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# HARDY'S INEQUALITY IN UNBOUNDED DOMAINS

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Abstract. The aim of this paper is to consider Hardy's inequality with weight on unbounded domains. In particular, using a decomposition lemma, we study the existence of a minimizer for

$$S_\varepsilon(\Omega) := \inf_{u \in D^{1,2}_\varepsilon(\Omega)} \frac{\int_\Omega |\nabla u|^2 \delta^\varepsilon \, dx}{\int_\Omega |u|^2 \delta^{\varepsilon-2} \, dx}.$$

## 1. Introduction

We shall consider an extension of the following one-dimensional Hardy's inequality:

$$\left( \int_0^\infty |u(t)|^2 t^{\varepsilon-2} \, dt \right)^{1/2} \leq \frac{2}{1-\varepsilon} \left( \int_0^\infty |u'(t)|^2 t^\varepsilon \, dt \right)^{1/2},$$

where $u' = du/dt$ and $0 \leq \varepsilon < 1$ (see [6]). Let us denote $D^{1,2}_\varepsilon(\Omega)$, the closure of $D(\Omega)$ for the following inner product:

$$(u, v) := \int_\Omega (\delta(x))^\varepsilon \nabla u \cdot \nabla v \, dx,$$

where $\Omega$ is a domain in $\mathbb{R}^N$ with non-empty boundary and where

$$\delta(x) := \operatorname{dist}(x, \partial\Omega) \quad \text{for all } x \in \mathbb{R}^N.$$

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In the rest of the present paper, we assume that $\Omega$ is a domain (bounded or not), but with compact $C^2$-boundary. Let us denote by $S_\varepsilon(\Omega)$ the best possible constant for Hardy's inequality

$$S_\varepsilon(\Omega) := \inf_{u\in D_\varepsilon^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 \delta^\varepsilon\,dx}{\int_\Omega |u|^2 \delta^{\varepsilon-2}\,dx} \tag{1}$$

and $c_\varepsilon$ the quantity $c_\varepsilon := (1-\varepsilon)^2/4$ for $0 \le \varepsilon < 1$.

The main theme of this paper is the connection between the value of $S_\varepsilon(\Omega)$ and the existence of minimizer for (1). Our main result states that $S_\varepsilon(\Omega)$ is achieved provided $S_\varepsilon(\Omega) < c_\varepsilon$. The corresponding result for $\varepsilon = 0$ is due to Marcus, Mizel and Pinchover ([5]). However our approach is completely different. As in [3], [7]–[11], we shall use a decomposition lemma. Because of the weight $\delta^\varepsilon$, we shall introduce a quantity that allows us to take simultaneously concentrations at boundary and at infinity in account. Of course, if we deal with bounded domains, the last concentration is irrelevant.

## 2. Preliminaries

Our first step will be to present four lemmas that will be required to define our functional space.

Lemma 1. *Let $u$ in $D(\mathbb{R}^+)$. Then we have the following Hardy's inequality:*

$$\int_0^\infty |u(t)|^2 t^{\varepsilon-2}\,dt \le c_\varepsilon^{-1} \int_0^\infty |u'(t)|^2 t^\varepsilon\,dt.$$

(See [6].) We also need this inequality due to Caffarelli, Kohn and Nirenberg [2]. In addition, the best constant is given in [11], using the method of Garcia and Peral ([4]).

Lemma 2. *For $N \ge 3$, $0 \le \varepsilon < 1$ we have, for all $u$ in $D(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} |u|^2 |x|^{\varepsilon-2}\,dx \le \left(\frac{2}{N-2+\varepsilon}\right)^2 \int_{\mathbb{R}^N} |\nabla u|^2 |x|^\varepsilon\,dx.$$

Throughout the rest of this paper we shall assume that

(H) $\Omega$ is a domain with compact boundary of class $C^2$.

We give now some auxiliary results due to Brezis and Marcus ([1]) that will be needed later on. For $\beta > 0$ let,

$$\Omega_\beta := \{x \in \Omega : \delta(x) < \beta\}, \quad \Sigma_\beta := \{x \in \Omega : \delta(x) = \beta\}.$$

Assuming that $\beta$ is sufficiently small, say $\beta < \beta_0$, for every $x \in \Omega_\beta$ there exists a unique point $\sigma(x) \in \Sigma := \partial\Omega$ such that $\delta(x) = |x - \sigma(x)|$. Let $\Pi : \Omega_\beta \to$

$(0,\beta)\times\Sigma$ be the mapping defined by $\Pi(x)=(\delta(x),\sigma(x))$. This mapping is a $C^2$ diffeomorphism and its inverse is given by

$$\Pi^{-1}(t,\sigma)=\sigma+t\nu(\sigma)\quad\text{for all }(t,\sigma)\in(0,\beta)\times\Sigma,$$

where $\nu(\sigma)$ is the inward unit vector normal to $\Sigma$ at $\sigma$. For $0<t<\beta_0$, let $H_t$ denote the $C^2$ diffeomorphism $\Pi^{-1}(t,\cdot)$ of $\Sigma$ onto $\Sigma_t$. We know that its Jacobian satisfies

$$|\operatorname{Jac}H_t(\sigma)-1|\le ct\quad\text{for all }(t,\sigma)\in(0,\beta_0)\times\Sigma,\tag{2}$$

where c is a constant relying only on $\Sigma$, $\beta_0$ and the choice of local coordinates. Since $\nu(\sigma)$ is orthogonal to $\Sigma_t=\Pi^{-1}(t,\Sigma)$ at $\sigma+t\nu(\sigma)$, it follows that, for every integrable non-negative function $f$ in $\Omega_\beta$,

$$\int_{\Omega_\beta}f=\int_0^\beta dt\int_{\Sigma_t}f\,d\sigma_t=\int_0^\beta dt\int_\Sigma f(t,H_t(\sigma))(\operatorname{Jac}H_t)\,d\sigma,$$

where $d\sigma$, $d\sigma_t$ denote surface elements on $\Sigma$, $\Sigma_t$, respectively. Consequently, by (2),

$$\begin{aligned}\int_\Sigma d\sigma\int_0^\beta f(t,H_t(\sigma))(1-ct)\,dt&\le\int_{\Omega_\beta}f\,dt\\&\le\int_\Sigma d\sigma\int_0^\beta f(t,H_t(\sigma))(1+ct)\,dt.\end{aligned}\tag{3}$$

Now, we are able to present the following generalization of an inequality due to Brezis and Marcus ([1]).

Lemma 3. *Under assumption* (H), *for* $0<\beta<\beta_0$, *we have the following inequality*

$$\int_{\Omega_\beta}|\nabla u|^2\delta^\varepsilon\,dx\ge(c_\varepsilon+o(1))\int_{\Omega_\beta}|u|^2\delta^{\varepsilon-2}\,dx\qquad\text{for all }u\in D(\Omega),$$

*where* $o(1)$ *is a quantity which tends to zero as* $\beta\to0$.

Proof. By (3), Lemma 1 and the fact that $|\nabla u(t,H_t(\sigma))|\ge|u'(t,H_t(\sigma))|$,

$$\begin{aligned}\int_{\Omega_\beta}|\nabla u|^2\delta^\varepsilon\,dx&\ge\int_\Sigma d\sigma\int_0^\beta|\nabla u(t,H_t(\sigma))|^2t^\varepsilon(1-ct)\,dt\\&\ge c_\varepsilon\int_\Sigma d\sigma\int_0^\beta\frac{|u(t,H_t(\sigma))|^2}{t^{2-\varepsilon}}(1-ct)\,dt\\&=c_\varepsilon\int_\Sigma d\sigma\int_0^\beta\frac{|u(t,H_t(\sigma))|^2}{t^{2-\varepsilon}}(1+ct)\,dt\\&\quad-c_\varepsilon\int_\Sigma d\sigma\int_0^\beta\frac{|u(t,H_t(\sigma))|^2}{t^{2-\varepsilon}}2ct\,dt\end{aligned}$$

$$\begin{aligned}&\geq c_\varepsilon \int_\Sigma d\sigma \int_0^\beta \frac{|u(t,H_t(\sigma))|^2}{t^{2-\varepsilon}}(1+ct)\,dt\\&\quad - 2c\beta c_\varepsilon \int_\Sigma d\sigma \int_0^\beta \frac{|u(t,H_t(\sigma))|^2}{t^{2-\varepsilon}}\,dt\\&\geq c_\varepsilon \int_{\Omega_\beta} |u|^2\delta^{\varepsilon-2}\,dx + \frac{(-2c\beta c_\varepsilon)}{(1-c\beta)}\int_{\Omega_\beta}|u|^2\delta^{\varepsilon-2}\,dx,\end{aligned}$$

using (2) and again (3). $\square$

Lemma 4. *There exists $c > 0$ such that*

$$\int_\Omega |u|^2\delta^{\varepsilon-2}\,dx \leq c\int_\Omega |\nabla u|^2\delta^\varepsilon\,dx \quad \textit{for all } u \in D(\Omega).$$

Proof. As in Lemma 3, take $0 < \beta < \beta_0$ and let us divide $\Omega$ into three parts:

$$\Omega_\beta, \Omega_R := \Omega \setminus \overline{B(0,5R)} \quad \text{and} \quad K := \Omega \setminus (\Omega_\beta \cup \Omega_R),$$

where $R$ is taken sufficiently large to have $\partial\Omega \subset B(0,R)$. On $\Omega_R$ and $\Omega_\beta$ desired inequalities follow respectively from Lemma 2 and Lemma 3. On the compact set $K$, we just have to use the Poincare's inequality (see [9] or [10]) and the fact that the minimum value of $\delta(x)$ is achieved in K. $\square$

Definition 1. For $N \geq 3$ and for $\Omega \subset \mathbb{R}^N$, let $D^{1,2}_\varepsilon(\Omega)$ be the completion of $D(\Omega)$ with respect to the inner product

$$(u,v) := \int_\Omega \delta^\varepsilon \nabla u \cdot \nabla v\,dx.$$

Finally, let us recall that

$$S_\varepsilon(\Omega) := \inf_{u\in D^{1,2}_\varepsilon(\Omega)} \frac{\int_\Omega |\nabla u|^2\delta^\varepsilon\,dx}{\int_\Omega |u|^2\delta^{\varepsilon-2}\,dx}$$

and, by Lemma 4, $S_\varepsilon(\Omega) > 0$.

## 3. Minimizing sequences for $S_\varepsilon(\Omega)$

In order to prove that $S_\varepsilon(\Omega)$ is achieved if $S_\varepsilon(\Omega) < c_\varepsilon$, we can consider an arbitrary minimizing sequence $(u_n) \subset D^{1,2}_\varepsilon(\Omega)$:

$$|\delta^{(\varepsilon-2)/2}u_n|_2 = 1, \quad |\delta^{\varepsilon/2}\nabla u_n|_2^2 \to S_\varepsilon(\Omega), \quad n\to\infty. \tag{4}$$

Going if necessary to a subsequence, we may assume $u_n \rightharpoonup u$ in $D^{1,2}_\varepsilon(\Omega)$, so that

$$|\delta^{\varepsilon/2}\nabla u|_2^2 \leq \underline{\lim}|\delta^{\varepsilon/2}\nabla u_n|_2^2 = S_\varepsilon(\Omega).$$

Hence u is a minimizer provided $|\delta^{(\varepsilon-2)/2}u|_2 = 1$. But we know only that $|\delta^{(\varepsilon-2)/2}u|_2 \leq 1$.

Lemma 5. *Under assumption* (H), *let* $(u_n) \subset D^{1,2}_\varepsilon(\Omega)$ *be a sequence such that*

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } D^{1,2}_\varepsilon(\Omega), \\ |\nabla(u_n - u)|^2\delta^\varepsilon &\rightharpoonup \mu \quad \text{in } M(\Omega), \\ u_n &\to u \quad \text{a.e. on } \Omega. \end{aligned}$$

*Define*

$$\mu_{B,\infty} := \lim_{\beta\to 0^+} \overline{\lim_{n\to\infty}} \bigg( \int_{\Omega_\beta} |\nabla u_n|^2 \delta^\varepsilon\,dx + \int_{|x|>\beta^{-1}} |\nabla u_n|^2\delta^\varepsilon\,dx \bigg), \tag{5}$$

$$\nu_{B,\infty} := \lim_{\beta\to 0^+} \overline{\lim_{n\to\infty}} \bigg( \int_{\Omega_\beta} |u_n|^2 \delta^{\varepsilon-2}\,dx + \int_{|x|>\beta^{-1}} |u_n|^2\delta^{\varepsilon-2}\,dx \bigg). \tag{6}$$

*It follows that*

$$\nu_{B,\infty} \le c_\varepsilon^{-1}\mu_{B,\infty}, \tag{7}$$

$$\overline{\lim_{n\to\infty}} |\delta^{\varepsilon/2}\nabla u_n|_2^2 = |\delta^{\varepsilon/2}\nabla u|_2^2 + \mu_{B,\infty} + \|\mu\|, \tag{8}$$

$$\overline{\lim_{n\to\infty}} |\delta^{(\varepsilon-2)/2} u_n|_2^2 = |\delta^{(\varepsilon-2)/2} u|_2^2 + \nu_{B,\infty}. \tag{9}$$

Proof. (a) Assume first $u = 0$. Take $0 < \beta < \beta_0/2$ such that $\partial\Omega \subseteq B(0,\beta^{-1})$ and let $\Psi_\beta \in C^\infty(\mathbb{R}^N)$, $0 \le \Psi_\beta(x) \le 1$ on $\Omega$, be such that

- $\Psi_\beta(x) = 1$, for $|x| > 5\beta^{-1} + 1$ and for $x \in \Omega_\beta$,
- $\Psi_\beta(x) = 0$, for $x \in \Omega \setminus \Omega_{2\beta}$ such that $|x| < 5\beta^{-1}$.

$$\begin{aligned} \int |\Psi_\beta(x) u_n|^2 \delta^{\varepsilon-2}\,dx &= \int_{\Omega_{2\beta}} |\Psi_\beta u_n|^2\delta^{\varepsilon-2}\,dx + \int_{|x|\ge 5\beta^{-1}} |\Psi_\beta u_n|^2 \delta^{\varepsilon-2}\,dx \\ &\le \int_{\Omega_{2\beta}} |\Psi_\beta u_n|^2\delta^{\varepsilon-2}\,dx + \int_{|x|\ge 5\beta^{-1}} |\Psi_\beta u_n|^2 (|x| - \beta^{-1})^{\varepsilon-2}\,dx \end{aligned}$$

because if $|x| \ge 5\beta^{-1}$ then $\delta(x)/2 \le |x| - \beta^{-1} \le \delta(x)$. So we have

$$\begin{aligned} \int |\Psi_\beta u_n|^2\delta^{\varepsilon-2}\,dx &\le (c_\varepsilon + o(1))^{-1} \int_{\Omega_{2\beta}} |\nabla \Psi_\beta u_n|^2 \delta^\varepsilon\,dx \\ &\quad + \bigg(\frac{2}{N-2+\varepsilon}\bigg)^2 \int_{|x|\ge 5\beta^{-1}} |\nabla\Psi_\beta u_n|^2\delta^\varepsilon\,dx \\ &\le (c_\varepsilon + o(1))^{-1} \int |\nabla \Psi_\beta u_n|^2\delta^\varepsilon\,dx, \end{aligned}$$

by Lemmas 2 and 3. On the other hand, we have

$$\begin{aligned} &\overline{\lim_{n\to\infty}} \int |\nabla\Psi_\beta u_n|^2\delta^\varepsilon\,dx \\ &= \overline{\lim_{n\to\infty}} \bigg( \int |\nabla u_n|^2 \Psi_\beta^2 \delta^\varepsilon\,dx + \int |\Psi_\beta|^2 u_n^2\delta^\varepsilon\,dx + 2\int (\nabla\Psi_\beta\cdot\nabla u_n)\Psi_\beta u_n\delta^\varepsilon\,dx \bigg) \\ &= \overline{\lim_{n\to\infty}} \int |\nabla u_n|^2\Psi_\beta^2\delta^\varepsilon\,dx \end{aligned}$$

because $u_n \to 0$ in $L^2_{\mathrm{Loc}}$. Consequently, we obtain

$$\overline{\lim_{n\to\infty}} \int |u_n|^2 \Psi_\beta^2 \delta^{\varepsilon-2}\, dx \le (c_\varepsilon + o(1))^{-1} \overline{\lim_{n\to\infty}} \int |\nabla u_n|^2 \Psi_\beta^2 \delta^\varepsilon\, dx. \tag{10}$$

Letting $\omega_1 := \Omega_\beta \cup (\mathbb{R}^N \setminus \overline{B(0, 5\beta^{-1}+1)})$ and $\omega_2 := \Omega_{2\beta} \cup (\mathbb{R}^N \setminus \overline{B(0, 5\beta^{-1})})$, it becomes obvious that

$$\int_{\omega_1} |\nabla u_n|^2 \delta^\varepsilon\, dx \le \int |\nabla u_n|^2 \Psi_\beta^2 \delta^\varepsilon\, dx \le \int_{\omega_2} |\nabla u_n|^2 \delta^\varepsilon\, dx,$$
$$\int_{\omega_1} |u_n|^2 \delta^{\varepsilon-2}\, dx \le \int |u_n|^2 \Psi_\beta^2 \delta^{\varepsilon-2}\, dx \le \int_{\omega_2} |u_n|^2 \delta^{\varepsilon-2}\, dx.$$

We obtain from (5) and (6)

$$\mu_{B,\infty} = \lim_{\beta\to 0^+} \overline{\lim_{n\to\infty}} \int |\nabla u_n|^2 \Psi_\beta^2 \delta^\varepsilon\, dx,$$
$$\nu_{B,\infty} = \lim_{\beta\to 0^+} \overline{\lim_{n\to\infty}} \int |u_n|^2 \Psi_\beta^2 \delta^{\varepsilon-2}\, dx.$$

Inequality (7) follows directly from (10).

(b) Let us now consider the general case. For more convenience, let us write $\Omega'_\beta := \Omega_\beta \cup (\mathbb{R}^N \setminus \overline{B(0, \beta^{-1})})$ and $v_n := u_n - u$. Since

$$\overline{\lim_{n\to\infty}} \int_{\Omega'_\beta} |\nabla v_n|^2 \delta^\varepsilon\, dx = \overline{\lim_{n\to\infty}} \int_{\Omega'_\beta} |\nabla u_n|^2 \delta^\varepsilon\, dx - \int_{\Omega'_\beta} |\nabla u|^2 \delta^\varepsilon\, dx,$$

we obtain

$$\lim_{\beta\to 0^+} \overline{\lim_{n\to\infty}} \int_{\Omega'_\beta} |\nabla v_n|^2 \delta^\varepsilon\, dx = \mu_{B,\infty}.$$

By the Brézis–Lieb Lemma (see [9] or [10]), we have

$$\int_{\Omega'_\beta} |u|^2 \delta^{\varepsilon-2}\, dx = \lim_{n\to\infty} \left( \int_{\Omega'_\beta} |u_n|^2 \delta^{\varepsilon-2}\, dx - \int_{\Omega'_\beta} |v_n|^2 \delta^{\varepsilon-2}\, dx \right),$$

so

$$\lim_{\beta\to 0^+} \overline{\lim_{n\to\infty}} \int_{\Omega'_\beta} |v_n|^2 \delta^{\varepsilon-2}\, dx = \nu_{B,\infty}.$$

Inequality (7) follows directly from the corresponding inequality for $(v_n)$.

(c) Since $v_n \rightharpoonup 0$ in $D^{1,2}_\varepsilon(\Omega)$, we have

$$|\nabla u_n|^2 \delta^\varepsilon \rightharpoonup \mu + |\nabla u|^2 \delta^\varepsilon \quad \text{in } M(\Omega).$$

Again by Brézis–Lieb Lemma, we have for every nonnegative $h \in D(\Omega)$

$$\int |hu|^2 \delta^{\varepsilon-2}\, dx = \lim_{n\to\infty} \left( \int |hu_n|^2 \delta^{\varepsilon-2}\, dx - \int |hv_n|^2 \delta^{\varepsilon-2}\, dx \right)$$

hence

$$|u_n|^2 \delta^{\varepsilon-2} \rightharpoonup |u|^2 \delta^{\varepsilon-2} \quad \text{in } M(\Omega),$$

because $u_n \to 0$ in $L^2_{\mathrm{Loc}}$. Let us take $\beta$ as in (a). We have

$$\begin{aligned}\overline{\lim_{n\to\infty}}\int|\nabla u_n|^2\delta^\varepsilon\,dx &= \overline{\lim_{n\to\infty}}\bigg(\int\Psi_\beta^2|\nabla u_n|^2\delta^\varepsilon\,dx+\int(1-\Psi_\beta^2)|\nabla u_n|^2\delta^\varepsilon\,dx\bigg)\\ &= \overline{\lim_{n\to\infty}}\int\Psi_\beta^2|\nabla u_n|^2\delta^\varepsilon\,dx+\int(1-\Psi_\beta^2)\,d\mu\\ &\quad+\int(1-\Psi_\beta^2)|\nabla u|^2\delta^\varepsilon\,dx.\end{aligned}$$

When $\beta \to 0$, we obtain by Lebesgue's theorem

$$\overline{\lim_{n\to\infty}}\int|\nabla u_n|^2\delta^\varepsilon\,dx=\mu_{B,\infty}+\|\mu\|+|\delta^{\varepsilon/2}\nabla u|_2^2.$$

The proof of (9) is similar. $\square$

THEOREM 6. *Let $N \geq 3$, $\Omega \subseteq \mathbb{R}^N$ satisfying assumption* (H) *and* $(u_n) \subset D^{1,2}_\varepsilon(\Omega)$ *be a minimizing sequence for $S_\varepsilon(\Omega)$ satisfying* (4). *If $S_\varepsilon(\Omega) < c_\varepsilon$ then $(u_n)$ contains a convergent subsequence. In particular, there exists a minimizer for $S_\varepsilon(\Omega)$.*

PROOF. Since $(u_n)$ is bounded in $D^{1,2}_\varepsilon(\Omega)$ we may assume, going if necessary to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u &&\text{in } D^{1,2}_\varepsilon(\Omega),\\ |\nabla(u_n-u)|^2\delta^\varepsilon &\rightharpoonup \mu &&\text{in } M(\Omega),\\ u_n &\to u &&\text{a.e. on } \Omega.\end{aligned}$$

By the preceding lemma

$$S_\varepsilon(\Omega)=|\delta^{\varepsilon/2}\nabla u|_2^2+\mu_{B,\infty}+\|\mu\|, \tag{11}$$

$$1=|\delta^{(\varepsilon-2)/2}u|_2^2+\nu_{B,\infty}. \tag{12}$$

We deduce from (7), (11) and the definition of $S_\varepsilon(\Omega)$

$$S_\varepsilon(\Omega)\geq S_\varepsilon(\Omega)|\delta^{(\varepsilon-2)/2}u|_2^2+c_\varepsilon\nu_{B,\infty}.$$

It follows by (12) that $\nu_{B,\infty} = 0$ because $S_\varepsilon(\Omega) < c_\varepsilon$. Hence we have proved that $|\delta^{(\varepsilon-2)/2}u|_2^2 = 1$ and so

$$|\delta^{\varepsilon/2}\nabla u|_2^2=S_\varepsilon(\Omega)=\lim|\delta^{\varepsilon/2}\nabla u_n|_2^2. \qquad\square$$

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