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NABLA THEOREMS AND MULTIPLE SOLUTIONS FOR SOME NONCOOPERATIVE ELLIPTIC SYSTEMS

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Abstract. We study some variational principles which imply the existence of multiple critical points for a functional f, using the properties of both f and ∇f on some suitable sets. We derive some multiplicity theorems for a certain class of strongly indefinite functionals and we apply these results for finding multiple solutions of an elliptic system of reaction-diffusion type.

1. Introduction

In the study of nonlinear differential partial equations and systems of variational type it sometimes happens that the topological structure of the sublevels of the functional associated with the problem gives satifactory estimates of the number of solutions only in the "generic case". Morse theory, for instance, allows to estimate in a precise way the number of critical points of a functional f provided it is a priori known that they are all non-degenerate. This condition is usually very hard to check, nevertheless, in some problems, other types of conditions on the gradient of f have proved themselves useful. In [9] the authors give some variatonal theorems of "mixed type" (which we now call ∇-theorems), which prove multiplicity results for critical points of a functional f by means of some properties both of the gradient of f and on the sublevels of f (the latter, by themselves, would not be enough). The main idea in proving such theorems is

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the constuction of another functional g which has more complex sublevels (hence the multiplicity result for g) and is such that its critical points are also critical poins for f (thanks to the assumptions on the gradient). These theorems have been applied in [9] to problems of elliptic equations with jumping nonlinearities or to variational inequalities (see [6]).

In this paper we are concerned with the following elliptic system

$$\text{(ES)} \qquad \begin{cases} -\Delta u = \alpha u - \delta v + F_r(x,u,v), \\ \Delta v = -\delta u - \gamma v + F_s(x,u,v), \\ u, v \in W_0^{1,2}(\Omega), \end{cases}$$

where Ω is a bounded subset of $\mathbb{R}^N$, $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function such that $F(x,0,0) = 0$, $F'(x,0,0) = 0$, $F''(x,0,0) = 0$ and F is superquadratic at infinity. Nontrivial solutions of (ES) have been found in [8], [2] (see also the references therein). As well known the solutions of (ES) are the critical points of the functional $I : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ defined by

$$I(u,v) = \frac{1}{2} \int_\Omega [(|Du|^2 - \alpha u^2) + 2\delta uv - (|Dv|^2 - \gamma v^2)]\, dx - \int_\Omega F(x,u,v)\, dx.$$

For suitable values of the parameters α, γ and δ we show that I satisfies a set of inequalities analogous to those which occur in jumping problems; in this case however an additional technical difficulty shows up, since the quadratic term in I has infinitely many positive eigenvalues and infinitely many negative ones (this is commonly referred as I being a "strongly indefinite" functional). To face this fact we have adopted the point of view of [1], [5], using the notion of limit relative category (which is recalled in the appendix) and we have developed a "limit" version of the ∇-theorems, which fits the situation we are confronted with. This is done in Section 2.

In Section 3 we present an abstract framework for functionals which are the sum of a quadratic form and a superquadratic nonlinear term. This framework is used in Section 4, and allows us to find other nontrivial solutions of (ES)

2. The ∇-theorems

In this section we formalize the ideas predented in the introduction. The main results are Theorems 2.5 and 2.10 which extend some of the ∇-theorems stated in [9]. In our case the properties of g are shown in the following theorem. First we need some notation.

Notation 2.1. As in the previous section we consider a Hilbert space H. Moreover, we consider three closed subspaces X_1, X_2 and X_3, such that $H = X_1 \oplus X_2 \oplus X_3$. We assume, for simplicity, X_1, X_2 and X_3 to be mutually orthogonal and denote by P_1, P_2 and P_3 the associated orthogonal projections.

We shall use the following sets

$$
\begin{aligned}
\Delta_{R,R_1,R_2} &= \{u \in X_1 \oplus X_2 \mid \|P_1 u\| \le R,\ R_1 \le \|P_2 u\| \le R_2\},\\
T_{R,R_1,R_2} &= \{u \in X_1 \oplus X_2 \mid \|P_1 u\| = R,\ R_1 \le \|P_2 u\| \le R_2\}\\
&\quad \cup \{u \in X_1 \oplus X_2 \mid \|P_1 u\| \le R,\ \|P_2 u\| = R_1\}\\
&\quad \cup \{u \in X_1 \oplus X_2 \mid \|P_1 u\| \le R,\ \|P_2 u\| = R_2\},\\
S_\rho &= \{u \in X_2 \oplus X_3 \mid \|u\| = \rho\},
\end{aligned}
$$

where $0 \le R_1 \le R_2$, $R \ge 0$ and $\rho > 0$ are real numbers. Furthermore, we consider the set

$$C = \{u \in H \mid \|P_2 u\| \ge 1\}.$$

Finally, given a non negative integer h we denote by $\mathcal{B}^{h+1}$ the ball in $\mathbb{R}^{h+1}$ and by $\mathcal{S}^h$ its boundary.

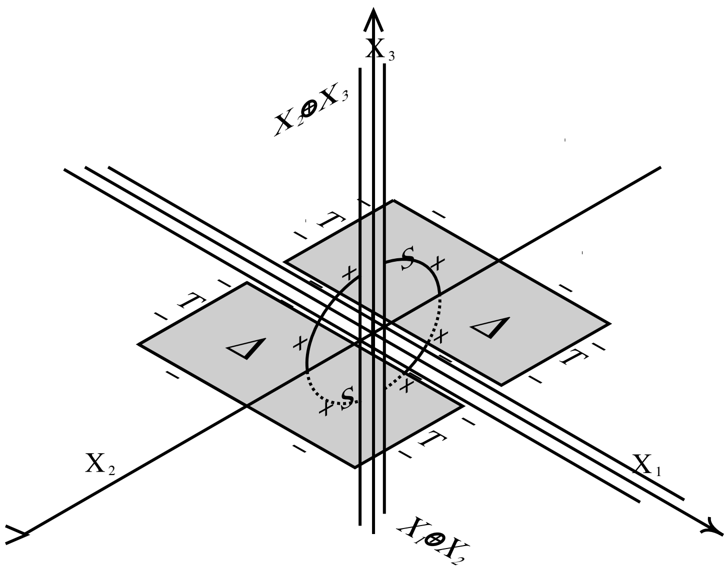

Figure 1. The topological situation of Teorems 2.2 and 2.7

Theorem 2.2. *Let $g : C \to \mathbb{R}$ be a $\mathcal{C}^{1,1}$-function. Let R, R_1, R_2 and ρ be such that* $1 \le R_1 < \rho < R_2$, $R > 0$ *and let* $\Delta = \Delta_{R,R_1,R_2}$, $T = T_{R,R_1,R_2}$, $S = S_\rho$. *Assume that* $\dim(X_1) < \infty$, $1 \le \dim(X_2) < \infty$ *and*

$$\sup g(T) < \inf g(S \cap C)$$

(*see Figure* 1). *Set* $a = \inf g(S)$, $b = \sup g(\Delta)$ *and suppose* $b < \infty$. *Then*

$$\operatorname{cat}_{C,T}(g^b) \ge 2, \tag{2.2.1}$$

$$\textit{if } \sup g(T) \le \beta < a, \textit{ then } \operatorname{cat}_{C,T}(g^\beta) = 0. \tag{2.2.2}$$

Finally, if $(\mathrm{PS})_c$ *holds for all* c *in* $[a,b]$, *then* g *has at least two lower critical points in* $g^{-1}([a,b])$.

Proof. It is easy to see that there exists a retraction from C to Δ, which keeps T fixed, hence

$$\mathrm{cat}_{C,T}(g^b) \geq \mathrm{cat}_{C,T}(\Delta) = \mathrm{cat}_{\Delta,T}(\Delta).$$

Moreover, the pair (Δ, T) is homeomorphic to the pair $(\mathcal{B}^{n+1} \times \mathcal{S}^m, \mathcal{S}^n \times \mathcal{S}^m)$ (as one can easily check). From Lemma 2.3 which follows, we obtain $\mathrm{cat}_{\Delta,T}(\Delta) = 2$, hence (2.2.1) holds.

Up to some standard work, it can be proved that T is a deformation retract of $C \setminus S$. If $\sup g(T) \leq \beta < a$, then $T \subset g^\beta \subset C \setminus S$, therefore $\mathrm{cat}_{C,T}(b^\beta) = 0$.

The remaining part of the thesis follows easily from Theorem 5.5. □

Lemma 2.3. *For any non negative integers* h, k *we have:*

$$\mathrm{cat}_{\mathcal{B}^{h+1} \times \mathcal{S}^k, \mathcal{S}^h \times \mathcal{S}^k}(\mathcal{B}^{h+1} \times \mathcal{S}^k) = 2.$$

Proof. As shown for instance in [5]

$$\mathrm{cuplength}\,(\mathcal{B}^{h+1} \times \mathcal{S}^k, \mathcal{S}^h \times \mathcal{S}^k) = \mathrm{cuplength}\,(\mathcal{B}^{h+1}, \mathcal{S}^h) + \mathrm{cuplength}\,(\mathcal{S}^k) = 1.$$

The conclusion follows from Theorem 5.2. □

Let $f : H \to \mathbb{R}$ be a $\mathcal{C}^1$-function. The property which allows to reduce the study of the critical points of f to the study of topologically richer functional is expressed by the following condition.

Definition 2.4. Let X be a closed subspace of H and c be a real number. We say that f satisfies the condition $\nabla(X, c)$ if there exists $\delta > 0$ such that

$$\inf\{\|P_{X\oplus[u]}\mathrm{grad}\, f(u)\| \mid u \in H,\ \mathrm{dist}(u, X) < \delta,\ |f(u) - c| < \delta\} > 0$$

where $[u] = \mathrm{span}\,(u)$ and by $P_{X\oplus[u]}$ we denote the orthogonal projection onto $X \oplus [u]$.

This conditions means that there are no critical points at level c for the restriction of f on X, *up to some uniformity*. Actually is not difficult to see that $\nabla(X, c)$ is equivalent to the following pair of conditions:

- $f|_X$ has no critical points u in X with $f(u) = c$,
- if $(u_n)_n$ is a sequence in H such that $\mathrm{dist}(u_n, X) \to 0$, $f(u_n) \to c$ and $P_{X\oplus[u_n]}\mathrm{grad}\, f(u_n) \to 0$, then $(u_n)_n$ admits a subsequence which converges.

We point out that the above condition is weaker than the one introduced in [9], due to the presence of $P_{X\oplus[u_n]}$ instead of P_X, so Theorem 2.5 which follows is more general than the corresponding one in [9]. This is useful, for instance, in applications where superlinear nonlinearities occur.

Theorem 2.5 (Torus-sphere linking). *Let $f : H \to \mathbb{R}$ be a $\mathcal{C}^1$-function. Assume that* $\dim(X_1 \oplus X_2) < \infty$, $\dim X_2 \geq 1$ *and*

$$\sup f(T) < \inf f(S)$$

where $T = T_{R,R_1,R_2}$, $S = S_\rho$, $0 \leq R_1 < \rho < R_2$ and $R > 0$ (see Figure 2). Let $a = \inf f(S)$ and $b = \sup f(\Delta)$ ($\Delta = \Delta_{R,R_1,R_2}$). Assume that $b < \infty$ and

- $(\mathrm{PS})_c$ *holds for any c in $[a,b]$,*
- $\nabla(X_1 \oplus X_3, c)$ *holds for any c in $[a,b]$.*

Then f has at least two critical points in $f^{-1}([a,b])$.

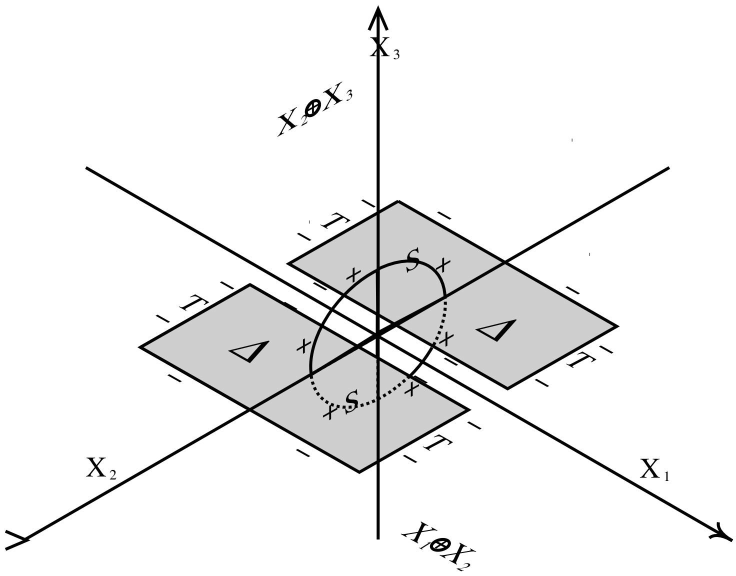

Figure 2. The topological situation of Teorems 2.5 and 2.10

Proof. The proof goes through several steps.

(I) We define the map $\Phi : H \setminus X_1 \oplus X_3 \to H$ setting

$$\Phi(z) = z - \frac{P_2 z}{\|P_2 z\|} \quad \text{for } z \notin X_1 \oplus X_3,$$

and the map $g : C \to \mathbb{R}$ by $g = f \circ \Phi$. It turns out that

$$
\text{(2.5.1)}\quad \operatorname{grad}^-_C g(z) = \begin{cases} P_z(\operatorname{grad} f)(\Phi(z)) \\ \quad + \left(1 - \dfrac{1}{\|P_2 z\|}\right) Q_z(\operatorname{grad} f)(\Phi(z)) & \text{if } z \in \operatorname{int}(C), \\ P_z(\operatorname{grad} f)(\Phi(z)) \\ \quad - \left\langle (\operatorname{grad} f)(\Phi(z)), \dfrac{P_2 z}{\|P_2 z\|} \right\rangle^+ \dfrac{P_2 z}{\|P_2 z\|} & \text{if } z \in \partial C, \end{cases}
$$

where P_z denotes the orthogonal projection onto $X_1 \oplus X_3 \oplus [z]$ and $Q_z = \mathrm{Id} - P_z$ (for the notion of lower gradient $\operatorname{grad}_C g$ see Definition 5.3).

It is clear that g satisfies to the inequalities of Theorem 2.2 on the sets $T' = \Phi^{-1}(T)$ and $S' = \Phi^{-1}(S)$.

(II) Now we prove that g verifies $(\mathrm{PS})_c$ for all c in $[a, b]$. Let $(z_n)_n$ be a sequence in C such that $g(z_n) \to c$ and $\operatorname{grad}^-_C g(z_n) \to 0$. Set $u_n = \Phi(z_n)$, it is clear that $f(u_n) \to c$. We claim that $\inf \|P_2 z_n\| > 1$. By contradiction if $\|P_2 z_n\| \to 1$, then $\operatorname{dist}(u_n, X_1 \oplus X_3) \to 0$ and, using (2.5.1), $P_{X_1 \oplus X_3 \oplus [u_n]} \operatorname{grad} f(u_n) \to 0$ (notice that if $u_n \in X_1 \oplus X_3$, then $P_{X_1 \oplus X_3 \oplus [u_n]}$ is just $P_{X_1 \oplus X_3}$). By $\nabla(X_1 \oplus X_3, c)$ this is impossible. This implies $\inf \operatorname{dist}(u_n, X_1 \oplus X_3) > 0$, hence $\operatorname{grad} f(u_n) \to 0$, that is $(u_n)_n$ is a Palais–Smale sequence for f. By $(\mathrm{PS})_c$ for f $(u_n)_n$ has a convergent subsequence and it is easy to check that the same property holds for $(z_n)_n$.

(III) Using Theorem 2.2 we find two lower critical points z_1 and z_2 for g, with $a \le g(z_i) \le b$, $i = 1, 2$. Using property ∇ again we get that $z_i \notin \partial C$, $i = 1, 2$; then (as one can easily prove) $u_i = \Phi(z_i)$, $i = 1, 2$, are critical points for f with $a \le f(u_i) \le b$. □

Now we need some versions of Theorem 2.2 and Theorem 2.5 in the case $\dim(X_1) = \infty$. To this aim we shall prove some limit version of the previous theorems, using Theorem 5.8 instead of Theorem 5.5.

Notation 2.6. We consider now a sequence $(H_n)_n$ of closed subspaces of H, of finite dimension and such that $X_2 \subset H_n$ for all n. We also assume $H_n \subset H_{n+1}$ and $\bigcup_{n \in \mathbb{N}} H_n$ to be dense in H. We denote by P_{H_n} the orthogonal projection onto H_n. We set also $C_n = C \cap H_n$. It is not difficult to see that, since $X_2 \subset H_n$, $P_2 P_{H_n} = P_{H_n} P_2 = P_2$, hence $C_n = \{u \in H_n \mid \|P_2 u\| \ge 1\}$.

Theorem 2.7. *Let $g : C \to \mathbb{R}$ be a $\mathcal{C}^{1,1}$-function. Let R, R_1, R_2 and ρ be such that $1 \le R_1 < \rho < R_2$, $R > 0$ and let $\Delta = \Delta_{R,R_1,R_2}$, $T = T_{R,R_1,R_2}$ and $S = S_\rho$. Assume that $1 \le \dim(X_2) < \infty$ and $\sup g(T) < \inf g(S \cap C)$. Set $a = \inf g(S)$, $b = \sup g(\Delta)$. Let $b < \infty$ and assume that $(\mathrm{PS})^*_c$ with respect to $(C_n)_n$ holds for all c in $[a, b]$. Then g has at least two lower critical points in $g^{-1}([a, b])$.*

Proof. Set $T_n = T \cap H_n$, $\Delta_n = \Delta \cap H_n$ and $S_n = S \cap H_n$; it turns out that $\sup g(T_n) < \inf g(S_n)$ for all n. Then, using (2.2.1) and (2.2.2), we obtain that

$$\operatorname{cat}_{C_n,T_n}(g^b \cap C_n) \geq 2,$$
$$\text{if } \sup g(T) \leq \beta < a \text{ then } \operatorname{cat}_{C_n,T_n}(g^\beta \cap C_n) = 0,$$

for all n. This implies

$$\operatorname{cat}^*_{C,T}(g^b) \geq 2,$$
$$\text{if } \sup g(T) \leq \beta < a \text{ then } \operatorname{cat}^*_{C,T}(g^\beta) = 0.$$

and applying Theorem 5.8 we get the conclusion. □

We want now to consider a theorem analogous to Theorem 2.5 with $\dim X_1 = \infty$. To this aim we introduce a suitable adaptation of the condition (∇).

Definition 2.8. Let X be a closed subspace of H such that $P_X P_{H_n} = P_{H_n} P_X$ for all n, where P_X denotes the orthogonal projection onto X. Let c be a real number. We say that f satisfies the condition $\nabla^*(X, c)$ with respect to $(H_n)_n$, if there exists $\delta > 0$ such that

$$\liminf_{n\to\infty}\{\|P_{H_n} P_{X\oplus[u]} \operatorname{grad} f(u)\| \mid u \in H_n,\ \operatorname{dist}(u, X) < \delta,\ |f(u) - c| < \delta\} > 0$$

(as before $[u] = \operatorname{span}(u)$ and $P_{X\oplus[u]}$ is the projection onto $X \oplus [u]$).

It can be easily seen that, under the above made assumptions, $\nabla^*(X, c)$ is equivalent to the following pair of conditions:

- $f|_X$ has no critical points u in X with $f(u) = c$,
- if $(h_n)_n$ is a sequence in $\mathbb{N}$ such that $h_n \to \infty$, $(u_n)_n$ is a sequence in H such that $u_n \in H_{h_n}$ for all n, $\operatorname{dist}(u_n, X) \to 0$, $f(u_n) \to c$ and $P_{H_{h_n}} P_{X\oplus[u_n]} \operatorname{grad} f(u_n) \to 0$, then $(u_n)_n$ admits a subsequence which converges.

Lemma 2.9. *Let X be a closed subspace of H and denote by P_X, Q_X the orthogonal projections onto $X, X^\perp$ respectively. Assume that $P_X P_{H_n} = P_{H_n} P_X$ for all n. Let c be a real number and let f satisfy* $(\mathrm{PS})^*_c$ *and $\nabla^*(X, c)$ with respect to $(H_n)_n$. Define the set $C = \{z \in H \mid \|Q_X z\| \geq 1\}$ and the map $\Phi : H\backslash X \to H$, by*

$$\Phi(z) = z - \frac{Q_X z}{\|Q_X z\|}.$$

Then

(1) *the function $g = (f \circ \Phi)|_C$ verifies the condition* $(\mathrm{PS})^*_c$ *with respect to the sequence $(C_n)_n = (C \cap H_n)_n$,*
(2) *there are no critical points z for g such that $g(z) = c$ and $z \in \partial C$.*

Proof. We prove the first claim. Let $(h_n)_n$ be a sequence in $\mathbb{N}$ with $h_n \to \infty$ and $(z_n)_n$ be a sequence in C with $z_n \in C_{h_n}$ for all n, $g(z_n) \to c$ and

$\operatorname{grad}^-_{C_{h_n}} g(z_n) \to 0$. Set $u_n = \Phi(z_n)$; obviously $f(u_n) \to c$. We first consider the case $z_n \notin \partial C_{h_n}$ for n large. Notice that, for n, h in $\mathbb{N}$:

$$\operatorname{grad}^-_{C_h} g(z_n) = P_{H_h}\bigg(P_{X\oplus[z_n]}\operatorname{grad} f(u_n) \\ + \bigg(1 - \frac{1}{\|Q_X(z_n)\|}\bigg)P_{(X\oplus[z_n])^\perp}\operatorname{grad} f(u_n)\bigg).$$

Using the commutation properties of the projections one can easily deduce that

$$\text{(2.9.1)} \qquad P_{H_{h_n}} P_{X\oplus[u_n]}\operatorname{grad} f(u_n) \to 0 \quad \text{and} \\ \bigg(1 - \frac{1}{\|Q_X(z_n)\|}\bigg) P_{H_{h_n}} P_{(X\oplus[u_n])^\perp}\operatorname{grad} f(u_n) \to 0.$$

It is not possible that $\|Q_X(z_n)\| \to 1$ because in this case $\operatorname{dist}(u_n, X) \to 0$ and the sequence $(u_n)_n$ would contradict $\nabla^*(X, c)$. So by (2.9.1) we get

$$P_{H_{h_n}}\operatorname{grad} f(u_n) \to 0.$$

Using $(\mathrm{PS})^*_c$ for f it follows that $(u_n)_n$ admits a subsequence $(u_{k_n})_n$ such that $u_{k_n} \to u$ for some u in $H \setminus X$. Since Φ is invertible in $\operatorname{int}(C)$, $z_{k_n} \to \Phi^{-1}(u)$. It remains to consider the case $z_n \in \partial C_{h_n}$ for infinitely many n. We claim that this case cannot occur. Actually if $z_n \in \partial C_{h_n}$ we have:

$$\operatorname{grad}^-_{C_{h_n}} g(z_n) = P_{H_{h_n}}(P_{X\oplus[z_n]}\operatorname{grad} f(u_n) - \langle\operatorname{grad} f(u_n), Q_X z_n\rangle^+ Q_X z_n).$$

Using again the properties of the projections we get $P_{H_{h_n}} P_X \operatorname{grad} f(u_n) \to 0$ and again this fact contradicts the property $\nabla^*(X, c)$.

To have the second claim proved, just notice that, for z in ∂C, $\operatorname{grad}^- g(z) = P_{X\oplus[z]}\operatorname{grad} f(\Phi(z)) - \langle\operatorname{grad} f(\Phi(z)), Q_X z\rangle\, Q_X z$. □

Now we can state the main theorem of this section.

Theorem 2.10. *Let $f : H \to \mathbb{R}$ be a $\mathcal{C}^{1,1}$ function. Assume that $1 \le \dim X_2 < \infty$ and*

$$\sup f(T) < \inf f(S),$$

where $T = T_{R,R_1,R_2}$, $S = S_\rho$, $0 \le R_1 < \rho < R_2$ and $R > 0$. Let $a = \inf f(S)$ and $b = \sup f(\Delta)$ ($\Delta = \Delta_{R,R_1,R_2}$). Assume that $b < \infty$ and

- $(\mathrm{PS})^*_c$ *holds for any c in $[a, b]$,*
- $\nabla^*(X_1 \oplus X_3, c)$ *holds for any c in $[a, b]$.*

Then f has at least two critical points in $f^{-1}([a, b])$.

Proof. We argue as in the proof of Theorem 2.5: let C, Φ and $g = (f \circ \Phi)|_C$ be as in the proof of Theorem 2.5. By (1) of Lemma 2.9, g verifies $(\mathrm{PS})^*_c$ for every c in $[a, b]$ with respect to the sequence $(C_n)_n = (C \cap H_n)_n$. By Theorem 2.7, g has two lower critical points z_1, z_2 in $g^{-1}([a, b])$. By (2) of Lemma 2.9, $z_i \notin \partial C$ for $i = 1, 2$. Therefore $u_i = \Phi(z_i)$ are critical points for f in $f^{-1}([a, b])$. □

3. An abstract framework

Let H be a separable Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ and norm $\|\cdot\|$. Let $L : H \to H$ be a continuous, symmetric linear operator of the form $L = J + K$, where J is an isomorphism and K is compact. Let $\omega : H \to \mathbb{R}$ be a $\mathcal{C}^{1,1}$-function. We denote by Q the quadratic form $Q(u) = \langle Lu, u\rangle$ and define the functional $f : H \to \mathbb{R}$ by

$$f(u) = \frac{1}{2}Q(u) - \omega(u).$$

In the sequel we shall use some of the following assumptions on ω.

$(\omega, 0)$ $\omega(0) = 0$, $\omega(u) > 0$ if $u \neq 0$, $\operatorname{grad}\omega(u) = o(\|u\|)$ as $u \to 0$,

(ω, ∞) there exist $c > 0$, $\mu > 2$ and b in $\mathbb{R}$ such that $\omega(u) \geq c\|u\|_0^\mu - b$ for all u in H, where $\|\cdot\|_0$ is a norm in H weaker than $\|\cdot\|$,

(K_ω) $\operatorname{grad}\omega$ is a compact map,

$(\omega, \omega', 0)$ if $\omega'(u)(u) - 2\omega(u) = 0$, then $\operatorname{grad}\omega(u) = 0$,

$(\omega, \omega', \infty)$ if $\|u_n\| \to \infty$ and $(\omega'(u_n)(u_n) - 2\omega(u_n))/\|u_n\| \to 0$, then there exists $(u_{hn})_n$ and w in H such that

$$\frac{\operatorname{grad}\omega(u_{h_n})}{\|u_{h_n}\|} \to w \quad \text{and} \quad \frac{u_{h_n}}{\|u_{h_n}\|} \rightharpoonup 0,$$

(ω, ω') there exist $\varphi, \psi : [0, \infty[\to \mathbb{R}$ continuous and such that $\psi(s)/s \to 0$ as $s \to 0$, $\varphi(s) > 0$ if $s > 0$,

$$\begin{aligned} \|\operatorname{grad}\omega(u)\|^2 &\leq \psi(\omega(u)) \quad && \text{for all } u, \\ \omega'(u)(u) - 2\omega(u) &\geq \varphi(u) && \text{for all } u. \end{aligned}$$

Remark 3.1. If $H = W_0^{1,2}(\Omega)$, Ω bounded subset of $\mathbb{R}^N$, then the function

$$\omega(u) = \int_\Omega |u|^p\,dx,$$

with $2 < p < 2^*$, verifies all the above conditions.

Remark 3.2. The assumption $L = J + K$, J isomorphism and K compact is equivalent to saying that there exist three closed subspaces H^-, H^0 and H^+ which are mutually orthogonal and two numbers $c^-, c^+ > 0$ such that $H^0 = \ker(L)$, $\dim(H^0) < \infty$, $H = H^- \oplus H^0 \oplus H^+$ and

$$\begin{aligned} Q(u) &\geq c^+\|u\|^2 \quad && \text{for all } u \text{ in } H^+, \\ Q(u) &\leq -c^-\|u\|^2 && \text{for all } u \text{ in } H^-. \end{aligned}$$

Definition 3.3. We shall consider H^-, H^0, H^+, c^+ and c^- as in the previous remark and denote by P^+ P^0 and P^- the relative orthogonal projections. Given a sequence $(H_n)_n$ of closed subspaces of H we shall consider the following assumption:

(L, H_n) $H_n = H_n^- \oplus H^0 \oplus H_n^+$ where $H_n^+ \subset H^+$, $H_n^- \subset H^-$ for all n (H_n^+ and H_n^- are subspaces of H), $\dim(H_n) < \infty$, $H_n \subset H_{n+1}$, $\bigcup_{n\in\mathbb{N}} H_n$ is dense in H.

In this situation we shall denote by P_{H_n} the ortogonal projections onto H_n.

Proposition 3.4. *Assume that* (ω, ∞) *holds. Then for every finite dimensional space* X *we have:*

$$\sup f(X \oplus H^0 \oplus H^-) < \infty, \qquad \limsup_{\substack{\|u\|\to\infty \\ u\in X\oplus H^0\oplus H^-}} \frac{f(u)}{\|u\|^2} < 0.$$

Proof. We first prove the second statement. We have:

$$\frac{f(u)}{\|u\|^2} \leq \|L\| \|P^+\widehat{u}\|^2 - c\|\widehat{u}\|_0^\mu \|u\|^{\mu-2} + \frac{b}{\|u\|^2} - c^-\|P^-\widehat{u}\|^2, \tag{3.4.1}$$

where $\widehat{u} = u/\|u\|$. Let $\|u_n\| \to \infty$, two possible cases arise.

- $\|\widehat{u}_n\|_0 \to 0$. In this case it follows that $\widehat{u}_n \rightharpoonup 0$, hence $\widehat{P}^+u \to 0$ and $\widehat{P}^0 u \to 0$ (since they lie a finite dimensional space). Then $\|P^-\widehat{u}\| \to 1$ and then
$$\limsup_{n\to\infty} \frac{f(u_n)}{\|u_n\|^2} \leq -c^-.$$
- $\|\widehat{u}_n\|_0 \geq \varepsilon > 0$. By (3.4.1) this implies
$$\lim_{n\to\infty} \frac{f(u_n)}{\|u_n\|^2} = -\infty.$$

In any case the conclusion follows. In particular $f \leq 0$ outside a suitable large ball. On the other side, by (3.4.1), f is bounded on any bounded set, so the first assertion follows. □

Lemma 3.5. *Let* $(H_n)_n$ *verify the assumptions* (L, H_n) *and let* $(z_n)_n$ *be a sequence such that*

$$z_n \in H_n \quad \textit{for all } n, \quad P_{H_n} L z_n \to w \quad (\textit{in } H)$$

for a point w *in* H. *Then* $Lz_n \to w$. *As a consequence, (since* $L = J + K$*) if* $(z_n)_n$ *is also bounded, then* $(z_n)_n$ *has a subsequence which converges to a point* z *such that* $Lz = w$.

Proof. Since P^+ P^0 and P^- commute with P_{H_n} and with L we get

$$P_{H_n} L P^+ z_n \to P^+ w, \quad P_{H_n} L P^- z_n \to P^- w, \quad P^0 w = 0.$$

Let z_+ be the unique point in H^+ such that $Lz_+ = P^+w$, we claim that $P^+z_n \to z_+$. We have

$$(0 \leftarrow)\ P_{H_n}LP^+z_n - Lz_+ = P_{H_n}L(P^+z_n - z_+) + (P_{H_n}Lz_+ - Lz_+)$$

hence

$$P_{H_n}L(P^+z_n - z_+) = P_{H_n}L(P^+z_n - P_{H_n}z_+) \to 0.$$

Then

$$\begin{aligned}&\big\langle P_{H_n}L(P_{H_n}z_+ - P^+z_n), P_{H_n}z_+ - P^+z_n\big\rangle \\ &\qquad = \big\langle L(P_{H_n}z_+ - P^+z_n), P_{H_n}z_+ - P^+z_n\big\rangle \geq c^+\|P_{H_n}z_+ - P^+z_n\|^2\end{aligned}$$

which gives

$$c^+\|P_{H_n}z^+ - P^+z_n\| \leq \|P_{H_n}L(P_{H_n}z_+ - P^+z_n)\| \to 0.$$

So $z_+ - P^+z_n = (z_+ - P_{H_n}z_+) + (P_{H_n}z_+ - P^+z_n) \to 0$. In the same way, if z_- is such that $Lz_- = P^-w$, it follows $P^-z_n \to z_-$ and the conclusion follows. □

Proposition 3.6. *Let $(H_n)_n$ verify (L, H_n) and assume that (K_ω) and $(\omega, \omega', \infty)$ hold. Then the functional f verifies* $(\mathrm{PS})_c^*$ *with respect to $(H_n)_n$ for every real number c.*

Proof. Let $c \in \mathbb{R}$ and $(u_n)_n$ be a sequence in $\mathbb{N}$ such that $h_n \to \infty$, $(u_n)_n$ be a sequence such that

$$u_n \in H_{h_n} \quad \text{for all } n, \quad f(u_n) \to c, \quad P_{H_{h_n}} \operatorname{grad} f(u_n) \to 0.$$

We claim that $(u_n)_n$ is bounded. If not, by contradiction, we can suppose $\|u_n\| \to \infty$ and set $\widehat{u}_n = u_n/\|u_n\|$. Then

$$\big\langle P_{H_{h_n}} \operatorname{grad} f(u_n), \widehat{u}_n\big\rangle = \langle \operatorname{grad} f(u_n), \widehat{u}_n\rangle = 2\frac{f(u_n)}{\|u_n\|} - \frac{\omega'(u_n)(u_n) - 2\omega(u_n)}{\|u_n\|},$$

hence $(\omega'(u_n)(u_n) - 2\omega(u_n))/\|u_n\| \to 0$. By $(\omega, \omega', \infty)$ this implies that

$$\operatorname{grad}\omega(u_n)/\|u_n\| \quad \text{converges and} \quad \widehat{u}_n \rightharpoonup 0.$$

We get

$$0 \leftarrow \frac{P_{H_{h_n}} \operatorname{grad} f(u_n)}{\|u_n\|} = P_{H_{h_n}} L\widehat{u}_n - P_{H_{h_n}} \frac{\operatorname{grad}\omega(u_n)}{\|u_n\|}$$

so $(P_{H_{h_n}} L\widehat{u}_n)_n$ converges. Using Lemma 3.5 and the fact that $(\widehat{u}_n)_n$ is bounded, we obtain that $(\widehat{u}_n)_n$ has a limit (up to subsequences). Since $\widehat{u}_n \rightharpoonup 0$ we get $\widehat{u}_n \to 0$ which is impossible because $\|\widehat{u}_n\| = 1$. So we have proved that $(u_n)_n$ is bounded.

We can now suppose that $u_n \rightharpoonup u$, for some u in H. By (K_ω) we obtain that $\operatorname{grad}\omega(u_n) \to \operatorname{grad}\omega(u)$, then $(P_{H_{h_n}} Lu_n)_n$ converges. Using Lemma 3.5 again, we deduce that, up to a subsequence, $(u_n)_n$ converges to some u and it is trivial to see that u is critical for I. □

We can now state a preliminar existence result.

Theorem 3.7. *Let* $(\omega, 0)$, (ω, ∞), $(\omega, \omega', \infty)$ *and* (K_ω) *hold. Then* f *has a nontrivial critical point.*

Proof. For R, ρ positive real numbers and e in H we set

$$\begin{aligned} S^+_\rho &= \{u \in H^+ \mid \|u\| = \rho\}, \\ \Sigma^-_R(e) &= \{u \mid u = se + v,\ s \geq 0,\ v \in H^0 \oplus H^+,\ \|u\| = R\} \\ &\qquad \cup \{v \in H^0 \oplus H^+,\ \|v\| \leq R\}, \\ \Delta^-_R(e) &= \{u \mid u = se + v,\ s \geq 0,\ v \in H^0 \oplus H^+,\ \|u\| \leq R\}. \end{aligned}$$

Then, using $(\omega, 0)$, we can find ρ small enough such that

$$a = \inf f(S^+_\rho) > 0 = \sup f(H^o \oplus H^+).$$

Let $e \in H^+$, using Proposition 3.4 and the fact that $\omega \geq 0$, we can find R large enough such that

$$\sup f(\Sigma^-_R) = 0.$$

Moreover, we set $b = \sup f(\Delta^-_R(e))$, it is clear that $b < \infty$.

Now let $(H_n)_n$ be a sequence of subspaces of H satisfying (L, H_n) (such a sequence exists because H is separable). Clearly we can suppose $e \in H_n$ for all n. We have, for all n in $\mathbb{N}$,

$$\sup f(\Sigma^-_R(e) \cap H_n) < \inf f(S^+_\rho \cap H_n).$$

Moreover, $(\mathrm{PS})_c$ holds for $f_n = f|_{H_n}$ for any c in $\mathbb{R}$, so, by linking arguments, there exists a critical point u_n for f_n with

$$a \leq \inf f(S^+_\rho \cap H_n) \leq f(u_n) \leq \sup f(\Delta^-_R(e) \cap H_n) \leq b.$$

Using the $(\mathrm{PS})^*_c$ condition, we obtain that, up to a subsequence, $u_n \to u$, with u a critical point for f such that $a \leq f(u) \leq b$ (hence $u \neq 0$). □

We introduce now a closed subspace X of H and denote by P_X the orthogonal projection onto X.

Lemma 3.8. *Let* $(\omega, \omega', \infty)$ *and* (K_ω) *hold and let* $(H_n)_n$ *be a sequence of subspaces of* H *satisfying* (L, H_n). *Moreover, assume that* X *has finite codimension in* H. *Then for any real number* c *and any sequence* $(u_n)_n$ *such that*

$$u_n \in H_n \quad \text{for all } n, \quad f(u_n) \to c, \quad P_{X \oplus [u_n]} P_{H_n} \operatorname{grad} f(u_n) \to 0,$$

there exists a subsequence of $(u_n)_n$ *which converges to a point* u *such that* $f(u) = 0$ *and* $P_{X \oplus [u]} \operatorname{grad} f(u) = 0$. *Here* $[u] = span\{u\}$ *and* $P_{X \oplus [u]}$ *is the orthogonal projection onto* $X \oplus [u]$ (*and the same with* u_n *instead of* u).

PROOF. We first prove that $(u_n)_n$ is bounded. If not, by contradiction, we can suppose $\|u_n\| \to \infty$ and we take $\widehat{u}_n = u_n/\|u_n\|$. We have:

$$\begin{aligned} 0 \leftarrow \big\langle P_{X\oplus[u_n]} P_{H_n} \operatorname{grad} f(u_n), \widehat{u}_n \big\rangle &= \langle \operatorname{grad} f(u_n), \widehat{u}_n \rangle \\ &= 2\frac{f(u_n)}{\|u_n\|} - \frac{\omega'(u_n)(u_n) - 2\omega(u_n)}{\|u_n\|}. \end{aligned}$$

Using (ω,ω',∞) we deduce that $(\operatorname{grad}\omega(u_n)/\|u_n\|)_n$ converges and $\widehat{u}_n \rightharpoonup 0$. Then $P_{X\oplus[u_n]} P_{H_n} L\widehat{u}_n$ converges and since P_X is the sum of the identity plus a compact operator (X has finite codimension), we get that $P_{H_n} L\widehat{u}_n$ converges. By Lemma 3.5, we obtain that $(\widehat{u}_n)_n$ converges and this is contradictory since $\widehat{u}_n \rightharpoonup 0$ and $\|\widehat{u}_n\| = 1$. So $(u_n)_n$ is bounded and we can suppose that $u_n \rightharpoonup u$ for some u in H. By (K_ω) $\operatorname{grad}\omega(u_n) \to \operatorname{grad}\omega(u)$ so, by difference, $(P_{X\oplus[u_n]} P_{H_n} Lu_n)_n$ converges. As before this implies that $(P_{H_n} Lu_n)_n$ converges and, by Lemma 3.5, we have the conclusion. $\square$

The following result is an immediate consequence of Lemma 3.8.

COROLLARY 3.9. *Under the same assumptions of Lemma* 3.8*, if c is such that $f|_X$ has no critical points u with $f(u) = c$, then f satisfies $\nabla^*(X, c)$.*

We can now state a multiplicity theorem wich will allow to prove the main results of the next section.

THEOREM 3.10. *Assume that $(\omega, 0)$, (ω,∞), (ω,ω',∞) and (K_ω) hold. Let Y be a closed subspace of H^+ with finite dimension. Suppose that*

$$\sup_{u\in H^-\oplus H^0\oplus Y} f(u) < \inf\{f(u) \mid f(u) > 0,\ u \in Y^\perp,\ \operatorname{grad} f|_{Y^\perp}(u) = 0\}. \tag{3.10.1}$$

Then f has at least two distinct critical points u_1, u_2 such that, for $i = 1, 2$,

$$\sup_{\rho>0} \inf_{u\in H^+,\ \|u\|=\rho} f(u) \le f(u_i) \le \sup_{u\in H^-\oplus H^0\oplus Y} f(u).$$

PROOF. Set $X_1 = H^- \oplus H^0$, $X_2 = Y$, $X_3 = (X_1 \oplus X_2)^\perp (\subset H^+)$; we use the notations introduced in Notations 2.1. We have that $X_2 \oplus X_3 = H^+$, then, by $(\omega, 0)$, there exists $\rho > 0$ such that $a = \inf f(S_\rho) > 0$.

By Proposition 3.4 it turns out that $\sup f(T) = 0$, where $T = T_{R,0,R_2}$ for R and R_2 large enough. Moreover, if $b = \sup f(\Delta)$ $(\Delta = \Delta_{R,0,R_2})$, then $b < \infty$.

Let now $(H_n)_n$ be a sequence of subspaces of H verifying (L, H_n). By (3.10.1) $f|_{X_1\oplus X_3}$ has no critical levels between a and b, hence, by Corollary 3.9, f satisfies $\nabla^*(X_1 \oplus X_3, c)$ for any c in $[a, b]$. By Proposition 3.6 the $(\mathrm{PS})_c$ condition with respect to $(H_n)_n$ holds, for any c in $[a, b]$. Using the ∇ Theorem 2.5 the conclusion follows. $\square$

We consider now some conditions under which we are able to prove that (3.10.1) holds. Such conditions will be useful in the next section. In the sequel the following two lemmas will be applied to the restriction of f on $Y^\perp$.

Lemma 3.11. *Assume that* $(\omega,0)$, $(\omega,\omega',0)$, (ω,ω',∞) *and* (K_ω) *hold. Suppose that* $H^0=\{0\}$. *Then there exists* ε *in* $\mathbb{R}$ *such that* $\varepsilon>0$ *and* f *has no critical points* u *with* $0<|f(u)|<\varepsilon$.

Proof. Since $H^0=\{0\}$, there exists a constant $c>0$ such that

$$\|Lu\|\geq c\|u\| \quad \text{for all } u \text{ in } H. \tag{3.11.1}$$

This fact and assumption $(\omega,0)$ imply that 0 is an isolated critical point for f, that is there exists $R>0$ such that 0 is the only critical point for f in the ball $B_R(0)$.

We now claim that 0 is the only critical point u of f such that $f(u)=0$. Actually if u is such a point:

$$\begin{aligned}0=\langle \operatorname{grad} f(u),u\rangle &= f'(u)(u)=2f(u)-(\omega'(u)(u)-2\omega(u))\\ &\Rightarrow\ (\omega'(u)(u)-2\omega(u))=0\ \Rightarrow\ \operatorname{grad}\omega(u)=0\ \Rightarrow\ Lu=0\ \Rightarrow\ u=0.\end{aligned}$$

To conclude we argue by contradiction and suppose that there exists a sequence $(u_n)_n$ such that $\operatorname{grad} f(u_n)=0$, $f(u_n)\neq 0$ for all n and $f(u_n)\to 0$. As in the proof of Theorem 3.6 we can easily show that $(u_n)_n$ converges to a critical point u with $f(u)=0$. Then $u_n\to 0$ but this contradicts the fact that 0 is an isolated critical point. □

Lemma 3.12. *Assume that* (ω,ω') *hold. Then there exists a real number* $\varepsilon=\varepsilon(\varphi,\psi,c^+)>0$ *such that, if* $f(u)\geq 0$ *and* $\operatorname{grad} f(u)=0$, *then* $f(u)\geq\varepsilon$.

Proof. Let u be a critical point for f such that $f(u)\geq 0$. Then

$$0\leq f(u)=\frac12\left\langle LP^+u,u\right\rangle-\frac12\left\langle LP^-u,u\right\rangle-\omega(u)\Rightarrow 2\omega(u)\leq\left\langle LP^+u,u\right\rangle.$$

Then

$$\begin{aligned}0=\operatorname{grad} f(u)&=LP^+u+LP^-u-\operatorname{grad}\omega(u)\\ \Rightarrow\psi(\omega(u))\ &\geq\|\operatorname{grad}\omega(u)\|^2=\|LP^+u\|^2+\|LP^-u\|^2\\ &\geq\|LP^+u\|^2\geq c^+\left\langle LP^+u,u\right\rangle\geq 2c^+\omega(u)\Rightarrow\omega(u)\geq\delta,\end{aligned}$$

where δ is such that

$$0<|s|<\delta\ \Rightarrow\ \frac{\psi(s)}{s}<2c^+.$$

Finally

$$f(u)=\frac{\omega'(u)(u)}{2}-\omega(u)\geq\frac12\varphi(\omega(u))\geq\varepsilon$$

provided $2\varepsilon=\inf\{\varphi(s)\mid s\geq\delta\}$ $(\varepsilon>0)$. □

4. Some non-cooperative elliptic systems

In this section we deal with the following elliptic system

$$\text{(ES)}\qquad \begin{cases} -\Delta u = \alpha u - \delta v + F_r(x,u,v), \\ \Delta v = -\delta u - \gamma v + F_s(x,u,v), \\ u, v \in W_0^{1,2}(\Omega), \end{cases}$$

where Ω is a bounded, connected open subset of $\mathbb{R}^N$, $N \geq 3$ (for the sake of simplicity), α, γ and δ are real numbers, $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which has continuous derivatives $F_r(x,r,s), F_s(x,r,s)$ with respect to r and s, for almost any x in Ω. We shall consider the following assumptions on the nonlinear term F:

(F.0) $F(x,0,0) = 0, F(x,r,s) > 0$ if $(r,s) \neq (0,0)$, $\displaystyle\inf_{\substack{x\in\Omega \\ r^2+s^2=R^2}} F(x,r,s) > 0$,

(F.1) $|F_r(x,r,s)| + |F_s(x,r,s)| \leq a(|r|^\nu + |s|^\nu)$ for all x, r, s,

(F.2) $rF_r(x,r,s) + sF_s(x,r,s) \geq \mu F(x,r,s)$ for all x, r, s,

(F.3) $|F_r(x,r,s)| + |F_s(x,r,s)| \leq c(F(x,r,s)^{\delta_1} + F(x,r,s)^{\delta_2})$,

where $a \geq 0$, $R > 0$, $\mu \in \,]2, 2^*[$, $\nu \leq 2^* - 1 - (2^* - \mu)(1 - 2^{*'}/2^*)$ and $1/2 < \delta_1 \leq \delta_2 \leq 1/2^{*\prime}$.

Remark 4.1. Assume (F.0) and (F.2) to hold. Then there exist a_0, b_0 in $\mathbb{R}$, with $a_0 > 0$ such that

$$F(x,r,s) \geq a_0(|r|^\mu + |s|^\mu) - b_0 \quad \text{for all } x, r, s.$$

Proof. Let r, s be such that $r^2 + s^2 \geq R^2$. For $t \geq 1$ we set $\varphi(t) = F(x,tr,ts)$. Then

$$\varphi'(t) = rF_r(x,tr,ts) + sF_s(x,tr,ts) \geq \frac{\mu}{t}\varphi(t).$$

Multiplying by $t^{-\mu}$, we get $(t^{-\mu}\varphi(t))' \geq 0$, hence $\varphi(t) \geq \varphi(1)t^\mu$ for $t \geq 1$. It follows

$$\begin{aligned} F(x,r,s) &\geq F\bigg(x, \frac{Rr}{\sqrt{r^2+s^2}}, \frac{Rs}{\sqrt{r^2+s^2}}\bigg)\bigg(\frac{\sqrt{r^2+s^2}}{R}\bigg)^\mu \\ &\geq c_0\bigg(\frac{\sqrt{r^2+s^2}}{R}\bigg)^\mu \end{aligned}$$

where $c_0 = \inf\{F(x,r,s) \mid x \in \Omega,\ r^2 + s^2 = R^2\}$. This implies the thesis. $\square$

We observe that (ES) has the the solution $u = v = 0$ whatever α,γ and δ are. To investigate the existence of other solutions we start by noticing that (ES) has a variational structure. If we define the functional $I : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$I(u,v) = I_{(\alpha,\gamma,\delta)}(u,v) = Q_{(\alpha,\gamma,\delta)}(u,v) - \int_\Omega F(x,u,v)\,dx,$$

where

$$Q_{(\alpha,\gamma,\delta)}(u,v) = \frac{1}{2}\int_\Omega \left(|Du|^2 - \alpha u^2\right)\,dx + \delta \int_\Omega uv\,dx - \frac{1}{2}\int_\Omega \left(|Dv|^2 - \gamma v^2\right)\,dx,$$

then is is straightforward to see that I is of class $\mathcal{C}^{1,1}$ and that the solutions of (ES) are exactly the critical points of I.

Using this framework it is possible to re-prove (see [2]) the following theorem, by means of Theorem 3.7.

Theorem 4.2. *Assume that* (F.0)–(F.2) *hold. Then for any α, γ and δ the system* (ES) *has a nontrivial solution.*

We want now to show that, using Theorem 3.10 based on the ∇-theorems of Section 3 the existence of additional nontrivial solutions can be proved for suitable α, γ, δ. We first need to introduce some notations.

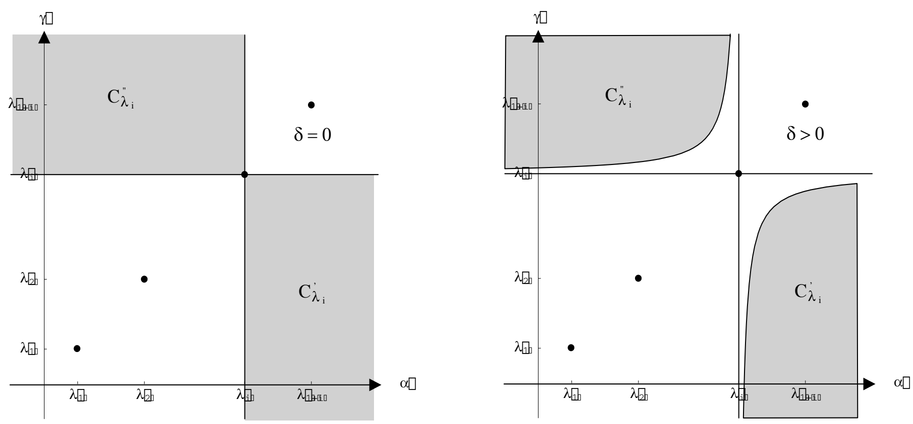

Figure 3. The sets C_{λ_i}

Notations 4.3. We denote by $(\lambda_i)_{i\in\mathbb{N}}$ the sequence of the eigenvalues of $-\Delta$ in $W_0^{1,2}(\Omega)$ ($\lambda_1 < \lambda_2 \le \lambda_3 \le \ldots$) and by $(e_i)_{i\in\mathbb{N}}$ the associated eigenfunctions. Let λ_i be given; we denote by H_{λ_i} the subspace spanned by all e_j with $\lambda_j = \lambda_i$. We also set:

$$\begin{aligned} q_{\lambda_i}(\alpha,\gamma,\delta) &= (\alpha-\lambda_i)(\gamma-\lambda_i)+\delta^2,\\ C_{\lambda_i} &= \{(\alpha,\gamma,\delta)\in\mathbb{R}^3 \mid q_{\lambda_i}(\alpha,\gamma,\delta)\le 0\},\\ C'_{\lambda_i} &= C_{\lambda_i}\cap\{\alpha\ge\gamma\}, \quad C''_{\lambda_i} = C_{\lambda_i}\cap\{\alpha\le\gamma\} \end{aligned}$$

(see Figure 3). Moreover, we denote by $\mu^+_{\lambda_i}$ and $\mu^-_{\lambda_i}$ the eigenvalues of the 2×2 matrix $\left(\begin{smallmatrix}\lambda_i-\alpha & \delta\\ \delta & \gamma-\lambda_i\end{smallmatrix}\right)$, namely

$$\mu^{\pm}_{\lambda_i} = \frac{1}{2}(\gamma-\alpha\pm\sqrt{(\gamma-\alpha)^2+4q_{\lambda_i}(\alpha,\gamma,\delta)}).$$

Moreover, we denote by $(c^+_{\lambda_i}, d^+_{\lambda_i})$ and $(c^-_{\lambda_i}, d^-_{\lambda_i})$ the eigenvectors of $\left(\begin{smallmatrix}\lambda_i-\alpha & \delta\\ \delta & \gamma-\lambda_i\end{smallmatrix}\right)$ corresponding to $\mu^+_{\lambda_i}$ and $\mu^-_{\lambda_i}$ respectively (if $\mu^+_{\lambda_i}=\mu^-_{\lambda_i}$ they can be choosen at will, provided they are orthogonal).

Finally we denote by W the space $W^{1,2}_0(\Omega)\times W^{1,2}_0(\Omega)$, set

$$\begin{aligned}
E_{\lambda_i} &= \{(ce,de)\in W \mid (c,d)\in\mathbb{R}^2,\ e\in H_{\lambda_i}\},\\
E^+_{\lambda_i} &= \{(c^+_{\lambda_i}e, d^+_{\lambda_i}e)\in W \mid e\in H_{\lambda_i}\},\\
E^-_{\lambda_i} &= \{(c^-_{\lambda_i}e, d^-_{\lambda_i}e)\in W \mid e\in H_{\lambda_i}\},
\end{aligned}$$

and denote by $H^+_{\alpha,\gamma,\delta}$, $H^+_{\alpha,\gamma,\delta}$ and $H^0_{\alpha,\gamma,\delta}$ the positive, negative and null space relative to the quadratic form Q in W.

The following proposition can be proved with easy computations (see [2]).

Proposition 4.4. *Let $(\alpha,\gamma,\delta)\in\mathbb{R}^3$.*

(1) *$E^+_{\lambda_i}$ and $E^-_{\lambda_i}$ are eigenspaces for the operator $L_{\alpha,\gamma,\delta}: W\to W$ associated with $Q_{\alpha,\gamma,\delta}$, with eigenvalues $\mu^+_{\lambda_i}/\lambda_i$ and $\mu^-_{\lambda_i}/\lambda_i$, respectively. These spaces generate W as i spans $\mathbb{N}$. Notice that*

$$\lim_{i\to\infty}\mu^-_{\lambda_i}/\lambda_i=-1,\qquad \lim_{i\to\infty}\mu^+_{\lambda_i}/\lambda_i=1,$$

and $\lim_{(\alpha,\gamma,\delta)\to(\alpha_0,\gamma_0,\delta_0)}\mu^{\pm}_{\lambda_i}(\alpha,\gamma,\delta)=\mu^{\pm}_{\lambda_i}(\alpha_0,\gamma_0,\delta_0)$ uniformly, with respect to i in $\mathbb{N}$.

(2) *We have:*

$$\begin{aligned}
H^-_{\alpha,\gamma,\delta} &= \Big(\bigoplus_{\mu^-_{\lambda_i}<0} E^-_{\lambda_i}\Big)\oplus\Big(\bigoplus_{\mu^+_{\lambda_i}<0} E^+_{\lambda_i}\Big),\\
H^0_{\alpha,\gamma,\delta} &= \Big(\bigoplus_{\mu^-_{\lambda_i}=0} E^-_{\lambda_i}\Big)\oplus\Big(\bigoplus_{\mu^+_{\lambda_i}=0} E^+_{\lambda_i}\Big),\\
H^+_{\alpha,\gamma,\delta} &= \Big(\bigoplus_{\mu^-_{\lambda_i}>0} E^-_{\lambda_i}\Big)\oplus\Big(\bigoplus_{\mu^+_{\lambda_i}>0} E^+_{\lambda_i}\Big).
\end{aligned}$$

In particular $H^0_{\alpha,\gamma,\delta}\neq\{0\}\Leftrightarrow(\alpha,\gamma,\delta)\in\bigcup_{i\in\mathbb{N}}\partial C_{\lambda_i}$ and in any case $\dim(H^0_{\alpha,\gamma,\delta})<\infty$.

The following lemma is a straightforward consequence of the above relations.

Lemma 4.5. *Let $i \in \mathbb{N}$. Let $(\alpha_0, \gamma_0, \delta_0) \in \partial C'_{\lambda_i}$. Then there exists a neighbourhood U' of $(\alpha_0, \gamma_0, \delta_0)$ in $\mathbb{R}^3$ such that:*

(1) *for all (α, γ, δ) in $U' \setminus C'_{\lambda_i}$ $H^+_{\alpha,\gamma,\delta}$ can be split as an orthogonal sum $X_2(\alpha, \gamma, \delta) \oplus X_3(\alpha, \gamma, \delta)$, where $\dim(X_2(\alpha, \gamma, \delta)) < \infty$ and*

$$\inf\{Q_{\alpha,\gamma,\delta}(w) \mid \|w\| = 1,\ w \in X_3(\alpha, \gamma, \delta),\ (\alpha, \gamma, \delta) \in U' \setminus C'_{\lambda_i}\} = K^+ > 0, \tag{4.5.1}$$

$$\lim_{\substack{(\alpha,\gamma,\delta)\to(\alpha_0,\gamma_0,\delta_0)\\ (\alpha,\gamma,\delta)\in U'\setminus C'_{\lambda_i}}} \sup\{Q_{\alpha,\gamma,\delta}(w) \mid \|w\| = 1,\ w \in X_2(\alpha, \gamma, \delta)\} = 0,$$

(2) *if $(\alpha_0, \gamma_0, \delta_0) \neq (\lambda_i, \lambda_i, 0)$ and*

$$(\alpha, \gamma, \delta) \in U' \setminus \bigcup_{\substack{j\in\mathbb{N}\\ (\alpha_0,\gamma_0,\delta_0)\in\partial C'_{\lambda_j}}} C'_{\lambda_j}, \tag{4.5.2}$$

then

$$H^0 = \{0\}, \tag{4.5.3}$$

$$\sup\{Q_{\alpha,\gamma,\delta}(w) \mid \|w\| = 1,\ w \in H^-_{\alpha,\gamma,\delta},\ (4.5.2) \text{ holds}\} < 0.$$

Now let $(\alpha_0, \gamma_0, \delta_0) \in \partial C''_{\lambda_i}$. Then there exists a neighbourhood U'' of $(\alpha_0, \gamma_0, \delta_0)$ in $\mathbb{R}^3$ such that:

(3) *conclusion (1) holds with $U' \setminus C'_{\lambda_i}$ replaced by $U'' \cap \operatorname{int}(C''_{\lambda_i})$;*

(4) *conclusion (2) holds with $U' \setminus \bigcup_{j\in\mathbb{N},(\alpha_0,\gamma_0,\delta_0)\in\partial C'_{\lambda_j}} C'_{\lambda_j}$, replaced by*

$$U'' \cap \bigg(\bigcap_{\substack{j\in\mathbb{N}\\ (\alpha_0,\gamma_0,\delta_0)\in\partial C''_{\lambda_j}}} \operatorname{int}(C''_{\lambda_j})\bigg),$$

and without requiring $(\alpha_0, \gamma_0, \delta_0) \neq (\lambda_i, \lambda_i, 0)$.

Also the following remark is easy to check.

Remark 4.6. Let $i \in \mathbb{N}$ and let (F.0)–(F.2) hold. If $(\alpha_0, \gamma_0, \delta_0) \in \partial C'_{\lambda_i}$, using the notations of Lemma 4.5, we have

$$\lim_{\substack{(\alpha,\gamma,\delta)\to(\alpha_0,\gamma_0,\delta_0)\\ (\alpha,\gamma,\delta)\in U'\setminus C'_{\lambda_i}}} \sup\{I_{\alpha,\gamma,\delta}(w) \mid w \in H^-_{\alpha,\gamma,\delta} \oplus H^0_{\alpha,\gamma,\delta} \oplus X_2(\alpha, \gamma, \delta)\} = 0.$$

Moreover, there exists $r > 0$ such that

$$\lim_{\substack{(\alpha,\gamma,\delta)\to(\alpha_0,\gamma_0,\delta_0)\\ (\alpha,\gamma,\delta)\in U'\setminus C'_{\lambda_i}}} \inf\{I_{\alpha,\gamma,\delta}(w) \mid w \in X_3(\alpha, \gamma, \delta),\ \|w\| = r\} > 0.$$

In the same fashion, if $(\alpha_0,\gamma_0,\delta_0)\in\partial C''_{\lambda_i}$, then

$$\lim_{\substack{(\alpha,\gamma,\delta)\to(\alpha_0,\gamma_0,\delta_0)\\(\alpha,\gamma,\delta)\in U''\cap\operatorname{int}(C''_{\lambda_i})}}\sup\{I_{\alpha,\gamma,\delta}(w)\mid w\in H^-_{\alpha,\gamma,\delta}\oplus H^0_{\alpha,\gamma,\delta}\oplus X_2(\alpha,\gamma,\delta)\}=0$$

and there exists $r>0$ such that

$$\lim_{\substack{(\alpha,\gamma,\delta)\to(\alpha_0,\gamma_0,\delta_0)\\(\alpha,\gamma,\delta)\in U''\cap\operatorname{int}(C''_{\lambda_i})}}\inf\{I_{\alpha,\gamma,\delta}(w)\mid w\in X_3(\alpha,\gamma,\delta),\ \|w\|=r\}>0.$$

We can finally state our multiplicity results for (ES). The following theorem can be compared with the results of [8] and shows that under the same assumptions more solutions can be found by means of the ∇-theorems.

Theorem 4.7. *Assume that* (F.0)–(F.2) *hold and let* $i\in\mathbb{N}$.

(1) *For any* $(\alpha_0,\gamma_0,\delta_0)$ *in* $\partial C'_{\lambda_i}\setminus\{(\lambda_i,\lambda_i,0)\}$ *there exists a neighbourhood* U' *of* $(\alpha_0,\gamma_0,\delta_0)$ *such that for* (α,γ,δ) *in* $U'\setminus\bigcup\{C'_{\lambda_j}\mid(\alpha_0,\gamma_0,\delta_0)\in\partial C'_{\lambda_j}\}$ *problem* (ES) *has at least three nontrivial solutions and* I *has at least two nontrivial critical levels.*
(2) *For any* $(\alpha_0,\gamma_0,\delta_0)$ *in* $\partial C''_{\lambda_i}$ *there exists a neighbourhood* U'' *of* $(\alpha_0,\gamma_0,\delta_0)$ *such that for* (α,γ,δ) *in* $U''\cap\bigcap\{\operatorname{int}(C''_{\lambda_j})\mid(\alpha_0,\gamma_0,\delta_0)\in\partial C''_{\lambda_j}\}$ *problem* (ES) *has at least three nontrivial solutions and* I *has at least two nontrivial critical levels.*

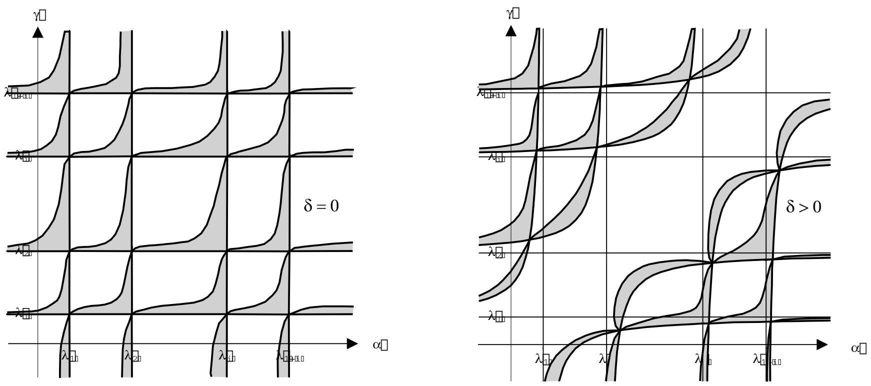

Figure 4. The three solutions zones of Theorem 4.7

Proof. By Lemma 4.9 which follows $(\omega,0)$, (ω,∞), $(\omega,\omega',0)$, (ω,ω',∞) and (K,ω) hold. It is also clear that the operator $L_{\alpha,\gamma,\delta}$ associated with $Q_{\alpha,\gamma,\delta}$ is the sum of an isomorphism and of a compact operator. By Proposition 4.4 it is possible to constuct a sequence $(H_n)_n$ such that (L,H_n) is verified.

Let for instance $(\alpha_0,\gamma_0,\delta_0)\in\partial C'_{\lambda_i}\setminus\{(\lambda_i,\lambda_i,0)\}$ and let U' as in Lemma 4.5. For (α,γ,δ) in $U'\setminus\bigcup\{C'_{\lambda_j}\mid(\alpha_0,\gamma_0,\delta_0)\in\partial C'_{\lambda_j}\}$ we set $X_1=H^-_{\alpha,\gamma,\delta}$, $X_2=$

$X_2(\alpha,\gamma,\delta)$, $X_3 = X_3(\alpha,\gamma,\delta)$, with the notations of Lemma 4.5. With this splitting we can apply Lemma 3.11 (on $X_1 \oplus X_3$) and Remark 4.6 to get that, up to shrinking U', (3.10.1) holds true (with $Y = X_2$). Applying Theorem 3.10 we obtain that there exist two distinct solutions u_1, u_2 of (ES) such that

$$0 < I(u_j) \le \sup I(X_1 \oplus X_2) \quad \text{for } j = 1,2. \tag{4.7.1}$$

Moreover, by Remark 4.6, up to shrinking U', there exists $r > 0$ such that

$$\sup I(X_1 \oplus X_2) < \inf\{I(w) \mid w \in X_3, \|w\| = r\} = a.$$

Let e be in X_3, $e \neq 0$. By Proposition 3.4

$$\lim_{\substack{w \in X_1 \oplus X_3 \oplus [e] \\ \|w\| \to \infty}} I(w) = -\infty.$$

Arguing as in the proof of Theorem 3.7 one can find a third critical point u_3 such that $I(u_3) \ge a$, so the conclusion follows. $\square$

Theorem 4.8. *Assume that* (F.0)–(F.3) *hold. Then for any i in $\mathbb{N}$ there exist a neighbourhood U'_{λ_i} of $\partial C'_{\lambda_i}$ and a neighbourhood U''_{λ_i} of $\partial C''_{\lambda_i}$ such that for (α,γ,δ) in $(U'_{\lambda_i} \setminus C'_{\lambda_i}) \cup (U''_{\lambda_i} \cap \operatorname{int}(C''_{\lambda_i}))$ problem* (ES) *has at least three nontrivial solutions.*

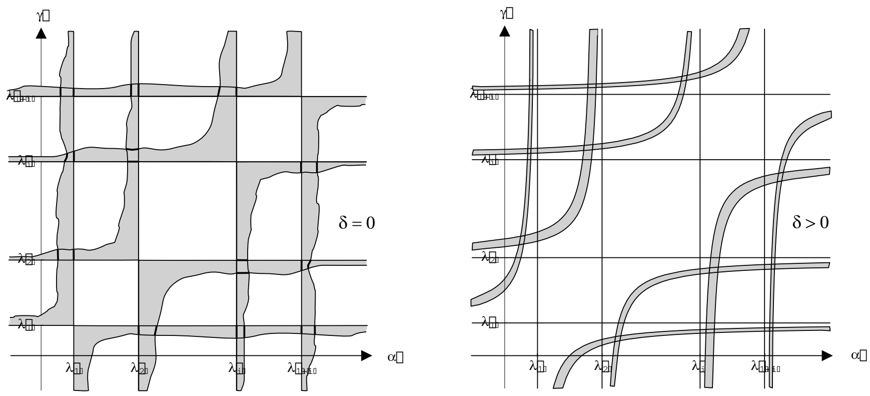

Figure 5. The three solutions zones of Theorem 4.8

Proof. By Lemma 4.9 and Lemma 4.11 $(\omega,0)$, (ω,∞), (ω,ω'), (ω,ω',∞) and (K,ω) hold. Let $(\alpha_0,\gamma_0,\delta_0) \in \partial C'_{\lambda_i} \setminus \{(\lambda_i,\lambda_i,0)\}$, let U' and K^+ be as in Lemma 4.5 and, for (α,γ,δ) in $U' \setminus C'_{\lambda_i}$, $X_1 = H^-_{\alpha,\gamma,\delta}$, $X_2 = X_2(\alpha,\gamma,\delta)$ and $X_3 = X_3(\alpha,\gamma,\delta)$. Then, for w in X_3, $Q_{\alpha,\gamma,\delta}(w) K^+ \ge \|w\|^2$. By Lemma 3.12, there exists $\varepsilon > 0$ such that

$$\inf I(w) \mid w \neq 0,\ \operatorname{grad} I|_{X_1 \oplus X_3}(w) = 0 \ge \varepsilon$$

for all (α,γ,δ) in $U'\setminus C'_{\lambda_i}$. By Remark 4.6, up to shrinking U', condition (3.10.1) is fulfilled (with $Y = X_2$). Therefore for all (α,γ,δ) in $U'\setminus C'_{\lambda_i}$ there exist two distinct critical points u_1 and u_2 such that (4.7.1) holds. The existence of the third critical point can be proved exactly as in Theorem 4.7. □

It remains to prove the following lemmas, mentioned in the above proofs.

LEMMA 4.9. *Assume that* (F.0)–(F.2) *hold. Then the function* $\omega : W \to \mathbb{R}$ *defined by*

$$\omega(u,v) = \int_\Omega F(x,u,v)\,dx$$

verifies the conditions $(\omega,0)$, (ω,∞), $(\omega,\omega',0)$, (ω,ω',∞) *and* (K,ω).

PROOF. Condition $(\omega,0)$ follows from (F.0) and (F.1), since $1<\nu<2^*-1$. Condition (ω,∞) is a consequence of Remark 4.1, taking $\|u\|_0 = \|u\|_{\mathrm{L}^\mu}$. Condition $(\omega,\omega',0)$ is implied by (F.2) and the fact that $F(x,r,s)>0$ for $(r,s)\neq(0,0)$.

To check (ω,ω',∞) notice that

$$\omega'(w)(w) - 2\omega(w) \ge (\mu-2)\int_\Omega F(x,w)\,dx \ge (\mu-2)(a_0\|w\|^\mu_{\mathrm{L}^\mu} - b_0)$$

($w=(u,v)$, we are using (F.2) and Remark 4.1. On the other hand, using (F.1), we have

$$\|\operatorname{grad}\omega(w)\| \le C_1\|F_w(x,w)\|_{\mathrm{L}^{2^{*\prime}}} \le C_2\||w|^\nu\|_{\mathrm{L}^{2^{*\prime}}},$$

(for suitable constants C_1, C_2). To get the conclusion it suffices to estimate $\||w|^\nu/\|w\|\|_{\mathrm{L}^{2^{*\prime}}}$ in terms of $\|w\|^\mu_{\mathrm{L}^\mu}/\|w\|$. If $\mu \ge 2^{*\prime}\nu$ this is an easy consequence of Hölder's inequality, otherwise we use Lemma 4.10 which follows, noticing that, the assumptions on μ and ν imply

$$\nu \le 2^* - 1 - (2^*-\mu)\left(1 - \frac{2^{*\prime}}{2^*}\right). \tag{4.9.2}$$

Condition (K,ω) is easily obtained with standard arguments. □

The following lemma is easily proved by standard interpolation arguments.

LEMMA 4.10. *If μ and ν are two numbers such that $1<\mu\le 2^{*\prime}\nu<2^*$, then there exists a constant C such that*

$$\left\|\frac{|w|^\nu}{\|w\|}\right\|_{\mathrm{L}^{2^{*\prime}}} \le C\left(\frac{\|w\|^\mu_{\mathrm{L}^\mu}}{\|w\|}\right)^{\nu\alpha/\mu}\|w\|^\beta,$$

where α is such that $\alpha/\mu + 1 - \alpha/2^* = 1/2^{*\prime}\nu$ for $\alpha>0$ and $\beta = (1-\alpha)\nu - 1$ $-\nu\alpha/\mu$. If (4.9.2) *holds, then $\beta\le 0$.*

PROOF. This is a consequence of standard interpolation inequalities. □

Lemma 4.11. *Let the whole set of assumptions* (F.0)–(F.3) *holds. Then* ω *satisfies condition* (ω, ω').

Proof. Set $w = (u, v)$ as before. By (F.3), for all w:

$$\begin{aligned} \|\mathrm{grad}\,\omega(w)\| &\leq \|F_w(x,w)\|_{\mathrm{L}^{2^*\prime}} \\ &\leq \mathrm{const}\|F(x,w)^{\delta_1} + F(x,w)^{\delta_2}\|_{\mathrm{L}^{2^*\prime}} \\ &\leq \mathrm{const}(\|F(x,w)^{\delta_1}\|_{\mathrm{L}^{2^*\prime}} + \|F(x,w)^{\delta_2}\|_{\mathrm{L}^{2^*\prime}}) \\ &\leq \mathrm{const}(\|F(x,w)^{\delta_1}\|_{\mathrm{L}^{1/\delta_1}} + \|F(x,w)^{\delta_2}\|_{\mathrm{L}^{1/\delta_2}}) \\ &\leq \mathrm{const}(\|F(x,w)\|_{\mathrm{L}^1}^{\delta_1} + \|F(x,w)\|_{\mathrm{L}^1}^{\delta_2}) \\ &= \mathrm{const}(\omega(w)^{\delta_1} + \omega(w)^{\delta_2}), \end{aligned}$$

which gives the first condition in (ω, ω'), since $\delta_1, \delta_2 > 1/2$. The second condition in (ω, ω') follows immediately from (4.9.1). □

5. Appendix

We briefly recall here the notion of relative category. Several slightly different definitions can be found in the literature; we shall use the version of [5], although any other one would serve to our porpouse as well.

Let X be a topological space and Y be a closed subspace of X.

Definition 5.1. Let A be a closed subset of X with $Y \subset A$. We define the relative category of A in (X, Y), which we denote by $\mathrm{cat}_{X,Y}(A)$, as the least integer h such that there exist $h+1$ closed subsets $U_0, \ldots, U_h$ with the following properties:

- $A \subset U_0 \cup \ldots \cup U_h$,
- $U_1, \ldots, U_h$ are contractible in X,
- $Y \subset U_0$ and there exists a continuous map $H : U_0 \times [0,1] \to X$ such that

$$\begin{aligned} H(x,0) &= x \quad \text{for all } x \text{ in } U_0, \\ H(x,t) &\in Y \quad \text{for all } x \text{ in } Y \text{ and all } t \text{ in } [0,1], \\ H(x,1) &\in Y \quad \text{for all } x \text{ in } U_0. \end{aligned}$$

If such an h does not exist, we say that $\mathrm{cat}_{X,Y}(A) = \infty$.

Relative category is connected with homology and cohomology groups as shown by the following theorem (see e.g. [5]).

Theorem 5.2. *Assume that there exist $h+1$ integers $p_0, \ldots, p_h$, with $p_i \geq 1$ for $i = 1, \ldots, h$, and there exist α_0 in $H^{p_0}(X, Y)$, α_i in $H^{p_i}(X)$, for $i = 1, \ldots, h$, such that $\alpha_0 \cup \ldots \cup \alpha_h \neq 0$ (in this case one says that the relative cuplength of (X, Y) is greater than or equal to h). Then* $\mathrm{cat}_{X,Y}(X) \geq h + 1$.

Now we recall, in a suitable form, a theorem which gives an estimate of the number of critical points of a functional, in terms of the relative category of its sublevels. Let H be a Hilbert space, with inner product $\langle \cdot\,,\,\cdot\rangle$ and norm $\|\cdot\|$ and let M be a $\mathcal{C}^{1,1}$, complete submanifold with boundary. Let $g : M \to \mathbb{R}$ be a $\mathcal{C}^{1,1}$-function.

Definition 5.3. If $u \in M$ we define the lower gradient of g at u, denoted by $\operatorname{grad}^-_M g(u)$, as

$$\operatorname{grad}^-_M g(u) = \begin{cases} P_{T_u(M)}\operatorname{grad}\widetilde{g}(u) & \text{if } u \notin \partial M, \\ P_{T_u(M)}\operatorname{grad}\widetilde{g}(u) + \langle \operatorname{grad}\widetilde{g}(u), \nu(u)\rangle\, \nu(u) & \text{if } u \in \partial M, \end{cases}$$

where $\widetilde{g}$ is any $\mathcal{C}^{1,1}$ extension of g to a neighbourhood of M, $P_{T_u(M)}$ denotes the orthogonal projection onto the tangent space $T_u(M)$ to M at u and $\nu(u)$ (which is an element of $T_u(M)$) is the unit normal vector to M at u pointing outwards. As well known this definition does not depend on the way $\widetilde{g}$ is choosen.

Definition 5.4. Let $c \in \mathbb{R}$. We say that g satisfies the Palais–Smale condition at level c, briefly $(\mathrm{PS})_c$ holds, if for any sequence $(u_n)_n$ in M such that $g(u_n) \to c$ and $\operatorname{grad}^-_M g(u_n) \to 0$ there exists a subsequence $(u_{n_k})_k$ which converges to a point u in M such that ($g(u) = c$ and) $\operatorname{grad}^-_M g(u) = 0$.

Theorem 5.5. *Let Y be a closed subset of M. For any integer i we set*

$$c_i = \inf\{\sup g(A) \mid A \textit{ is closed},\ Y \subset A,\ \operatorname{cat}_{M,Y}(A) \geq i\}.$$

Assume that $(\mathrm{PS})_c$ holds for $c = c_i$ and that $\sup g(Y) < c_i < \infty$. Then c_i is a lower critical level for g, that is there exists u in M such that $g(u) = c_i$ and $\operatorname{grad}^-_M g(u) = 0$. Moreover, if $c_i = \ldots = c_{i+j} = c$, then

$$\operatorname{cat}_M(\{u \in M \mid g(u) = c,\ \operatorname{grad}^-_M g(u) = 0\}) \geq j + 1.$$

Proof. The theorem can be proved repeating the classical arguments (see e.g. [5]), using a deformation lemma for functions on manifolds with boundary. The latter can be obtained, for example, by means of the theory of $C(p,q)$ functions (see e.g. [3], [4]). $\square$

We need in the following a version of the previous theorem suited to treat *strongly indefinite* functionals. In this case the notion of *limit* relative category turns out to be a very useful tool (see [5]). We recall here, briefly, a simplified version of it.

In the following we denote by $(M_n)_n$ a sequence of submanifolds of M.

Definition 5.6. Let Y be a closed subset of M. For any closed subset A of M such that $Y \subset A$ we define the limit relative category of A in (M, Y), with respect to $(M_n)_n$, as

$$\operatorname{cat}^*_{M,Y}(A) = \limsup_{n\to\infty} \operatorname{cat}_{M_n, Y\cap M_n}(A \cap M_n).$$

Definition 5.7. Let $c \in \mathbb{R}$. We say that g satisfies the limit Palais–Smale condition at level c, briefly $(\mathrm{PS})^*_c$ holds, with respect to the sequence $(M_n)_n$, if for any sequences $(h_n)_n$ in $\mathbb{N}$ with $h_n \to \infty$ and $(u_n)_n$ in M such that $u_n \in M_{h_n}$ for all n, $g(u_n) \to c$ and $\operatorname{grad}^-_{M_{h_n}} g(u_n) \to 0$ there exists a subsequence $(u_{n_k})_k$ which converges to a point u in M such that ($g(u) = c$ and) $\operatorname{grad}^-_M g(u) = 0$.

Also the following theorem can be proved like the classical one.

Theorem 5.8. *Assume that for all n in $\mathbb{N}$ there exists a retraction $r_n : M \to M_n$. Let Y be a closed subset of M. For any integer i we set*

$$c^*_i = \inf\{\sup g(A) \mid A \text{ is closed, } Y \subset A,\ \operatorname{cat}^*_{M,Y}(A) \geq i\}.$$

Assume that $(\mathrm{PS})^*_c$ holds for $c = c^*_i$, with respect to $(M_n)_n$ and that

$$\sup g(Y) < c^*_i < \infty.$$

Then c^*_i is a lower critical level for g. Moreover, if $c^*_i = \ldots = c^*_{i+j} = c$, then

$$\operatorname{cat}_M(\{u \in M \mid g(u) = c,\ \operatorname{grad}^-_M g(u) = 0\}) \geq j + 1.$$

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