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INEQUALITIES IN METRIC SPACES WITH APPLICATIONS

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Abstract. We prove the parallelogram inequalities in metric spaces and use them to obtain the fixed points of involutions.

1. Introduction and preliminaries

The parallelogram law is the one of fundamental property of Hilbert spaces which distinguishes them from general Banach spaces. This law is used in solving many problems in Hilbert spaces. Recently several authors have tried this idea for solving problems in Banach spaces by establishing equalities and usually inequalities analogous to the parallelogram law, see for example K. Goebel and W. A. Kirk [8], S. Reich [17], T. C. Lim [14], C. Zalinescu [23], I. E. Poffald and S. Reich [15], B. Prus and R. Smarzewski [16], H. K. Xu [22] and J. Górnicki [11]. W. Takahashi [21] introduced the notion of convexity in metric spaces and proved that all normed spaces and their convex subsets are convex metric spaces. Moreover, W. Takahashi also gave many examples of convex metric spaces which are not embedded in any normed/Banach space. Subsequently M. D. Guay, K. L. Singh and J. H. M. Whitfield [12], T. Shimizu and W. Takahashi [18], L. Gajič and M. Stojakovič [6], L. Ciric [5], I. Beg et al [2]–[4] and many other authors have studied fixed point theorems on convex metric spaces. Recently T. Shimizu and W. Takahashi [19] introduced the concept of uniform

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convexity in convex metric spaces, studies its properties and constructed example of a uniformly convex metric space which is not normed space. The purpose of this paper is to establish some inequalities in uniformly convex complete metric spaces analogous to the parallelogram law in Hilbert spaces. Applications of the inequalities obtained to fixed points of k-Lipschitzian involutions are also shown.

DEFINITION 1.1 ([21]). Let (X, d) be a metric space. A mapping $W : X\times X\times [0, 1] \to X$ is said to be a *convex structure* on X if for each $(x, y, \lambda) \in X\times X\times[0, 1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).$$

Metric space X together with the convex structure W is called a *convex metric space*. Obviously $W(x, x, \lambda) = x$.

Let X be a convex metric space. A nonempty subset K of X is said to be *convex* if $W(x, y, \lambda) \in K$ whenever $(x, y, \lambda) \in K \times K \times [0, 1]$. W. Takahashi [21] has shown that open spheres $B(x, r) = \{y \in X : d(x, y) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(x, y) \leq r\}$ are convex.

DEFINITION 1.2. A convex metric space X is said to have *property* (B) if it satisfies:

$$d(W(x, a, \alpha), W(y, a, \alpha)) = \alpha\, d(x, y).$$

Taking $x = a$ property (B) implies $\alpha\, d(a, y) = d(a, W(y, a, \alpha))$.

DEFINITION 1.3. A convex complete metric space X is said to be *uniformly convex* if for all $x, y, a \in X$,

$$[d(a, W(x, y, 1/2))]^2 \\ \leq \frac{1}{2}\left(1 - \delta\left(\frac{d(x, y)}{\max\{d(a, x), d(a, y)\}}\right)\right)([d(a, x)]^2 + [d(a, y)]^2),$$

where the function δ is a strictly increasing function on the set of strictly positive numbers and $\delta(0) = 0$.

Uniformly convex Banach spaces are uniformly convex metric spaces.

DEFINITION 1.4. A uniformly convex metric space X is said to be 2-*uniformly convex* if there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c\varepsilon^2$.

DEFINITION 1.5. Let K be a nonempty subset of a metric space X. A mapping $T : K \to K$ is call k-Lipschitzian if for all x, y in K, $d(Tx, Ty) \leq k\ d(x, y)$. A mapping $T : K \to K$ is called an involution if $T^2 = I$, where I denotes the identity map (see K. Goebel and W. A. Kirk [9]).

Beg Theorem 2.3 from [1], can be reformulated as:

THEOREM 1.6 ([1]). *Let X be a uniformly convex metric space having property* (B) *and K be a nonempty closed convex bounded subset of X. If $T : K \to K$ is a nonexpansive mapping then the set of fixed points of T is nonempty.*

REMARK 1.7. Let X be an uniformly convex metric space. If $d(x, z) = r_1, d(y, z) = r_2$ and $d(x, y) = r_1 + r_2$ then $z = W(x, y, r_2/(r_1 + r_2))$.

2. The inequalities

In this section we establish the parallelogram inequalities in metric spaces which are analogous to the parallelogram law in Hilbert spaces. Applications of these inequalities will be given in Section 3.

THEOREM 2.1. *Let (X, d) be a uniformly convex metric space having property* (B). *Then X is 2-uniformly convex if and only if there exists a number $c > 0$ such that, for all a, x, y in X,*

$$[d(a, W(x, y, 1/2))]^2 \leq 1/2([d(a, x)]^2 + [d(a, y)]^2 - c[d(x, y)]^2). \tag{1}$$

PROOF. *Necessity.* Let X be a 2-uniformly convex then $\delta(\varepsilon) \geq c\varepsilon^2$. Now for every $a, x, y \in X$,

$$\begin{aligned}
&[d(a,W(x, y, 1/2))]^2 \\
&\quad\leq \frac{1}{2}\left[1 - \delta\left(\frac{d(x, y)}{\max\{d(a, x), d(a, y)\}}\right)\right]([d(a, x)]^2 + [d(a, y)]^2) \\
&\quad\leq \frac{1}{2}([d(a, x)]^2 + [d(a, y)]^2) - \frac{c}{2}[d(x, y)]^2 \frac{[d(a, x)]^2 + [d(a, y)]^2}{[\max\{d(a, x), d(a, y)\}]^2} \\
&\quad\leq \frac{1}{2}([d(a, x)]^2 + [d(a, y)]^2 - c[d(x, y)]^2).
\end{aligned}$$

Sufficiency. Assume that inequality (1) holds. For each $\alpha > 0$ define

$$\xi(\alpha) := \inf\left\{\frac{1}{2}[d(a, x)]^2 + \frac{1}{2}[d(a, y)]^2 - [d(a, W(x, y, 1/2))]^2 : a, x, y \text{ in } X \text{ and } d(x, y) = \alpha\right\}.$$

and $\xi(0) = 0$. Then (i) $\xi(\alpha) > 0$, (ii) $\xi(\alpha\beta) = \alpha^2\xi(\beta) = \beta^2\xi(\alpha)$ for all $\alpha, \beta > 0$. (i.e. $\xi(\alpha) = \alpha^2\xi(1)$ for $\alpha > 0$).

By definition of ξ, for a, x, y in X, we have

$$\begin{aligned}
[d(a, W(x, y, 1/2))]^2 &\leq \frac{1}{2}[d(a, x)]^2 + \frac{1}{2}[d(a, y)]^2 - \xi(\alpha) \quad \text{(using (ii))} \\
&= \frac{1}{2}[d(a, x)]^2 + \frac{1}{2}[d(a, y)]^2 - \alpha^2\xi(1) \\
&= \frac{1}{2}([d(a, x)]^2 + [d(a, y)]^2 - c[d(x, y)]^2)
\end{aligned}$$

$$= \frac{1}{2}\left(1 - \frac{c[d(x,y)]^2}{[d(a,x)]^2 + [d(a,y)]^2}\right)([d(a,x)]^2 + [d(a,y)]^2),$$

where $c = 2\xi(1)$. It further implies

$$[d(a, W(x,y,1/2))]^2 \leq \frac{1}{2}\left(1 - \delta\left(\frac{d(x,y)}{\max\{d(a,x), d(a,y)\}}\right)\right)([d(a,x)]^2 + [d(a,y)]^2)$$

where δ is defined by

$$\delta\left(\frac{d(x,y)}{\max\{d(a,x), d(a,y)\}}\right) := 2\xi(1) \ \left(\frac{[d(x,y)]^2}{[d(a,x)]^2 + [d(a,y)]^2}\right),$$

It can be easily shown that δ is strictly increasing function on the set of strictly positive numbers with $\delta(0) = 0$ and $\delta(\varepsilon) \geq c\varepsilon^2$ where $c = 2\xi(1)$. Hence X is a 2-uniformly convex metric space. □

Theorem 2.2. *Let X be a 2-uniformly convex metric space having property* (B)*, K a nonempty closed convex subset of X. Let $\{x_n\}$ be a bounded sequence. Then there exists a unique point z in K such that*

$$\limsup_{n\to\infty}[d(x_n, z)]^2 \leq \limsup_{n\to\infty}[d(x_n, x)]^2 - c[d(x,z)]^2, \tag{2}$$

for every x in K, where c is the constant given in (1).

Proof. Let $p(x) = \limsup_{n\to\infty}[d(x_n, x)]^2$, $x \in X$. By Theorem 2.1,

$$p(W(x,y,1/2)) \leq \frac{1}{2}p(x) + \frac{1}{2}p(y) - \frac{c}{2}[d(x,y)]^2.$$

Thus there is a unique point z in K such that $p(z) = \inf_{x\in K} p(x)$. It follows from inequality (1), that

$$p(W(x,z,1/2)) \leq \frac{1}{2}p(x) + \frac{1}{2}p(z) - \frac{c}{2}[d(x,z)]^2$$

for x in K, also $p(z) \leq p(W(x,z,1/2))$. It implies that,

$$0 \leq p(x) - p(z) - c[d(x,z)]^2,$$

and the inequality (2) follows. □

3. Fixed points

Let K be a nonempty closed convex subset of a convex complete metric space X. Let $T : K \to K$ be a mapping. For $x_0 \in K$, we define,

$$x_{n+1} = W(x_n, Tx_n, 1/2).$$

If there exists a $c, 0 \leq c < 1$ such that

$$d(x_{n+2}, x_{n+1}) \leq cd(x_{n+1}, x_n), \quad n = 0, 1, 2, \ldots$$

Then the sequence $\{x_n\}$ converges in K. Indeed, from (4), it follows that $d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$ and $\{x_n\}$ converges to a point p (say) in K.

Theorem 3.1. *Let X be a convex complete metric space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a k-Lipschitzian map. Assume that there exists real constants a, b such that $0 \leq a < 1$ and $b > 0$. If for arbitrary $x \in K$ there exists $u \in k$ such that*

(i) *$d(Tu, u) \leq ad(Tx, x)$, and*
(ii) *$d(u, x) \leq bd(Tx, x)$.*

Then T has a fixed point in K.

Proof. Let $x_0 \in K$ be an arbitrary point. Consider a sequence $\{x_n\} \subset K$ which satisfies the following conditions

$$d(Tx_{x+1}, x_{n+1}) \leq ad(Tx_n, x_n),$$

and

$$d(x_{n+1}, x_n) \leq bd(Tx_n, x_n), \quad n = 0, 1, 2, \ldots$$

It implies that

$$d(x_{n+1}, x_n) \leq ba^n d(Tx_0, x_0).$$

If further implies that $\{x_n\}$ is a convergent sequence. Let $\lim_{n\to\infty} x_n = x$. Then,

$$\begin{aligned} d(Tx, x) &\leq d(Tx, Tx_n) + d(Tx_n, x_n) + d(x_n, x) \\ &\leq (1 + k)d(x, x_n) + a^n d(Tx_0, x_0) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence $Tx = x$. □

Theorem 3.2. *Let X be a convex complete metric space and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a k-Lipschitzian involution. If $1 \leq k < 2$ then T has a fixed point in K.*

Proof. For any $x \in K$, let $u = W(x, Tx, 1/2)$ Then,

$$d(u, x) = d(W(x, Tx, 1/2), x) \leq \frac{1}{2} d(Tx, x),$$

and

$$\begin{aligned} d(u, Tu) &= d(W(x, Tx, 1/2), Tu) \leq \frac{1}{2}[d(x, Tu) + d(Tx, Tu)] \\ &= \frac{1}{2}[d(T^2x, Tu) + d(Tx, Tu)] \leq \frac{k}{2}[d(Tx, u) + d(x, u)] \\ &= \frac{k}{2}[d(Tx, W(x, Tx, 1/2)) + d(x, W(x, Tx, 1/2))] \leq \frac{k}{2} d(Tx, x). \end{aligned}$$

Where by assumption $k/2 < 1$.

Now, for arbitrary $x_0 \in K$, define inductively a sequence $\{x_n\} \subset K$ by

$$x_{n+1} = W(x_n, Tx_n, 1/2),$$

for $n = 0, 1, 2, \ldots$ By Theorem 3.1 this sequence is convergent $\lim_{n\to\infty} x_n = x$ (say) and $Tx = x$. $\square$

Remark 3.3. As immediate corollary to Theorem 3.4 we have K. Goebel [7].

Theorem 3.4. *Let X be a 2-uniformly convex metric space having property (B) and K be a nonempty closed convex subset of X. Let $T : K \to K$ be a k-Lipschitzian involution. If $0 \leq k < \sqrt{4+2c}$, then T has a fixed point.*

Proof. For any $x \in K$, let $u = W(x, Tx, 1/2)$. Then Theorem 2.1, implies that

$$\begin{aligned}
[d(u,Tu)]^2 &= [d(Tu, W(x,Tx,1/2)]^2 \\
&\leq \frac{1}{2}([d(Tu,x)]^2 + [d(Tu,Tx)]^2 - c[d(x,Tx)]^2) \\
&= \frac{1}{2}([d(Tu,T^2x)]^2 + [d(Tu,Tx)]^2 - c[d(x,Tx)]^2) \\
&\leq \frac{1}{2}([kd(Tx,u)]^2 + [kd(u,x)]^2 - c[d(x,Tx)]^2) \\
&= \frac{1}{2}([k\ d(Tx, W(x,Tx,1/2))]^2 \\
&\qquad + [k\ d(W(x,Tx,1/2),x)]^2 - c[d(x,Tx)]^2) \\
&\leq [(k^2-2c)/4][d(x,Tx)]^2,
\end{aligned}$$

where by assumption $(k^2 - 2c)/4 < 1$. For arbitrary $x_0 \in K$, defining inductively a sequence $\{x_n\} \subset K$ by $x_{n+1} = W(x_n, Tx_n, 1/2)$, $n = 0, 1, 2, \ldots$. Theorem 3.1 implies that this sequence is convergent, let $\lim_{n\to\infty} x_n = x$. Then $Tx = x$. $\square$

Theorem 3.4. *Let X be a 2-uniformly convex metric space having property (B) and let K be a nonempty closed convex bounded susbet of X.If $T : K \to K$ satisfies for every x, y in K,*

(i) $d(T^2x, T^2y) \leq d(x,y)$, *and*
(ii) $d(Tx,Ty) \leq k\ d(x,y)$,

with $0 \leq k < \sqrt{4+2c}$, then T has a fixed point in K.

Proof. By Theorem 1.6, $K_1 = \{x \in K : T^2x = x\}$ is nonempty and closed. Also K_1 is convex. To prove this fact let $x_1, x_2 \in K_1$ and $t \in (0,1)$, then we have

$$\begin{aligned}
d(x_1,x_2) &\leq d(x_1, T^2(W(x_1,x_2,t))) + d(T^2(W(x_1,x_2,t)), x_2) \\
&\leq d(T^2x_1, T^2(W(x_1,x_2,t))) + d(T^2(W(x_1,x_2,t)), T^2x_2) \\
&\leq d(x_1, (W(x_1,x_2,t)) + d(W(x_1,x_2,t), x_2) \\
&\leq (1-t)d(x_1,x_2) + td(x_1,x_2) = d(x_1,x_2).
\end{aligned}$$

It implies that

$$(5)\quad d(x_1, T^2(W(x_1, x_2, t))) + d(T^2(W(x_1, x_2, t)), x_2) = d(x_1, (W(x_1, x_2, t)) + d(W(x_1, x_2, t), x_2) = d(x_1, x_2).$$

Since $d(T^2x, T^2y) \le d(x, y), T^2x_1 = x_1$ and $T^2x_2 = x_2$ therefore

$$(6)\qquad d(x_1, T^2(W(x_1, x_2, t))) = d(x_1, W(x_1, x_2, t)) = r_1$$

and

$$(7)\qquad d(x_2, T^2(W(x_1, x_2, t))) = d(x_2, W(x_1, x_2, t)) = r_2.$$

Now, using equality (5), we obtain

$$(8)\qquad \frac{r_2}{r_1 + r_2} = \frac{d(x_2, W(x_1, x_2, t))}{d(x_1, W(x_1, x_2, t)) + d(x_2, W(x_1, x_2, t))} = \frac{td(x_1, x_2)}{d(x_1, x_2)} = t.$$

Equalities (5)–(8) together with Remark 1.7 imply

$$T^2(W(x_1, x_2, t)) = W\left(x_1, x_2, \frac{r_1}{r_1 + r_2}\right) = W(x_1, x_2, t).$$

It further implies that K_1 is convex. Moreover, $T(K_1) = K_1$ and $T^2 = I$ on K_1. Hence Theorem 3.4 implies that T has a fixed point in K_1. $\square$

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