximum principle in Lemma 2.5 that $\zeta_n^{(2)} \geq \zeta_n^{(1)}$ in $\Omega$. Since $u_\lambda^{(e)}$ is the minimal solution of (2.4) in $[z_\lambda, b(\epsilon)]$, thus, $u_\lambda \geq u_\lambda^{(e)}$ in $\Omega$.

3. Asymptotic behaviour of large positive solutions of (1.1) when $\lambda$ is large

In this Section we shall study the asymptotic behaviour of the positive solutions of (1.1) when $\lambda$ is large. We first consider the following ordinary differential equations

\[
(3.1) \quad -y'' = f(y) - \epsilon, \quad y(0) = 0, \quad y(\infty) = b(\epsilon),
\]

\[
(3.2) \quad -y'' = f(y), \quad y(0) = 0, \quad y(\infty) = b.
\]

By the first integrals of the equations, we have that each of (3.1) and (3.2) has a unique positive solution $y_\epsilon(t)$ and $y(t)$ respectively which satisfies $(y_\epsilon)'(t) > 0$ for $t \in [0, \infty)$ and $y'(t) \geq 0$ for $t \in [0, \infty)$ (see [5], [15]). To show $(y_\epsilon)'(t) > 0$ for $t \in [0, \infty)$, we use the fact that $|f'(s)|$ is bounded for $s \in [0, b(\epsilon)]$. Now we show that there exists $t_0 > 0$ such that $y'(t) > 0$ for $t \in (0, t_0)$ and $y'(t_0) = 0$ and $y(t) \equiv b$ for $t \in [t_0, \infty)$. In fact, from the first integral of (3.2), we have

\[|y'(t)|^2 + 2F(y(t)) \equiv C, \quad t \in (0, \infty),\]

where $F(s) = \int_0^s f(\xi) d\xi$. Therefore,

\[|y'(t)|^2 = 2(F(b) - F(y)).\]
Since \( F(b) > F(s) \) for \( 0 < s < b \), we have
\[
\int_0^{y(t)} (F(b) - F(s))^{-1/2} ds = 2^{1/2} t.
\]
Since \( f(s) \sim (b - s)^\omega \) for \( s \) near \( b \), \( 0 < \omega < 1 \), we know that \( F(b) - F(s) \geq \rho(b - s)^{1+\omega} \), where \( \rho > 0 \). Thus, \( \int_0^b (F(b) - F(s))^{-1/2} ds = A < \infty \). Let
\[
t_0 = 2^{-1/2} A.
\]
Then the first integral of (3.2) implies that \( y'(t) > 0 \) for \( t \in (0, t_0) \), \( y'(t_0) = 0 \) and \( y \equiv b \) in \( [t_0, \infty) \). On the other hand, by the first integrals of the equations (3.1)–(3.2), we also know that
\[
(y_\varepsilon)'(0) = \left( 2 \int_0^{b(\varepsilon)} [f(s) - \varepsilon] ds \right)^{1/2},
\]
\[
y'(0) = \left( 2 \int_0^b f(s) ds \right)^{1/2}.
\]
Thus \( (y_\varepsilon)'(0) \to y'(0) \) as \( \varepsilon \to 0 \). Therefore,
\[
y_\varepsilon \to y \quad \text{in } C_{\text{loc}}^1(0, \infty).
\]

If \( x \in \Omega \) and \( x \) is near \( \partial \Omega \), \( x \) can be uniquely written in the form \( x = s + t\nu(s) \), where \( s \in \partial \Omega \), \( \nu(s) \) denotes the inward unit normal vector to \( \partial \Omega \) at \( s \), and \( t \) is small and positive. We will make frequent use of these coordinates. If \( \lambda > 0 \), define \( \eta_\lambda(x) = y(\lambda^{1/2}t) \) if \( x \) is near \( \partial \Omega \) and \( \eta_\lambda(x) = b \) otherwise.

**Proposition 3.1.** Let \( f \) satisfy (f_1)–(f_3). For any \( \theta > 0 \) sufficiently small, there is \( \lambda = \lambda(\theta) > \lambda^{**} \) such that if \( \lambda > \lambda \) and \( u_\lambda \in [z_\lambda, b] \) is a positive solution of (1.1), then
\[
(1 - \theta)\eta_\lambda \leq u_\lambda \leq (1 + \theta)\eta_\lambda.
\]

To prove this result, we first obtain the following sweeping out result.

**Proposition 3.2 (Sweeping Out Result).** Let \( f \) satisfy (f_1)–(f_3),
\[
u \in H_0^1(\Omega) \cap C^0(\overline{\Omega})
\]
with \( \max u \leq b \) be a solution of (1.1) and let \( A = \{v_t \mid t \in [0, 1]\} \) be a family of subsolutions of (1.1) satisfying \( v_t \in H_0^1(\Omega) \cap C^0(\overline{\Omega}) \), \( \max v_t \leq \tilde{b} < b \) and \( v_t \leq 0 \) on \( \partial \Omega \) for all \( t \in [0, 1] \). If
1. \( t \to (v_t - v_0) \in C^0(\overline{\Omega}) \) is continuous with respect to the \( \| \cdot \|_0 \)-norm,
2. \( u \geq v_0 \) in \( \overline{\Omega} \), and
3. \( u \not\equiv v_t \), for all \( t \in [0, 1] \) and \( x \) near \( \partial \Omega \), then \( u \geq v_t \) in \( \overline{\Omega} \) for \( t \in [0, 1] \).
Proof. Define $G = \{ x \in \Omega \mid u(x) = \bar{b} \}$. We know that $G$ depends upon $\lambda$, we shall omit the subscript $\lambda$ here and below for simplicity. In the following, we only consider the case $G \neq \emptyset$. The case $G = \emptyset$ can be studied similarly.

Set $E = \{ t \in [0, 1] \mid u \geq v_t \text{ in } \Omega \}$. By (ii), $E$ is nonempty. Moreover, $E$ is closed. We easily know that $G \subset \subset \Omega$ is closed. Since $\max v_t \leq \tilde{b} < b$, then, for $0 < \tau < b - \tilde{b}$, we can choose a neighbourhood $O$ of $G$ such that $G \subset O \subset \subset \Omega$ and $u \geq v_t + \tau$ in $\Omega \setminus O$ for any $t \in [0, 1]$. Moreover, there exists $M > 0$ sufficiently large such that for any $t \in E$,

$$f(u) + Mu \geq f(v_t) + M v_t \quad \text{for } x \in \Omega \setminus \overline{O}.$$  

Thus, for $t \in E$, we can easily show that $u > v_t$ in $\Omega \setminus \overline{O}$. In fact, suppose that there exists $x_0 \in \Omega \setminus \overline{O}$ and $u - v_t$ vanishes at $x = x_0$, then $u - v_t$ attains its minimum at $x = x_0$ in $\Omega \setminus \overline{O}$. On the other hand,

$$-\Delta (u - v_t) + \lambda M (u - v_t) \geq 0 \quad \text{in } \Omega \setminus \overline{O}.$$  

The Hopf’s maximum principle implies that $u \equiv v_t$ in $\Omega \setminus \overline{O}$. This contradicts the assumption (iii).

Now applying the strong maximum principle, we have

$$\frac{\partial (u - v_t)}{\partial \nu} < 0 \quad \text{on } \partial \Omega.$$  

Thus, $u - v_t \geq c \phi$ on $\Omega \setminus \overline{O}$ (see [14]), where $c > 0$ and $\phi$ is the unique positive solution of the problem

$$-\Delta \phi = 1 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial \Omega.$$  

This and the arguments above imply that $u - v_t \geq c_1 \phi$ for $x \in \Omega$, where $c_1 > 0$. By (i), we have that $E$ is an open set. Thus, $E = [0, 1]$. \[\square\]

Proof of Proposition 3.1. To prove this proposition, we first construct sub- and supersolutions of (1.1), then obtain (3.4) by sweeping out results. We only consider the case that $u_\lambda$ has flat core, i.e. $G_\lambda = \{ x \in \Omega \mid u_\lambda(x) = \bar{b} \} \neq \emptyset$ in the proof. If flat core of $u_\lambda$ does not exist, the proof is similar but is simpler. (The key step in the proof below is to establish the sweeping out results. If $\max u_\lambda < \bar{b}$ for all $\lambda$ large, for a fixed $\lambda$ large, we can choose $M_\lambda > 0$ such that $|f'(s)| \leq M_\lambda$ for $s \in (0, \max_\Omega u_\lambda]$. Assuming $G_\lambda = \emptyset$ in the proof below, we obtain our conclusion in this case by the similar arguments.)

Near $\partial \Omega$, we use the $s, t$ coordinates. In these variables, 

$$\Delta u_\lambda = \frac{\partial^2 u_\lambda}{\partial t^2} + b(s, t) \frac{\partial u_\lambda}{\partial t} + \text{terms involving } s \text{ derivatives}.$$
If \( \pi_\varepsilon < (y_\varepsilon)'(0) \) but is close, using the first integral of (3.1) we easily prove that the solution \( \tilde{y}_\varepsilon \) of (3.1) with the initial conditions:

\[
\tilde{y}_\varepsilon(0) = 0, \quad \tilde{y}'_\varepsilon(0) = \pi_\varepsilon,
\]

first increases to a number near \( b(\varepsilon) \) but less than \( b(\varepsilon) \), and then decreases to zero (see [5]). Hence there is \( \tilde{l}_\varepsilon \) near \( b(\varepsilon) \) and \( \tilde{l}_\varepsilon \) sufficiently large such that

\[
\tilde{y}_\varepsilon(\tilde{l}_\varepsilon) = \tilde{t}_\varepsilon, \quad \tilde{y}'_\varepsilon(\tilde{l}_\varepsilon) = 0.
\]

We know that there exists \( M_\varepsilon > 0 \) sufficiently large such that \( h_\varepsilon(s) := f(s) - \varepsilon + M_\varepsilon s \) is strictly increasing for \( s \in (0, b(\varepsilon)] \) (since \( b(\varepsilon) < b \)). Hence if \( \tilde{\mu} \) is close to 1 and \( \beta \) is small, the solution \( \Gamma_\varepsilon \) of

\[
-x'' - \beta x' + M_\varepsilon x = \tilde{\mu} h_\varepsilon(x(t)), \quad x(0) = 0, \quad x'(0) = \pi_\varepsilon
\]

increases until \( \tilde{l}_\varepsilon \), where \( \Gamma_\varepsilon(\tilde{l}_\varepsilon) \) is close to \( b(\varepsilon) \) but less than \( b(\varepsilon) \). Moreover, \( \tilde{l}_\varepsilon \) is sufficiently large.

Let \( t_0 \) be the number such that \( y'(t) > 0 \) for \( 0 < t < t_0 \) and \( y(t) \equiv b \) for \( t \geq t_0 \), where \( y \) is the unique positive solution of (3.2). We know that \( \tilde{l}_\varepsilon > t_0 \) for any \( \varepsilon > 0 \) sufficiently small and \( \Gamma_\varepsilon(t_0) < b(\varepsilon) \). Define

\[
\tilde{\eta}_\lambda^{(c)}(x) = \begin{cases} 
\Gamma_\varepsilon(\lambda^{1/2} t) & \text{if } x \text{ is close to } \partial \Omega \text{ and } 0 \leq t \leq \lambda^{-1/2} t_0, \\
\Gamma_\varepsilon(t_0) & \text{otherwise},
\end{cases}
\]

where \( x = s + t \nu(s) \) if \( x \) is near \( \partial \Omega \). (Thus \( \tilde{\eta}_\lambda^{(c)} \) is constant except near \( \partial \Omega \).) We know that \( \tilde{\eta}_\lambda^{(c)} \) is in \( C^1(\Omega) \) except the points \( x = s + t \nu(s) \) with \( t = \lambda^{-1/2} t_0 \).

Suppose we can show that, for \( \lambda \) large and \( u_\lambda \) is a positive solution of (1.1), then

\[
u_{\lambda} \geq \tilde{\eta}_\lambda^{(c)} \quad \text{for all } \varepsilon > 0 \text{ sufficiently small.}
\]

Since \( \Gamma_\varepsilon \) is close to \( y_\varepsilon \) on compact intervals if \( \pi_\varepsilon \) is near \( (y_\varepsilon)'(0) \); \( \tilde{\mu} \) is near 1 and \( \beta \) is small, \( u_\lambda \geq (1 - \theta/2) \tilde{\eta}_\lambda^{(c)} \), where

\[
\tilde{\eta}_\lambda^{(c)}(x) = \begin{cases} 
y_\varepsilon(\lambda^{1/2} t) & \text{if } x \text{ is close to } \partial \Omega \text{ and } 0 \leq t \leq \lambda^{-1/2} t_0, \\
y_\varepsilon(t_0) & \text{otherwise},
\end{cases}
\]

Since \( y_\varepsilon \to y \) in \( C^0([0, t_0]) \), as \( \varepsilon \to 0 \), we have \( \Gamma_\varepsilon^{(c)} \to \eta_\lambda \) in \( C^0(\Omega) \) as \( \varepsilon \to 0 \), where

\[
\eta_\lambda(x) := \begin{cases} 
y(\lambda^{1/2} t) & \text{if } x \text{ is close to } \partial \Omega \text{ and } 0 \leq t \leq \lambda^{-1/2} t_0, \\
b & \text{otherwise}.
\end{cases}
\]

Thus \( u_\lambda \geq (1 - \theta) \eta_\lambda \). This will prove half of Proposition 3.1.
Now we show (3.6). By choosing $\beta < 0$ and $\widetilde{\mu} < 1$, we have $\tilde{\eta}_j^{(e)}$ is in $C^0(\overline{\Omega}) \cap H^1_0(\Omega)$. Moreover, we can find $e \in (0,1)$ such that $u_\lambda \geq \tilde{\eta}_j^{(e)}$ by Theorem 2.3 and Remark 2 after its proof. Now we deduce that

$$u_\lambda \geq \tilde{\eta}_j^{(e)} \quad \text{for } j \in [e, 1].$$

We first show that $\tilde{\eta}_j^{(e)}$ are subsolutions of (1.1) for $j \in [e, 1]$. We only need to check this for $\Omega_0 := \{x = s + \nu(s) \in \Omega \mid 0 \leq t \leq (\lambda \lambda)^{-1/2}t_0\}$. Since

$$\begin{align*}
-\Delta \tilde{\eta}_j^{(e)} &= -(\tilde{\eta}_j^{(e)})'' - b(s,t)(\tilde{\eta}_j^{(e)})' \\
&\leq \lambda[j(\lambda j)(\lambda j)^{1/2}t)] + j(\lambda j) \lambda j \beta ((\lambda j)^{1/2}t) \\
&+ (j(\lambda j))^{1/2}(\beta(\lambda j)^{1/2} - b(s,t))((\lambda j)^{1/2}t) \\
&\leq \lambda f(\lambda j)^{1/2}t) + j(\lambda j - 1)(f(\lambda j)^{1/2}t) \\
&+ (j(\lambda j))^{1/2}(\beta(\lambda j)^{1/2} - b(s,t))((\lambda j)^{1/2}t) \\
&+ (j(\lambda j))^{1/2}(\beta(\lambda j)^{1/2} - b(s,t))((\lambda j)^{1/2}t).
\end{align*}$$

Define $m_{\varepsilon,j}(s) = f(s) + [j(\lambda j - 1)/J(\lambda j - 1)]M_s$. Since $j(\lambda j - 1)/J(\lambda j - 1) \geq \beta > 0$ for $j \in [e, 1]$, if we choose $M_s$ sufficiently large, we have that $m_{\varepsilon,j}$ is also strictly increasing in $(0,b(\varepsilon))$ for all $j \in [e, 1]$. Thus, $m_{\varepsilon,j}(\lambda j)^{1/2}t)$ is strictly in $\Omega$ and $\tilde{\eta}_j^{(e)}$ is a subsolution of (1.1) for each $j \in [e, 1]$ provided $\tilde{\mu} < 1$ and $\beta > 0$. On the other hand, we also know that max $\tilde{\eta}_j^{(e)} < b(\varepsilon) < b$ in $\Omega$ for any $\varepsilon > 0$ sufficiently small. Then the sweeping out result (see Proposition 3.2) implies that

$$u_\lambda \geq \tilde{\eta}_j^{(e)} \quad \text{for all } j \in [e, 1].$$

Now we construct supersolutions of (1.1) to prove the right hand side of (3.4). If $\bar{\eta}_1 > y'(0)$ and close, it is easy to show from the first integral that the solution $\tilde{\eta}_1$ of (3.2) such that $\tilde{\eta}_1(0) = 0$, $y_1'(0) = \bar{\eta}_1$, increases till it hits $y = b$. Hence if $\tilde{\mu} > 1$ close to 1 and $\beta > 0$ is small, the solution $\bar{\eta}_1$ of

$$-x'' - \beta x' + M_\alpha x = \mu[f(x(t) + M_\alpha x(t)], \quad x(0) = 0, \quad x'(0) = \bar{\eta}_1$$

increases until $\bar{t}_1$, where $\bar{\eta}_1(\bar{t}_1) = b$. Where $M_\alpha > 0$ satisfies $|f'(s)| \leq \mu/(\lambda - 1)M_\alpha$ for $s \in [0, a]$. Clearly $\bar{t}_1 < t_0$ provided that $\mu$ is near 1 and $\beta$ is small. We define

$$\eta_\lambda(x) = \begin{cases} 
\bar{\eta}_1(\lambda^{1/2}t) & \text{if } 0 < t < \lambda^{-1/2}\bar{t}_1, \\
\bar{t}_1 & \text{otherwise}.
\end{cases}$$

Choosing $\tilde{\mu} > 1$ and $\beta > 0$, we shall show that

$$u_\lambda \leq \eta_\lambda \quad \text{for } j \in [1, e]$$

provided it is possible to choose $e > 1$ such that $u_\lambda \leq \eta_\lambda$ for $\lambda$ large.
Define \( E = \{ j \in [1, e] \mid u_\lambda \leq \overline{\eta}_{j, \lambda} \} \). We know that \( e \in E \) and \( E \) is closed. Let
\[
G = \{ x \in \Omega \mid u_\lambda(x) = b \}, \quad F_j = \{ x \in \Omega \mid \overline{\eta}_{j, \lambda}(x) = b \},
\]
and
\[
\Omega_j = \{ x = s + t\nu(s) \in \Omega \mid 0 < t < (j\lambda)^{-1/2} \tau \}.
\]
(Note that \( G, F_j \) and \( \Omega_j \) depend upon \( \lambda \), we omit the subscript \( \lambda \) here and below.) We shall prove that for each \( j_0 \in E \), there is a neighbourhood \( J_0 \) of \( j_0 \) such that
\[
(3.9) \quad G \subset F_j \quad \text{for all} \ j \in J_0.
\]
Notice that \( G \) is closed, we first show that for a sufficiently small neighbourhood \( Q \) of \( G \) such that \( G \subset Q \Subset \Omega \), there exists \( \tau > 0 \) (depending upon \( Q \)) such that
\[
(3.10) \quad \overline{\eta}_{j_0, \lambda} \geq u_\lambda + \tau \quad \text{on} \ \partial Q.
\]
(Note that both \( Q \) and \( \tau \) depend upon \( \lambda \).) On the contrary, there exists \( x_0 \in \partial Q \) such that
\[
\overline{\eta}_{j_0, \lambda}(x_0) = u_\lambda(x_0).
\]
Since \( x_0 \notin G \), it is clear that \( x_0 \notin F_{j_0} \). Setting \( \delta = \text{dist}(x_0, F_{j_0})/2 \), we have \( B_\delta(x_0) \cap F_{j_0} = \emptyset \). Since \( \max_{\overline{B}_\delta(x_0)} \overline{\eta}_{j_0, \lambda} < b \), we can find \( M_{j_0} > 0 \) such that \( \overline{\eta}(s) := f(s) + M_{j_0}s \) is strictly increasing for \( s \in [0, \max_{\overline{B}_\delta(x_0)} \overline{\eta}_{j_0, \lambda}] \). This also implies that \( B_\delta(x_0) \subset \Omega_{j_0} \).

On the other hand, for \( \lambda \) sufficiently large,
\[
(3.11) \quad -\Delta(\overline{\eta}_{j_0, \lambda} - u_\lambda) + \lambda M_{j_0}(\overline{\eta}_{j_0, \lambda} - u_\lambda) = \lambda(\overline{\eta}(\overline{\eta}_{j_0, \lambda} - \overline{\eta}(u_\lambda)) + \lambda(j_0\mu - 1) f(\overline{\eta}_{j_0, \lambda} + j_0(\mu - 1) M_\mu \overline{\eta}_{j_0, \lambda})
\]
\[
+ (\lambda j_0)^{1/2}[\beta(j_0 \lambda)^{1/2} - b(s, t)] \|\eta\|_1 > 0 \quad \text{in} \ B_\delta(x_0)
\]
provided \( \mu > 1 \) and \( \beta > 0 \), where we use the facts that \( j_0(\mu - 1)/(j_0\mu - 1) > (\mu - 1)/\mu \) and that \( |f'(s)| \leq [\mu/\mu - 1]M_\mu \) for \( s \in [0, a] \) (we can easily see that the second term on the right hand side of (3.11) is positive). Therefore, the strong maximum principle implies \( \overline{\eta}_{j_0, \lambda} \equiv u \) in \( B_\delta(x_0) \). This contradicts (3.11).

Since \( \overline{\eta}_{j_0, \lambda} \) is continuous in the norm \( \| \cdot \|_0 \) about \( j \), we have from (3.10) that there exists \( \delta > 0 \) sufficiently small such that
\[
(3.12) \quad \overline{\eta}_{j_0, \lambda} - u_\lambda \geq 0 \quad \text{on} \ \partial Q
\]
for \( j \in J := (j_0 - \delta, j_0 + \delta) \). (If \( j_0 = e \), we choose \( J = (j_0 - \delta, j_0) \).)

Now we show that (3.9) holds for a neighbourhood \( J_0 \) of \( j_0 \) with \( J_0 \subset J \). On the contrary, we have sequences \( \{ j_n \} \subset J \) and \( \{ x_n \} \subset G \) with \( j_n \to j_0 \) as \( n \to \infty \) such that \( \overline{\eta}_{j_n, \lambda}(x_n) < u_\lambda(x_n) \). Define \( m_n = \inf_{x \in \overline{Q}} [\overline{\eta}_{j_n, \lambda}(x) - u_\lambda(x)] \). We have
that \( m_n \) can be achieved at \( \xi_n \in Q \) and \( m_n < 0 \) for \( n \) sufficiently large. Now, setting
\[
H_n = \{ x \in \overline{Q} | \eta_{j_n,\lambda}(x) - u_\lambda(x) \geq 0 \},
\]
we know that \( H_n \) is closed and \( \xi_n \notin H_n \). Let \( \hat{\omega}_n = \text{dist}(\xi_n, H_n) \) and \( B_{\hat{\omega}_n}(\xi_n) \) be the ball with center at \( \xi_n \) and radius \( \hat{\omega}_n \). One easily knows that \( B_{\hat{\omega}_n}(\xi_n) \subset Q \) and
\[
(3.13) \quad \eta_{j_n,\lambda}(x) - u_\lambda(x) < 0 \quad \text{for} \quad x \in B_{\hat{\omega}_n}(\xi_n)
\]
and there is at least one point \( \eta_n \in \partial B_{\hat{\omega}_n}(\xi_n) \), where \( \eta_{j_n,\lambda} - u_\lambda \) vanishes.

On the other hand, choosing \( Q \) such that
\[
(3.14) \quad u_\lambda \geq b - \delta/4 \quad \text{in} \quad \overline{Q},
\]
where \( \delta > 0 \) is as in (f1), we have
\[
(3.15) \quad \eta_{j_n,\lambda} \geq u_\lambda \geq b - \delta/4 \quad \text{in} \quad \overline{Q}.
\]
The continuity on the \( C^0 \)-norm of \( \eta_{j,\lambda} \) about \( j \) implies that for \( n \) sufficiently large,
\[
(3.16) \quad \eta_{j_n,\lambda} \geq b - \delta/2 \quad \text{on} \quad \overline{Q}.
\]
Thus, for \( n \) sufficiently large,
\[
(3.17) \quad \eta_{j_n,\lambda} \geq b - \delta/2 \quad \text{on} \quad B_{\hat{\omega}_n}(\xi_n).
\]
Therefore, for \( x \in B_{\hat{\omega}_n}(\xi_n) \),
\[
-\Delta(\eta_{j_n,\lambda} - u_\lambda) = \lambda(j_n\hat{\mu}f(\eta_{j_n,\lambda}) - f(u_\lambda)) \\
+ (\lambda j_n)^{1/2}[\beta(\lambda j_n)^{1/2} - b(s, t)]y_1 + \lambda j_n(\hat{\mu} - 1)M_0 \eta_{j_n,\lambda}.
\]
Since \( f'(s) < 0 \) for \( s \in (b - \delta, b) \), we have that \( f(\eta_{j_n,\lambda}) \geq f(u_\lambda) \) in \( B_{\hat{\omega}_n}(\xi_n) \). Therefore,
\[
(3.18) \quad -\Delta(\eta_{j_n,\lambda} - u_\lambda) \geq 0 \quad \text{on} \quad B_{\hat{\omega}_n}(\xi_n)
\]
provided \( \beta > 0, \hat{\mu} > 1 \) and \( \lambda \) sufficiently large. It follows from (3.18) and the Hopf’s maximum principle that
\[
(3.19) \quad \eta_{j_n,\lambda} - u_\lambda \equiv m_n < 0 \quad \text{in} \quad B_{\hat{\omega}_n}(\xi_n).
\]
But (3.19) contradicts the fact that \( \eta_{j_n,\lambda} - u_\lambda \) has a zero point on \( \partial B_{\hat{\omega}_n}(\xi_n) \). This shows (3.9).
Now we show that there exists a neighbourhood $J_1$ of $j_0$ in $J_0$ such that $J_1 \subset E$. Since $j_0 \in E$ and $u_\lambda \leq \eta_{j_0 \lambda} \in \Omega$, we can choose a small neighbourhood $Q_1$ of $F_{j_0}$ such that $G \subset F_{j_0} \subset \subset Q_1$ and
\begin{align}
(3.20) & \quad u_\lambda \leq \eta_{j_0 \lambda} \quad \text{in } Q_1, \\
(3.21) & \quad u_\lambda < \eta_{j_0 \lambda} \quad \text{on } \partial Q_1.
\end{align}

By the property of $F_j$; the fact that $G \subset F_j$ for all $j \in J_0$ and the continuity of $\eta_{j,\lambda}$ in the $C^0$-norm about $j$, we have that there exists a neighbourhood $J_1$ of $j_0$ in $J_0$ such that $G \subset F_j \subset \subset Q_1$ for all $j \in J_1$ and (3.20)–(3.21) hold for all $j \in J_1$. The existence of $J_1$ can be obtained by the arguments similar to that in the proof of (3.9). Without loss of generality, we assume $Q \subset Q_1$.

Now we consider the domain $\Omega^1 := \Omega \setminus \overline{Q_1}$. Since $\Omega^1 \subset \Omega_{j_0}$, we know that $\max u_\lambda \leq \max \eta_{j_0 \lambda} < b \in \overline{\Omega^1}$. Thus, assuming that for $M_{j_0} > 0$ and $\eta(s)$ as above, we have
$$
\eta(\eta_{j_0 \lambda}) - \eta(u_\lambda) \geq 0 \quad \text{in } \Omega^1.
$$

Therefore,
\begin{align*}
-\Delta(\eta_{j_0 \lambda} - u_\lambda) + \lambda M_{j_0} (\eta_{j_0 \lambda} - u_\lambda) \\
= \lambda (\eta(\eta_{j_0 \lambda}) - \eta(u_\lambda)) + \lambda (j_0 \Delta - 1) \left[ f(\eta_{j_0 \lambda}) + \frac{j_0 (\mu - 1)}{j_0 \Delta - 1} M_{j_0} \eta_{j_0 \lambda} \right] \\
+ (\lambda j_0)^{1/2} \left[ \beta(j_0\lambda)^{1/2} - b(s,t) \eta_{j_0 \lambda} \right] \geq 0 \quad \text{in } \Omega^1.
\end{align*}

In fact, since $j(\mu - 1)/(j \Delta - 1) \geq (\mu - 1)/\Delta$ for $j \geq 1$, then
$$
f(\eta_{j_0 \lambda}) + \frac{j_0 (\mu - 1)}{j_0 \Delta - 1} M_{j_0} \eta_{j_0 \lambda} \geq 0$$
in $\Omega^1$. The arguments similar to that in the proof of Proposition 3.2 imply that there exists $c > 0$ such that
\begin{align}
(3.22) & \quad \eta_{j_0 \lambda} - u_\lambda \geq c \phi \quad \text{in } \overline{\Omega^1},
\end{align}

where $\phi$ is as that in the proof of Proposition 3.2. The continuity of $\eta_{j,\lambda}$ in the $C^0$-norm about $j$ implies that
\begin{align}
(3.23) & \quad u_\lambda \leq \eta_{j_0 \lambda} \quad \text{in } \Omega^1
\end{align}
for all $j \in J_2$, where $J_2$ is a neighbourhood of $j_0$ in $J_1$. (3.23) and the claim immediately after (3.20)–(3.21) above give the fact that $J_2 \subset E$. This implies that $E = [1, e]$.

Now we show that it is possible to choose $e > 1$ such that $u_\lambda \leq \eta_{e, \lambda}$ for $\lambda$ large and all positive solutions $u_\lambda \in [z_\lambda, b]$ of (1.1). It is easy to see that this reduces to showing that there is $K > 0$ such that $u_\lambda(x) \leq K \lambda^{1/2} t$ if $u_\lambda$ is a positive solution of (1.1), $x$ is near $\partial \Omega$ and $\lambda$ is large. Obviously, it suffices to prove the result for
t ≤ K_1/\lambda^{-1/2} (K_1 > 0). Now for arbitrary x_0 ∈ ∂Ω, letting X = λ^{1/2}(x - x_0) and  \tilde{u}_\lambda(X) = u_\lambda(x), then

\[-\Delta \tilde{u}_\lambda = f(\tilde{u}_\lambda) \quad \text{in} \; \Omega_\lambda, \quad \tilde{u}_\lambda = 0 \quad \text{on} \; \partial\Omega_\lambda,\]

where Ω_\lambda = \{X | \lambda^{-1/2}X + x_0 ∈ Ω\}. By a blow up argument as in [5], the stretching only flattens the boundary as \lambda → ∞. Since 0 ∈ ∂Ω and \|\tilde{u}\|_\infty ≤ b, we apply the regularity result of \(-\Delta\) to obtain that \(\nabla \tilde{u}_\lambda\) is bounded on the bounded subsets of Ω_\lambda which contain neighbourhoods of 0 on ∂Ω_\lambda. Hence, in the original variables, \(\|\nabla u_\lambda\|_\infty ≤ K\lambda^{1/2}\) on the subsets of Ω which contain neighbourhoods of x_0 on ∂Ω. The required estimate for u_\lambda near ∂Ω now follows since ∂Ω is compact. This completes the proof.

4. Uniqueness results

In this Section we show that (1.1) has only one large positive solution u_\lambda when \lambda is sufficiently large.

First note that from the definition of the large positive solution of (1.1), there exists ξ ∈ (a, b) and a ball B(x_0, r) ⊂ Ω which is independent of \lambda such that u_\lambda ≥ ξ in B(x_0, r) for all \lambda sufficiently large. Let \(w_\lambda(x_0)\) be as in (2.3). We know that w_\lambda(x_0) is a subsolution of (1.1) with \(f\) replaced by \(F_\varepsilon\) for \(λ > \mu_{x_0}^{-2}d(x_0, ∂Ω)^{-2}\). Therefore, it follows from the monotone arguments as in Lemma 2.4 and Theorem 2.3 that for \(λ ≥ \lambda^{**}_{x_0}\) (with \(x^{**}\) replaced by \(x_0\)), (2.4); (1.1) has a positive solution \(u_{\lambda, x^{**}}\) in \([w_\lambda(x_0), b]\) respectively (both of them are minimal solutions) such that \(\tilde{u}_\lambda ≥ u_{\lambda, x^{**}}\) in Ω and

\[-\Delta \tilde{u}_\lambda = f(\tilde{u}_\lambda) \quad \text{in} \; \Omega, \quad \tilde{u}_\lambda = 0 \quad \text{on} \; \partial\Omega,\]

\[-\Delta \tilde{u}_\lambda = f(\tilde{u}_\lambda) \quad \text{in} \; \Omega, \quad \tilde{u}_\lambda = 0 \quad \text{on} \; \partial\Omega,\]

where \(u_{\lambda, x^{**}}\) is the minimal positive solution of (1.1). (This is known from Remark 2 after the proof of Theorem 2.3.)

Now we show that \(u_\lambda ≥ \tilde{u}_\lambda\) and hence \(u_\lambda ≥ u_{\lambda, x^{**}}\) in Ω for \(λ ≥ \lambda^{**}\). Therefore, the estimates in Proposition 3.1 are true for u_\lambda.

It is enough to prove \(u_\lambda ≥ w_\lambda(x_0)\) in Ω. It is known from the above that \(w_\lambda(x_0)\) is a subsolution of (2.4) for \(\varepsilon\) sufficiently small and

\[\tau = \max w(\lambda, x_0) < b(\varepsilon) < b.\]
Moreover, there exists $\sigma_\varepsilon > 0$ such that
\begin{equation}
(4.3) \quad f(s) \geq \sigma_\varepsilon (s - \xi) \quad \text{for } s \in [\xi, b(\varepsilon)]
\end{equation}
since $\xi > a$ and $f(s) > 0$ for $s \in [\xi, b(\varepsilon)]$. (We know that $\xi < b(\varepsilon)$ when $\varepsilon$ is sufficiently small).

**Lemma 4.1.** Let $f$ satisfy (4.3). Then, for $\lambda > \bar{\lambda}$,
\begin{equation}
(4.4) \quad u_\lambda \geq b(\varepsilon) \quad \text{in } B(x_0, r/2).
\end{equation}

**Proof.** We know that $u_\lambda \geq \xi$ in $B(x_0, r)$. Now for any $x_1 \in B(x_0, r/2)$, we set
\[ \theta(x_1, \lambda, t; x) = \xi + t\phi_1((\sigma_\varepsilon \lambda / \lambda_1)^{1/2}(x - x_1)) \quad \text{for } x \in \tilde{B} \text{ and } t \in [0, b(\varepsilon) - \xi], \]
where $\lambda_1$ and $\phi_1$ with $\|\phi_1\|_{\infty} = 1$ are the first eigenvalue and the corresponding eigenfunction of the eigenvalue problem of $-\Delta$ in the unit ball of $\mathbb{R}^N$ with the Dirichlet boundary condition; $\tilde{B} = B(x_1, \lambda_1(\sigma_\varepsilon \lambda)^{-1})$. It is well-known that $\phi_1$ is radially symmetric and $\phi_1(0) = 1$. Note that for $\lambda > \lambda_1(\sigma_\varepsilon r/2)^{-1}$, $\tilde{B} \subset B(x_0, r)$.

We assume that $\lambda > \max\{\lambda_1(\sigma_\varepsilon r/2)^{-1}, \bar{\lambda}\}$. We claim that the set $\{\theta(x_0, \lambda, t) \mid t \in [0, b(\varepsilon) - \xi]\}$ is a family of subsolutions of the problem
\begin{equation}
(4.5) \quad -\Delta v = \lambda f(v) \quad \text{in } \tilde{B}, \quad v = u_\lambda \quad \text{on } \partial \tilde{B},
\end{equation}
with the closure of $\tilde{B}$ is contained in $B(x_0, r)$. It is clear that $u_\lambda \geq \theta(x_0, \lambda, 0)$ in $\tilde{B}$ and $|f'(s)| \leq M_\varepsilon$ for $s \in [0, b(\varepsilon)]$. Thus, by the similar argument to that in the proof of Proposition 3.2, we obtain that
\begin{equation}
(4.6) \quad u_\lambda \geq \theta(x_0, \lambda, b(\varepsilon) - \xi) \quad \text{in } \tilde{B}
\end{equation}
and thus
\begin{equation}
(4.7) \quad u_\lambda(x_1) \geq b(\varepsilon) \quad \text{for all } x_1 \in B(x_0, r/2).
\end{equation}
This completes the proof of Lemma 4.1. \hfill \square

It is easily seen that when $\lambda > (r/2)^{-2}\mu$, $w(\lambda, x_0; x) \leq 0$ for $x \in \Omega \setminus B(x_0, r/2)$. We assume $\lambda > \bar{\lambda} := \max\{\lambda_1(\sigma_\varepsilon r/2)^{-1}, \bar{\lambda}, (r/2)^{-2}\mu\}$ in the follows. Then we obtain
\begin{equation}
(4.8) \quad u_\lambda \geq w(x_0, \lambda) \quad \text{in } \Omega,
\end{equation}
since $\tau < b(\varepsilon)$. By the fact that $\tilde{u}_\lambda$ is the minimal positive solution of (1.1) in $[w(x_0, \lambda), b]$, we have
\begin{equation}
(4.9) \quad u_\lambda \geq \tilde{u}_\lambda \quad \text{in } \Omega.
\end{equation}
This is our claim.
Theorem 4.2. Assume that $f$ satisfies $(f_1)$–$(f_3)$. Then (1.1) has only one large positive solution $u_\lambda$ for $\lambda$ sufficiently large satisfying $\max_{\Omega} u_\lambda \leq b$ and

$$u_\lambda \to b \text{ in compact sets of } \Omega \text{ as } \lambda \to \infty.$$ (4.10)

Proof. The existence of at least one large positive solution $u_\lambda$ of (1.1) for $\lambda$ sufficiently large has been obtained in Theorem 2.3. We only need to study the uniqueness of $u_\lambda$.

By the argument above, we know that if $u_\lambda$ and $u_\lambda^*$ are two large positive solutions of (1.1) for $\lambda$ sufficiently large, then $u_\lambda \leq u_\lambda^*$ or $u_\lambda^* \leq u_\lambda$ holds and the asymptotic behaviour in Proposition 3.1 holds for both $u_\lambda$ and $u_\lambda^*$ and $\lambda$ large. Without loss of generality, we assume $u_\lambda^* \leq u_\lambda$ in $\Omega$ in the follows.

Now we show that for $\lambda$ sufficiently large,

$$u_\lambda \equiv u_\lambda^* \text{ in } \Omega.$$ (4.11)

On the contrary, there exist sequences $\{\lambda_n\}$ with $\lambda_n \to \infty$ as $n \to \infty$ and $\{u_n\} \equiv \{u_{\lambda_n}\}$, $\{u_n^*\} \equiv \{u_{\lambda_n}^*\}$ such that $u_n \neq u_n^*$ for all $n$.

Define $v_n = (u_n - u_n^*)/\|u_n - u_n^*\|_\infty$. Then $v_n \geq 0$, $v_n \neq 0$ in $\Omega$ and $\max_{\Omega} v_n = 1$ for all $n$. Setting

$$H_n = \{x \in \Omega | u_n(x) = b\},$$
$$H_n^* = \{x \in \Omega | u_n^*(x) = b\},$$

we easily know that $H_n^* \subset H_n \subset \subset \Omega$ and that $v_n$ satisfies the problem

$$-\Delta v_n = \lambda_n f'(\xi_n) v_n \text{ in } \Omega \setminus H_n, \quad v_n = 0 \text{ on } \partial \Omega,$$ (4.12)

where $\xi_n \in (u_n^*, u_n)$. Now we show if $\eta_n \in \Omega$ such that $v_n(\eta_n) = 1$, then

$$\text{dist}(\eta_n, \partial \Omega) \to 0 \text{ as } n \to \infty.$$ (4.13)

(Note that $v_n(x) = 0$ for $x \in H_n^*$.) In fact, it is known from Proposition 3.1 that if $K \subset \subset \Omega$ is a compact set, then $u_n^* \to b$, $u_n \to b$ in $K$ as $n \to \infty$. If $\eta_n \in K$ for all $n$ large, we know that $\eta_n \in K \setminus H_n^*$. There are two cases here: (i) there exists a subsequence of $\{\eta_n\}$ (still denoted by $\{\eta_n\}$) such that $\eta_n \in K \setminus H_n$, (ii) there exists a subsequence of $\{\eta_n\}$ (still denoted by $\{\eta_n\}$) such that $\eta_n \in H_n \setminus H_n^*$. Since $H_n$ and $H_n^*$ are closed sets in $\Omega$, for the first case, there exists a small neighbourhood $B_{\eta_n}$ of $\eta_n$ in $\Omega$ such that $B_{\eta_n} \cap H_n = \emptyset$ and $f'(\xi_n) < 0$ in $B_{\eta_n}$ for all $n$ large. (We use the continuity of $\xi_n$ in $B_{\eta_n}$ here.) This is a contradiction since $v_n$ attains its maximum on $\Omega$ at $\eta_n$. For the second case, we also can choose a small neighbourhood $B_{\eta_n}$ of $\eta_n$ in $\Omega$ such that $B_{\eta_n} \cap H_n^* = \emptyset$. On the other hand, we write (4.12) in the form

$$-\Delta v_n = \lambda_n \frac{f(u_n) - f(u_n^*)}{u_n - u_n^*} v_n.$$
and easily know that \(-\Delta u_n < 0\) in \(B_{\eta_n}\). This is also a contradiction. Thus, (4.13) holds.

Now we use the blow up argument as in [12], [5] to show that (4.13) does not hold. We consider two cases here: (we can choose subsequences if necessary)

(i) \(\lambda_n^{1/2} \text{dist}(\eta_n, \partial \Omega) \to Z \geq t_0\) (\(Z\) can be \(\infty\)), as \(n \to \infty\),

(ii) \(\lambda_n^{1/2} \text{dist}(\eta_n, \partial \Omega) \leq Z < t_0\) for all \(n\) sufficiently large, where \(t_0 > 0\) is the number defined in (3.3).

For the first case, we have from Proposition 3.1 that \(u_n(\eta_n) \to b, u_n^*(\eta_n) \to b\) as \(n \to \infty\). Thus we derive contradictions by the arguments similar to that in the proof of (4.13).

For the second case, we make a change of variables, \(X_n = \lambda_n^{1/2}(x - \tilde{\eta}_n)\), where \(\tilde{\eta}_n\) is the point on \(\partial \Omega\) closest to \(\eta_n\). Let \(\tilde{u}_n(X_n) = u_n(x), \tilde{u}_n^*(X_n) = u_n^*(x), \tilde{\xi}^n(X_n) = \xi(x)\) and \(\tilde{v}_n(X_n) = v_n(x)\). We have that \(\tilde{v}_n\) satisfies the problem

\[
-\Delta \tilde{v}_n = f'(\tilde{\xi}(\tilde{x}))\tilde{v}_n \quad \text{in} \quad \tilde{\Omega}_n \setminus \tilde{H}_n, \quad \tilde{v}_n = 0 \quad \text{on} \quad \tilde{\partial} \tilde{\Omega}_n,
\]

where

\[
\tilde{\Omega}_n \equiv \{ X_n = \lambda_n^{1/2}(x - \tilde{\eta}_n) \mid x \in \Omega \}, \quad \tilde{H}_n \equiv \{ X_n = \lambda_n^{1/2}(x - \tilde{\eta}_n) \mid x \in H_n \}.
\]

Note that \(\tilde{v}_n(Z_n) = 1\), where \(Z_n = \lambda_n^{1/2}(\eta_n - \tilde{\eta}_n)\) is at distance at most \(Z\) from 0 and \(Z < t_0\). By the argument similar to that in the proof of Theorem 2 of [5], we have that \(\tilde{u}_n \to y(x_1)\) in \(C^1_{\text{loc}}(T_1)\), \(\tilde{u}_n^* \to y(x_1)\) in \(C^1_{\text{loc}}(T_1)\) as \(n \to \infty\).

Where \(T_1 = \{ x \in \mathbb{R}^N \mid x_1 \geq 0 \}\) and \(y\) is the unique solution of (3.2). Defining \(H = \{ x \in T_1 \mid x_1 \geq t_0 \}\), we easily know that \(\tilde{H}_n \to H\) and \(\tilde{\xi}_n \to y(x_1)\) in \(C^1_{\text{loc}}(T_1 \setminus H)\) as \(n \to \infty\). Moreover, \(\tilde{v}_n\) converges in \(C^1_{\text{loc}}(T_1 \setminus H)\) to a non-trivial non-negative bounded solution \(\tilde{v}\) of

\[
-\Delta \tilde{v} = f'(y(x_1))\tilde{v} \quad \text{in} \quad T_1 \setminus H, \quad \tilde{v} = 0 \quad \text{on} \quad \partial T_1.
\]

Here \(\tilde{v}\) is non-trivial because \(\tilde{v}_n(Z_n) = 1\) and \(\text{dist}(0, Z_n) \leq Z < t_0\).

Now we show that such \(\tilde{v}\) can not exist by the three steps as that in the proof of Proposition 2 of [5], but with a different definition domain of \(y(x_1)\).

**Step 1.** We find a solution \(q\) of

\[
-u'' = f'(y)u,
\]

which is positive on \([0, t_0]\) and is not bounded as \(x_1 \to t_0^+\).

By differentiating the equation satisfied by \(y\) ((3.2)) with respect to \(x_1\), we see that \(y'(x_1)\) is a solution of (4.16). Let \(Y\) be the solution of the initial value problem

\[
-Y'' = f'(y)Y \quad \text{in} \quad (0, t_0), \quad Y(0) = 0, \quad Y'(0) = 1.
\]
We claim that \( Y(x_1) \to \infty \) as \( x_1 \to t_0^- \). In fact, we know from a simple computation
\[
(y'Y' - Yy'')' \equiv 0 \quad \text{in } (0, t_0).
\]
This implies that
\[
y'Y' - Yy'' \equiv C \quad \text{in } [0, t_0],
\]
where \( C = y'(0) \). Our claim can be obtained from (4.19) and the facts that \( y'(x_1) \to 0 \) and \( y''(x_1) \to 0 \) as \( x_1 \to t_0^- \). Define \( q(x_1) = y'(x_1) + Y(x_1) \). We easily know that \( q \) satisfies our requirement.

**Step 2.** If (4.15) has a non-trivial bounded non-negative solution \( \tilde{v} \), the \( \tilde{v} \) can be chosen so that \( T(x_1) = \sup_{y \in \mathbb{R}^{N-1}} \tilde{v}(x_1, y) \) is continuous for \( x_1 > 0 \).

The proof of Step 2 is similar to that of Proposition 2 in [5].

**Step 3.** We show that \( \tilde{v} \) cannot exist. If \( \tilde{v} \) exists, using the notation of Step 2, we consider \( r(x) = \tilde{v}(x)/q(x_1) \). By Steps 1 and 2 and the boundedness of \( \tilde{v} \), it follows that \( \lim_{x_1 \to t_0^-} T(x_1)/q(x_1) = 0 \). Thus, since \( T(0) = 0 \), we can find \( 0 < \tilde{x}_1 < t_0 \) such that
\[
\sup\{T(x_1)/q(x_1) \mid 0 \leq x_1 < t_0\} = T(\tilde{x}_1)/q(\tilde{x}_1).
\]
By Step 2, \( \tilde{v} \) can be chosen so that \( \tilde{v}(\tilde{x}_1, y) \) achieves its maximum on \( \mathbb{R}^{N-1} \) at 0. By our construction, \( r(x) \) achieves its maximum on \( \{(x_1, y) \mid 0 \leq x_1 < t_0, y \in \mathbb{R}^{N-1}\} \) at the interior point \( (\tilde{x}_1, 0) \). However, since \( q \) satisfies (4.16), a simple calculation shows that \( r \) satisfies an elliptic equation
\[
r''_{x_1} + 2(q'/q)r'_{x_1} + \Delta_{N-1}r = 0,
\]
where \( \Delta_{N-1} \) denotes the Laplacian in the \( y \) variables. Hence, by applying the maximum principle on compact sets, we see that \( r(x_1, y) \) is constant of \( 0 \leq x_1 < t_0, y \in \mathbb{R}^{N-1} \). This is impossible since \( r = 0 \) when \( x_1 = 0 \). \( \square \)

We easily obtain the following corollary from Theorem 4.2.

**Corollary 4.3.** Let \( f \) satisfy \((f_1)-(f_3)\) and \( \Omega \) be an \( N \)-ball or an annulus. Then \((1.1)\) has exactly one large positive solution \( u_\lambda \) which is radially symmetric for \( \lambda \) sufficiently large. Moreover, \( u_\lambda \to b \) in compact subsets of \( \Omega \) as \( \lambda \to \infty \).

**Remark.** Corollary 4.3 implies that \((1.1)\) has no non-radial large positive solutions for \( \lambda \) sufficiently large.

### 5. Flat core of the large positive solution

In this section we shall give the asymptotic behaviour of the flat core \( G_\lambda \) of the unique large positive solution \( u_\lambda \) as \( \lambda \to \infty \). The existence of \( G_\lambda \) for \( u_\lambda \) was obtained in [21]. Our main result of this section is
Theorem 5.1. Let $f$ satisfy $(f_1)$–$(f_3)$. Then for $\lambda$ sufficiently large, $G_\lambda$ satisfies that if $\bar{d}^*(\lambda) = \text{dist}(G_\lambda, \partial \Omega)$, then

$$\lim_{\lambda \to \infty} \lambda^{1/2} \bar{d}^*(\lambda) = \frac{C(F)^{1/2}}{2},$$

where

$$C(F) = \frac{1}{2} \left( \int_0^b \frac{2ds}{(F(b) - F(s))^{1/2}} \right)^2$$

and $F(s) = \int_0^s f(\xi) d\xi$.

Moreover,

$$\lim_{\lambda \to \infty} \lambda^{1/2} \text{dist}(x, G_\lambda) = \frac{C(F)^{1/2}}{2}$$

for any $x \in \partial \Omega$.

To prove this theorem, we start the study from the simple case $N = 1$, i.e. the problem

$$(5.1) \quad -u'' = \lambda f(u) \quad \text{in} \quad (0, \ell), \quad u(0) = 0, \quad u(\ell) = 0,$$

where $\ell > 0$ is independent of $\lambda$. The main idea of this section is similar to that in [18] but with many modifications.

Lemma 5.2. Assume that $f$ satisfies $(f_1)$–$(f_3)$. Then there exists a unique positive solution $v_\lambda(x)$ of (5.1) satisfying

$$(5.2) \quad v_\lambda \to b \quad \text{uniformly on compact sets of} \quad (0, \ell) \quad \text{as} \quad \lambda \to \infty.$$ 

Moreover, for $\lambda \geq \tilde{\lambda} := (1/\ell^2)C(F)$,

$$(5.3) \quad E_\lambda = \{x \in (0, \ell) \mid v_\lambda(x) = b \} = [d^*(\lambda), \ell - d^*(\lambda)],$$

where

$$(5.4) \quad C(F) = \frac{1}{2} \left( \int_0^b \frac{2ds}{(F(b) - F(s))^{1/2}} \right)^2,$$

$$F(s) = \int_0^s f(\xi) d\xi,$$

$$(5.5) \quad d^*(\lambda) = \frac{1}{2} C(F)^{1/2} \lambda^{-1/2}.$$ 

Proof. We claim that if $v_\lambda \in C^1([0, \ell])$ is a positive solution of (5.1) with $\|v_\lambda\|_\infty \leq b$, then $v_\lambda$ is symmetric about $x = \ell/2$. In fact, the first integral of (5.1) implies that

$$(5.6) \quad |v'_{\lambda}(x)|^2 + 2\lambda F(v_\lambda) = C, \quad x \in (0, \ell).$$

Let $\overline{\lambda}_\lambda = \sup_{0 < x < \ell} v_\lambda(x)$. Then it follows from (5.6) that

$$(5.7) \quad |v'_{\lambda}(x)|^2 = 2\lambda(F(\overline{\lambda}_\lambda) - F(v_\lambda)).$$

On the other hand, we easily know from (5.6) that $\overline{\lambda}_\lambda$ is the only critical value of $v_\lambda = v_\lambda(x)$. Therefore, if $x_1^\lambda = \min\{x \mid v_\lambda = \overline{\lambda}_\lambda\}$, $x_2^\lambda = \max\{x \mid v_\lambda = \overline{\lambda}_\lambda\}$,
then $v_\lambda$ increases before $x_1^\lambda$, decreases after $x_2^\lambda$, while $v_\lambda \equiv \overline{v}_\lambda$ in $x_1^\lambda \leq x \leq x_2^\lambda$. Thus, it follows from (5.7) that
\begin{equation}
\int_0^{x_1^\lambda} \frac{ds}{(F(\overline{v}_\lambda) - F(s))^{1/2}} = (2\lambda)^{1/2}x, \quad 0 < x < x_1^\lambda,
\end{equation}
and
\begin{equation}
\int_0^{x_2^\lambda} \frac{ds}{(F(\overline{v}_\lambda) - F(s))^{1/2}} = (2\lambda)^{1/2}(\ell - x), \quad x_2^\lambda < x < \ell.
\end{equation}
(5.8) and (5.9) imply that $v_\lambda$ is symmetric with respect to $\ell/2$ and
\begin{equation}
\int_0^{x_1^\lambda} \frac{ds}{(F(\overline{v}_\lambda) - F(s))^{1/2}} = (2\lambda)^{1/2}x_1^\lambda.
\end{equation}
This implies our claim.

To prove the existence, we first notice that it follows from (f2) that
\begin{equation}
\int_0^{b} \frac{ds}{(F(b) - F(s))^{1/2}} < \infty.
\end{equation}
Defining $C(F)$ and $d^*(\lambda)$ as in (5.4) and (5.5) and
\[ \tilde{\lambda} = \frac{1}{\ell^2} C(F), \]
we have that if $\lambda > \tilde{\lambda}$, then $d^*(\lambda) < \ell/2$. Now we define $v_\lambda(x)$ by
\begin{equation}
\int_0^{d^*(\lambda)} \frac{ds}{(F(b) - F(s))^{1/2}} = (2\lambda)^{1/2}x, \quad 0 < x < d^*(\lambda)
\end{equation}
and
\begin{equation}
v_\lambda(x) \equiv b \quad \text{for} \quad x \in [d^*(\lambda), \ell/2].
\end{equation}
We can define $v_\lambda$ on $[\ell/2, \ell]$ such that $v_\lambda$ is symmetric about $x = \ell/2$. It is clear that $v_\lambda$ is the required positive solution of (5.1).

Now we show that $v_\lambda$ is the unique positive solution of (5.1) such that max $v_\lambda \to b$ as $\lambda \to \infty$. In fact, suppose $w_\lambda$ is a positive solution of (5.1) such that max $w_\lambda \to b$ as $\lambda \to \infty$, we can show that $w_\lambda(\ell/2) = b$ for $\lambda$ sufficiently large. On the contrary, we know that $w_\lambda(\ell/2) := \overline{w}_\lambda < b$ for all $\lambda$ large. Since $F(s) = \int_0^s f(\xi) \, d\xi$, we know that for $s < \overline{w}_\lambda$ and near $\overline{w}_\lambda$,
\[ F(s) = F(\overline{w}_\lambda) + f(\overline{w}_\lambda)(s - \overline{w}_\lambda) + \frac{1}{2} f'(\overline{w}_\lambda)(s - \overline{w}_\lambda)^2 + o((s - \overline{w}_\lambda)^2). \]
We know that $f(\overline{w}_\lambda) > 0$ and $f'(\overline{w}_\lambda) < 0$ for $\lambda$ sufficiently large (since $\overline{w}_\lambda \to b$ as $\lambda \to \infty$). Thus,
\[ F(\overline{w}_\lambda) - F(s) \geq \frac{1}{2} f(\overline{w}_\lambda)(\overline{w}_\lambda - s) \quad \text{for} \quad s \text{ near } \overline{w}_\lambda. \]
Therefore,

$$\int_{s_0}^{\overline{\sigma}} (F(\overline{\sigma}) - F(s))^{-1/2} \leq 2(f(\overline{\sigma}))^{-1/2} \int_{s_0}^{\overline{\sigma}} (\overline{\sigma} - s)^{-1/2} \, ds < \infty$$

for \( s_0 \) near \( \overline{\sigma} \) and \( \lambda \) sufficiently large. On the other hand, we know from a similar identity to (5.10) that

$$\int_{0}^{\overline{\sigma}} \frac{ds}{(F(\overline{\sigma}) - F(s))^{1/2}} = (2\lambda)^{1/2} \ell.$$

(Since \( \overline{\sigma} < b \), \( \overline{\sigma} \) can only attain at \( x = \ell/2 \).) We easily derive a contradiction from (5.13) and (5.14). Since \( w_\lambda \) can also be written to the forms same as (5.11) and (5.12), we have that \( w_\lambda \equiv v_\lambda \) in \((0, \ell)\). \( \square \)

Now we are dealing with the case \( \Omega = B_R = \{ x \in \mathbb{R}^N | \|x\| < R \} \).

**Lemma 5.3.** Let \( u_\lambda \) be the unique large positive (radial) solution of (1.1) for \( \lambda \) sufficiently large obtained in Corollary 4.3. Then \( u_\lambda \) has flat core \( G_{\lambda, B} \).

Moreover,

$$\lim_{\lambda \to \infty} \sup_{\lambda} \lambda^{1/2} d(\lambda, B) \leq \frac{C(F)^{1/2}}{2},$$

where \( d(\lambda, B) = \text{dist}(G_{\lambda, B}, \partial B_R) \).

**Proof.** We know that \( u_\lambda \) satisfies the problem

$$-(r^{N-1} u'_\lambda)' = \lambda r^{N-1} f(u_\lambda), \quad r \in (0, R), \quad u'_\lambda(0) = 0, \quad u_\lambda(R) = 0.$$  

Now we introduce a change

$$\rho = g(r) = \begin{cases} \frac{1}{2-N} (R^2 - r^2) & N \geq 3, \\ \log(R/r) & N = 2. \end{cases}$$

Observe that \( 0 < \rho < \infty \) if \( 0 < r < R \). Setting \( v_\lambda(\rho) = u_\lambda(g^{-1}(\rho)) \) in (5.16) leads to the problem

$$-v''_\lambda = \lambda(g^{-1}(\rho))^{2(N-1)} f(v_\lambda), \quad 0 < \rho < \infty, \quad v(0) = v'(\infty) = 0,$$

where \( ' = d/d\rho \). Moreover, \( v_\lambda \) is the unique large positive solution of (5.17) (see [18]).

If we fix \( 0 < \theta < \infty \) independent of \( \lambda \) and \( v = v_\lambda(\rho) \) stands for the unique large positive solution to (5.17), then we have that \( v(\theta) \to b \) as \( \lambda \to \infty \) and that there exists a unique \( 0 < \eta_\lambda < \theta \) such that \( v(\eta_\lambda) = a \) and

$$-v'' \geq \lambda(g^{-1}(\theta))^{2(N-1)} f(v)$$

provided that \( \eta_\lambda < \rho < \theta \) (since \( f(v(\rho)) \geq 0 \) for \( \eta_\lambda < \rho < \theta \)). The uniqueness of \( \eta_\lambda \) can be known from the structure of \( u_\lambda \). In fact, we can easily show that \( u'_\lambda \equiv 0 \) and \( u_\lambda \equiv b \) in \([0, \tilde{r}_\lambda] \) for some \( \tilde{r}_\lambda \geq 0 \) and \( u'_\lambda < 0 \) in \((\tilde{r}_\lambda, R) \) (see [15]).
Thus, \( \nu_\lambda \) has the similar property. We know that \( \eta_\lambda = g(r_\lambda) \), where \( u_\lambda(r_\lambda) = a \). It follows from Proposition 3.1 that \( \lambda^{1/2}(R - r_\lambda) \to t^0 \) as \( \lambda \to \infty \), where \( t^0 > 0 \) satisfies \( y(t^0) = a \) and \( y(t) \) is the unique solution of (3.2). (This can also be obtained from the arguments similar to that in the proof of Theorem 4.2 or that in the proof of Theorem A in [13]. In fact, if \( r \) is near \( R \), \( X^\lambda = \lambda^{1/2}(R - r) \) and \( \tilde{u}_\lambda(X^\lambda) = u_\lambda(r) \), we know \( \tilde{u}_\lambda(X^\lambda) \to y(t) \) in \( C^1_{\text{loc}}(0, \infty) \) as \( \lambda \to \infty \), where \( y \) is the unique solution of (3.2).) By the first integral arguments similar to that in the proof of Lemma 5.2, we easily know from the property of \( y \) that \( r_\lambda = R - (2\lambda)^{-1/2} R^{1-N} \int_0^a ds \frac{F(b) - F(s)}{(F(b) - F(s))^{1/2}} + o(\lambda^{-1/2}) \) for \( \lambda \) sufficiently large. Therefore, for \( N \geq 3 \),

\[
\eta_\lambda = g(r_\lambda) = (2\lambda)^{-1/2} R^{1-N} \int_0^a ds \frac{ds}{(F(b) - F(s))^{1/2}} + o(\lambda^{-1/2}).
\]

For \( N = 2 \), we also obtain

\[
\eta_\lambda = g(r_\lambda) = (2\lambda)^{-1/2} R^{-1} \int_0^a ds \frac{ds}{(F(b) - F(s))^{1/2}} + o(\lambda^{-1/2}).
\]

(Note that we use Taylor expansions here.)

Let us introduce now the auxiliary problem

(5.18) \[-v'' = \lambda (g^{-1}(2\theta))^2(2^{(N-1)} f(v), \quad \eta_\lambda < \rho < \theta, \quad v(\eta_\lambda) = a, \quad v'(\theta) = 0.\]

We observe now that (5.18) admits a unique positive solution \( v = v_\lambda(\rho, \theta) \) provided \( \lambda > \lambda_0 \) and \( v_\lambda(\theta) = b \), where

\[
\lambda_0 = \theta^{-1} \left[ 2^{-1/2} R^{1-N} \int_0^a ds \frac{ds}{(F(b) - F(s))^{1/2}} + 2 + (2g^{-1}(2\theta))^{2(N-1)} - 2^{1/2} \int_0^b ds \frac{ds}{(F(b) - F(s))^{1/2}} \right]^2.
\]

In fact, restricting to \( \eta_\lambda < \rho < \theta \) the unique positive solution \( v_\lambda \) of

(5.19) \[-v'' = \lambda (g^{-1}(2\theta))^2(2^{(N-1)} f(v), \quad \eta_\lambda < \rho < 2\theta - \eta_\lambda, \quad v(\eta_\lambda) = v(2\theta - \eta_\lambda) = a\]
with max \( \tilde{v}_\lambda = b \), we obtain \( v_\lambda \). Now we show that \( (5.19) \) has a unique positive solution \( \tilde{v}_\lambda(x) \) with max \( \tilde{v}_\lambda = b \). In fact, the arguments similar to that in the proof of Lemma 5.2 imply that, if

\[
\lambda > \left[ \theta^{-1} \left( \lambda^{1/2} \eta_\lambda + (2g^{-1}(2\theta))^{2(N-1)} - \frac{1}{2} \int_0^b \frac{ds}{(F(b) - F(s))^{1/2}} \right) \right]^2,
\]

\( (5.19) \) has a unique solution \( \tilde{v}_\lambda(x) \) with \( \tilde{v}_\lambda(\theta) = b \) satisfying

\[
\int_a^{\tilde{v}_\lambda(x)} \frac{ds}{(F(b) - F(s))^{1/2}} = (2\lambda g^{-1}(2\theta))^{2(N-1)}(x - \eta_\lambda) \quad \text{for } x \in (\eta_\lambda, \theta)
\]

and

\[
\int_a^{\tilde{v}_\lambda(x)} \frac{ds}{(F(b) - F(s))^{1/2}} = (2\lambda g^{-1}(2\theta))^{2(N-1)}(2\theta - \eta_\lambda - x)
\]

for \( x \in (\theta, 2\theta - \eta_\lambda) \). Define

\[
d(\lambda) = \eta_\lambda + (2\lambda g^{-1}(2\theta))^{2(N-1)} - \frac{1}{2} \int_0^b \frac{ds}{(F(b) - F(s))^{1/2}},
\]

\[
A = \int_0^a \frac{ds}{(F(b) - F(s))^{1/2}}, \quad B = \int_a^b \frac{ds}{(F(b) - F(s))^{1/2}}.
\]

We easily know \( d(\lambda) < \theta \) for \( \lambda > \lambda_0 \) and sufficiently large and thus

\[ \tilde{v}_\lambda \equiv b \quad \text{in } [d(\lambda), 2\theta - d(\lambda)]. \]

It is clear that \( \tilde{v}_\lambda(\rho) \) (for \( \eta_\lambda < \rho < 2\theta - \eta_\lambda \)) is a subsolution of the problem

\[
-\nu'' = \lambda(g^{-1}(\rho))^{2(N-1)}f(\nu), \quad \nu(\eta_\lambda) = a, \quad \nu(2\theta - \eta_\lambda) = v_\lambda(2\theta - \eta_\lambda).
\]

(Note that \( v_\lambda(2\theta - \eta_\lambda) \to b \) as \( \lambda \to \infty \).) Since \( b \) is a supersolution of \( (5.20) \), then we use the arguments similar to that in the proof of Theorem 2.3 to obtain a positive solution \( \tau_\lambda \) of \( (5.20) \) in \([\tilde{v}_\lambda, b] \). Since \( \tau_\lambda(\rho) \) is the unique large positive solution of \( (5.17) \) and \( v_\lambda(\rho) \) satisfies \( (5.20) \), we can conclude

\[
(5.21) \quad \tau_\lambda \equiv v_\lambda \quad \text{in } (\eta_\lambda, \theta),
\]

\[
(5.22) \quad a < \tau_\lambda(\rho, \theta) \leq v_\lambda(\rho) \quad \text{for } \eta_\lambda < \rho < \theta.
\]

(To show \( (5.21) \), we first notice that \( v_\lambda(2\theta - \eta_\lambda) = u_\lambda(g^{-1}(2\theta - \eta_\lambda)) \to b \) as \( \lambda \to \infty \). \( \tau_\lambda \) and \( v_\lambda \) are corresponding to the solutions \( \tau_\lambda \) and \( v_\lambda \) of the problem

\[ -(r^{-N-1}u')' = \lambda r^{-N}f(u) \quad \text{in } (g^{-1}(2\theta - \eta_\lambda), g^{-1}(\eta_\lambda)), \]

\[ u(g^{-1}(\eta_\lambda)) = u_\lambda(g^{-1}(\eta_\lambda)) = a, \quad u(g^{-1}(2\theta - \eta_\lambda)) = u_\lambda(g^{-1}(2\theta - \eta_\lambda)). \]

Since \( u_\lambda \) is the unique large positive solution of \( (5.16) \), extending \( \tau_\lambda \) to be \( u_\lambda \) in \([0, g^{-1}(2\theta - \eta_\lambda)] \) and \((g^{-1}(\eta_\lambda), R] \), we have that \( \tau_\lambda \) is also a large positive solution of \( (5.16) \). Thus, \( \tau_\lambda \equiv u_\lambda \) for \( \lambda \) sufficiently large. This shows \( (5.21) \).
Notice that $v_\lambda(\rho, \theta)$ develops a flat core for each $\lambda > \lambda_0$ and

$$v_\lambda(d(\lambda)) = b.$$  \hspace{1cm} (5.23)

Since $u_\lambda$ is decreasing, (5.23) implies $u_\lambda(r) \equiv b$ for $0 \leq r \leq g^{-1}(d(\lambda))$. Thus,

$$0 < \text{dist}(G_\lambda, \partial B_R) \leq R - g^{-1}(d(\lambda)).$$  \hspace{1cm} (5.24)

If we put $\hat{d}(\lambda) := R - g^{-1}(d(\lambda))$, it follows that

$$\hat{d}(\lambda) = R^{N-1}[(2\lambda)^{-1/2}R^{1-N}A + (2\lambda)^{-1/2}(g^{-1}(2\theta))^{1-N}B + o(\lambda^{-1/2})] + o(\lambda^{-1/2})$$  \hspace{1cm} for $\lambda$ sufficiently large. Since

$$\lim_{\lambda \to \infty} \lambda^{1/2} \hat{d}(\lambda) = 2^{-1/2}A + 2^{-1/2} \left( \frac{R}{g^{-1}(2\theta)} \right)^{N-1} B,$$  \hspace{1cm} (5.25)

it is obtained from (5.24) and (5.25), after passing to the limit as $\theta \to 0^+$, that

$$\lim_{\lambda \to \infty} \sup \lambda^{1/2} \text{dist}(G_\lambda, \partial B_R) \leq \frac{C(F)^{1/2}}{2},$$  \hspace{1cm} (5.26)

since $2^{-1/2}(A + B) = C(F)^{1/2}/2$. This completes the proof of Lemma 5.3. \hfill $\square$

**Lemma 5.4.** Let $\Omega = A(R_1, R_2) = \{x \in \mathbb{R}^N | 0 < R_1 < |x| < R_2\}$ and $u_\lambda(r)$ be the unique large positive solution of (1.1) in $\Omega$ for $\lambda$ sufficiently large. If $G_{\lambda,A} = \{x \in A(R_1, R_2) | u_\lambda(|x|) = b\}$, then

$$\lim_{\lambda \to \infty} \inf \lambda^{1/2} \text{dist}(G_{\lambda,A}, \partial A) \geq \frac{C(F)^{1/2}}{2},$$  \hspace{1cm} (5.27)

**Proof.** Setting

$$\rho = g(r) = \begin{cases} \frac{1}{2-N} \left[ r^{2-N} - R_2^{2-N} \right] & \text{for } N \geq 3, \\ \log \left( \frac{r}{R_2} \right) & \text{for } N = 2, \end{cases}$$

and

$$v_\lambda(\rho) = u_\lambda(g^{-1}(\rho)),$$

we can rewrite (1.1) as

$$-v'' = \lambda(g^{-1}(\rho))^{2(N-1)} f(v), \quad 0 < \rho < T, \quad v(0) = v(T) = 0,$$

where $' = d/d\rho$ and $T = g(R_2)$. Since $u_\lambda$ is the unique large positive solution of (1.1) with $r_1 = R_1$, $r_2 = R_2$, then $v_\lambda(\rho)$ is the unique large positive solution of this problem. Moreover, there exist $0 < \eta_1^\lambda < \eta_2^\lambda < T$ such that

$$v_\lambda(\eta_1^\lambda) = v_\lambda(\eta_2^\lambda) = a.$$
and \( \eta^1_\lambda = g(r^1_\lambda), \eta^2_\lambda = g(r^2_\lambda), \) where \( R_1 < r^1_\lambda < r^2_\lambda < R_2 \) such that \( u_\lambda(r^1_\lambda) = u_\lambda(r^2_\lambda) = a. \) By the arguments similar to that in the proof of Lemma 5.3, we have
\[
\lambda^{1/2}(r^1_\lambda - R_1) \to t^0 = 2^{-1/2}A, \quad \lambda^{1/2}(R_2 - r^2_\lambda) \to t^0 = 2^{-1/2}A \quad \text{as} \quad \lambda \to \infty,
\]
where \( t^0 \) and \( A \) are defined in the proof of Lemma 5.3. Thus, for \( \lambda \) sufficiently large,
\[
r^2_\lambda = R_1 + (2\lambda)^{-1/2}A + o(\lambda^{-1/2}), \quad r^2_\lambda = R_2 - (2\lambda)^{-1/2}A + o(\lambda^{-1/2}).
\]
Then, for \( N \geq 3, \)
\[
\eta^1_\lambda = g(r^1_\lambda) = (2\lambda)^{-1/2}R_1^{1-N}A + o(\lambda^{-1/2}),
\]
\[
\eta^2_\lambda = g(r^2_\lambda) = (N - 2)^{-1}(R_1^{2-N} - R_2^{2-N}) - (2\lambda)^{-1/2}AR_2^{1-N} + o(\lambda^{-1/2}).
\]
For \( N = 2, \) we have
\[
\eta^1_\lambda = \log(1 + (2\lambda)^{-1/2}AR_1^{-1} + o(\lambda^{-1/2})) = (2\lambda)^{-1/2}AR_1^{-1} + o(\lambda^{-1/2}),
\]
\[
\eta^2_\lambda = \log(R_2R_1^{-1}) - (2\lambda)^{-1/2}AR_2^{-1} + o(\lambda^{-1/2}).
\]
(Note that we use Taylor expansions in the calculations.) Thus
\[
\eta^1_\lambda + \eta^2_\lambda \geq [2(N - 2)]^{-1}(R_1^{2-N} - R_2^{2-N}) \quad \text{for} \quad N \geq 3 \quad \text{and} \quad \lambda \quad \text{large},
\]
\[
\eta^1_\lambda + \eta^2_\lambda \geq 2^{-1}(\log R_2 - \log R_1) \quad \text{for} \quad N = 2 \quad \text{and} \quad \lambda \quad \text{large}.
\]
Now we consider the problem
\[
(5.28) \quad -v'' = \lambda R_1^{2(N-1)}f(v), \quad \eta^1_\lambda < \rho < \eta^2_\lambda, \quad v(\eta^1_\lambda) = v(\eta^2_\lambda) = a.
\]
The arguments similar to that in the proof of Lemma 5.3 imply that, for \( \lambda > \lambda_0 \) with
\[
\lambda_0^{1/2} = \begin{cases} 
\frac{4(N - 2)[(2R_1^{2(N-1)})^{-1/2}B + 2^{-1/2}R_1^{1-N}A + 2]}{R_1^{2-N} - R_2^{2-N}} & \text{for} \quad N \geq 2, \\
\frac{4[2R_1^{-1/2}B + 2^{-1/2}R_1^{-1}A + 2]}{\log R_2 - \log R_1} & \text{for} \quad N = 2,
\end{cases}
\]
(5.28) has a unique solution \( v^-_\lambda \) such that
\[
v^-_\lambda \equiv b \quad \text{in} \quad [d(\lambda), (\eta^1_\lambda + \eta^2_\lambda) - d(\lambda)],
\]
where \( d(\lambda) := (2\lambda)^{-1/2}R_1^{1-N}B + \eta^1_\lambda < (\eta^1_\lambda + \eta^2_\lambda)/2 \) and \( B \) is defined in the proof of Lemma 5.3.

On the other hand, \( v^-_\lambda \) is a subsolution of the problem
\[
(5.29) \quad -v'' = \lambda (g^{-1}(\rho))^{2(N-1)}f(v), \quad \eta^1_\lambda < \rho < \eta^2_\lambda, \quad v(\eta^1_\lambda) = v(\eta^2_\lambda) = a.
\]
Since \( b \) is a supersolution of (5.29), we can obtain a positive solution \( \overline{v}_\lambda \) of (5.29) in \([v^-_\lambda, b]\) by the arguments similar to that in the proof of Theorem 2.3. It is
clear that \( v_\lambda \) is a large positive solution of (5.29) and hence \( v_\lambda \equiv v_\lambda \) in \((\eta_\lambda^1, \eta_\lambda^2)\).

This implies that
\[
(5.30) \quad a < v_\lambda^-(\rho) \leq v_\lambda(\rho), \quad \eta_\lambda^1 < \rho < \eta_\lambda^2.
\]

We know that for \( \lambda > \lambda_0 \) and sufficiently large,
\[
v_\lambda^-(\rho) = b \quad \text{for} \quad \rho \in [d(\lambda), (\eta_\lambda^1 + \eta_\lambda^2) - d(\lambda)].
\]

Then \( v_\lambda \) has flat core and thus \( u_\lambda \) has flat core.

Let us consider the auxiliary problem
\[
(5.31) \quad -w'' = \lambda(g^{-1}(\theta))^{2(N-1)}f(w), \quad \eta_\lambda^1 < \rho < \theta, \quad w(\eta_\lambda^1) = a, \quad w(\theta) = b,
\]
where \( \theta \) is again an arbitrary fixed number so that \( \eta_\lambda^1 < \theta < (\eta_\lambda^1 + \eta_\lambda^2)/2 \). By the arguments similar to that in the proof of Lemma 5.3, we know that (5.31) exhibits a unique positive solution \( w = w_\lambda(\rho, \theta) \) for \( \lambda \) large enough. Moreover, if \( \rho(\lambda) := (2\lambda)^{-1/2}(g^{-1}(\theta))^{-1-N}B + \eta_\lambda^1 \), then \( w_\lambda = b \) for \( \rho(\lambda) \leq \rho \leq \theta \), while \( a < w_\lambda < b \) in \( \eta_\lambda^1 < \rho < \rho(\lambda) \). On the other hand, since \( v_\lambda \) solves (5.29) in \( \eta_\lambda^1 < \rho < \eta_\lambda^2 \) then it defines a subsolution to (5.31) in \( \eta_\lambda^1 < \rho < \theta \). Since \( b \) is a supersolution of (5.31), we can use the arguments similar to that in the proof of Theorem 2.3 to obtain that there exists a solution \( w_\lambda \) of (5.31) between \( v_\lambda \) and \( b \). The uniqueness of \( w_\lambda \) implies that \( w_\lambda \equiv w_\lambda \) in \([\eta_\lambda^1, \theta]\). Thus,
\[
(5.32) \quad a < v_\lambda(\rho) \leq w_\lambda(\rho, \theta) \leq b \quad \text{for} \quad \eta_\lambda^1 < \rho < \theta.
\]

(5.32) implies that
\[
u_\lambda(r) \leq w_\lambda(g(r), \theta) < b
\]
provided that \( r \in A(R_1, R_2) \) and \( R_1 < r < g^{-1}(\rho(\lambda)) \). This means that
\[
(5.33) \quad \text{dist}(G_{\lambda,A}, \Gamma_1) \geq g^{-1}(\rho(\lambda)) - R_1,
\]
where \( \Gamma_1 = \{ x \in \partial A \mid |x| = R_1 \} \). Observing that
\[
g^{-1}(\rho(\lambda)) - R_1 = [(2 - N)\rho(\lambda) + R_1^{2-N}]^{1/(2-N)} - R_1
= R_1^{N-1}\rho(\lambda) + o(\rho(\lambda))
= R_1^{N-1}[(2\lambda)^{-1/2}(g^{-1}(\theta))^{-1-N}B
+ (2\lambda)^{-1/2}R_1^{-N}A + o(\lambda^{-1/2})] + o(\rho(\lambda))
= (2\lambda)^{-1/2}\left( \frac{R_1}{g^{-1}(\theta)} \right)^{N-1} B + (2\lambda)^{-1/2}A + o(\lambda^{-1/2}),
\]
for \( \lambda \) sufficiently large, we conclude from (5.33) that
\[
\lim_{\lambda \to \infty} \inf \lambda^{1/2}\text{dist}(G_{\lambda,A}, \Gamma_1) \geq 2^{-1/2}\left( \frac{R_1}{g^{-1}(\theta)} \right)^{N-1} B + 2^{-1/2}A.
\]
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Such estimate rapidly leads, by letting $\theta \to 0^+$, to the desired result. Namely,

$$\lim_{\lambda \to \infty} \inf \lambda^{1/2} \text{dist}(G_{\lambda A}, \Gamma_1) \geq \frac{C(F)^{1/2}}{2}.$$  

We can use the same idea to claim that

$$\lim_{\lambda \to \infty} \inf \lambda^{1/2} \text{dist}(G_{\lambda A}, \Gamma_2) \geq \frac{C(F)^{1/2}}{2},$$  

where $\Gamma_2 = \{x \in \partial A \mid |x| = R_2\}$. In fact, considering the auxiliary problem

$$-w'' = \lambda (g^{-1}(T))^{2(N-1)} f(w), \quad \theta < \rho < \eta, \quad w(\theta) = b, \quad w(\eta) = a,$$

where $\theta$ is an arbitrary fixed number so that $(\eta_2^\lambda + \eta_1^\lambda)/2 < \theta < \eta^\lambda_2$, we have that this problem has a unique positive solution $w = w_\lambda(\rho, \theta)$ for $\lambda$ large enough. Moreover, if $\rho(\lambda) := \eta_2^\lambda - (2\lambda)^{-1/2}(g^{-1}(T))^{1-N} B$, then $w_\lambda = b$ for $\theta \leq \rho \leq \rho(\lambda)$, while $a < w_\lambda(\rho) < b$ for $\rho(\lambda) < \rho < \eta^\lambda_2$. The same arguments as the above imply that

$$\text{dist}(G_{\lambda A}, \Gamma_2) \geq R_2 - g^{-1}(\rho(\lambda)).$$

Our claim can be obtained by simple calculations. (Note that we need to use the formulae of $\eta^\lambda_2$ for $N \geq 3$ and $N = 2$ given above respectively in the calculations. Moreover, we know that $\eta^\lambda_2 \to T$ as $\lambda \to \infty$ and $g^{-1}(T) \to R_2$.)

**Proof of Theorem 5.1.** For any $x_0 \in \partial \Omega$ and a ball $B$ being chosen to be tangent to $\partial \Omega$ at $x_0$ and $B \subset \Omega$, we consider the problem

$$(5.34) \quad -\Delta z = \lambda f(z) \quad \text{in } B, \quad z = 0 \quad \text{on } \partial B.$$  

The arguments similar to that in the proof of Corollary 4.3 imply that (5.34) has a unique large positive (radial) solution $z_\lambda$ for $\lambda$ sufficiently large. Lemma 5.3 implies that for $\lambda$ sufficiently large, flat core $G_{\lambda B}$ of $z_\lambda$ exists. On the other hand, we know that $z_\lambda$ is a subsolution of (1.1) by extending it to be 0 on $\Omega \setminus B$, $b$ is a supersolution of (1.1). By the arguments similar to that in the proof of Theorem 2.3, we obtain a positive solution $u_\lambda \in [z_\lambda, b]$ of (1.1). It is clear that $u_\lambda$ is the unique large positive solution of (1.1). Therefore, $u_\lambda \geq z_\lambda$ in $\Omega$ and

$$\text{dist}(x_0, G_{\lambda}) \leq \text{dist}(x_0, G_{\lambda B}).$$

Since $\text{dist}(x_0, G_{\lambda B}) = \text{dist}(G_{\lambda B}, \partial B)$, thus Lemma 5.3 implies

$$(5.35) \quad \lim_{\lambda \to \infty} \sup \lambda^{1/2} \text{dist}(x_0, G_{\lambda}) \leq \frac{C(F)^{1/2}}{2}.$$

This implies

$$(5.36) \quad \lim_{\lambda \to \infty} \sup \lambda^{1/2} \max_{x \in \partial \Omega} \text{dist}(x, G_{\lambda}) \leq \frac{C(F)^{1/2}}{2}.$$
To get the estimate of $\lim_{\lambda \to \infty} \inf \lambda^{1/2} \min_{x \in \partial \Omega} \text{dist}(x, G_\lambda)$, we construct an annulus $A_\varepsilon = \{ x \in \mathbb{R}^N \mid \hat{a} < |x - y_\varepsilon| < R_\varepsilon \}$ with $y_\varepsilon \in \mathbb{R}^N$ such that $\Omega \subset A_\varepsilon$ and $A_\varepsilon$ tangent to $\partial \Omega$ at $x_0$. Now we consider the problem

\begin{equation}
-\Delta z = \lambda f(z) \quad \text{in } A_\varepsilon, \quad z = 0 \quad \text{on } \partial A_\varepsilon.
\end{equation}

Corollary 4.3 implies that (5.37) has a unique large positive (radial) solution $z_\lambda$ for $\lambda$ sufficiently large. Lemma 5.4 implies that for $\lambda$ sufficiently large, flat core $G_{\lambda,A_\varepsilon}$ of $z_\lambda$ exists. By extending $u_\lambda$ to be 0 in $A_\varepsilon \setminus \Omega$, we easily know that $u_\lambda$ is a subsolution of (5.37). Moreover, $b$ is a supersolution of (5.37). Thus the arguments similar to that in the proof of Theorem 2.3 imply that there exists a positive solution of (5.37) in $[u_\lambda, b]$. It is clear that this solution is the unique positive large solution $z_\lambda$ of (5.37). Thus $u_\lambda \leq z_\lambda$ in $\Omega$ for $\lambda$ sufficiently large. Thus,

$$\text{dist}(x_0, G_{\lambda,A_\varepsilon}) \leq \text{dist}(x_0, G_\lambda).$$

Moreover, Lemma 5.4 implies

\begin{equation}
\lim_{\lambda \to \infty} \inf \lambda^{1/2} \text{dist}(x_0, G_\lambda) \geq \frac{C(F)^{1/2}}{2}.
\end{equation}

This also implies that

\begin{equation}
\lim_{\lambda \to \infty} \inf \lambda^{1/2} \min_{x \in \partial \Omega} \text{dist}(x, G_\lambda) \geq \frac{C(F)^{1/2}}{2}.
\end{equation}

Now our conclusions of Theorem 5.1 can be easily obtained from (5.36) and (5.39). 

\qed

References

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