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ON SOME CLASSES OF OPERATOR INCLUSIONS WITH LOWER SEMICONTINUOUS NONLINEARITIES

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Abstract. We consider a class of multimaps which are the composition of a superposition multioperator $\mathcal{P}_F$ generated by a nonconvex-valued almost lower semicontinuous nonlinearity F and an abstract solution operator S. We prove that under some suitable conditions such multimaps are condensing with respect to a special vector-valued measure of noncompactness and construct a topological degree theory for this class of multimaps yielding some fixed point principles. It is shown how abstract results can be applied to semilinear inclusions, inclusions with m-accretive operators and time-dependent subdifferentials, nonlinear evolution inclusions and integral inclusions in Banach spaces.

1. Introduction

Differential inclusions with lower semicontinuous right-hand sides are the object of the constant interest of many researchers in the recent years (see, for example [12]–[16], [10], [7] and others). Much of the importance of this class stems from the fact that in this case convexity of the values of the multivalued nonlinearity is redundant that allows to cover a large number of important applications.

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In the case of inclusions with convex-valued multis the methods based on the topological degree theory proved their high efficiency. The non-convexity does not allow to apply the machinery of the classical degree theory directly. From the other side, while study semilinear differential inclusions in Banach spaces, in many cases which are important for applications, the semigroup generated by linear part is neither compact nor analytic. This leads to the case when the corresponding solution operator is non-compact and this circumstance also does not permit the application of topological degree for compact maps.

It is known that mild solution of semilinear type differential inclusion may be interpreted as a fixed point:

$$x \in S \circ \mathcal{P}_F(x) \tag{$*$}$$

where $\mathcal{P}_F$ is a superposition multioperator generated by multivalued nonlinearity F and S is a solution operator of inclusion. In a pure semilinear case operator S may be written explicitly in terms of a semigroup generated by a linear part of the differential inclusion (see, for example [12], [13], [17]). In a nonlinear case the solution operator S was studied by many authors (see [2], [17], [18] and other works).

In a recent paper [12] it was mentioned that the regularity condition for the nonlinearity F with respect to the Hausdorff measure of noncompactness implies that the multioperator $S \circ \mathcal{P}_F$ is condensing with respect to a special measure of noncompactness in a functional space. In a present work we consider the case of an abstract solution operator satisfying conditions (S1) and (S2) below and assume that multivalued nonlinearity F is almost lower semicontinuous and nonconvex-valued. We prove that the multioperator $S \circ \mathcal{P}_F$ is condensing with respect to the vector-valued measure of noncompactness ψ and this allows to construct a special topological degree theory allowing to study not only the semilinear version of the inclusion ($*$) but to consider also significantly more wide classes of differential and integral inclusions in a Banach space.

The paper is organized in the following way. After preliminaries, we construct the topological degree theory for a special class of condensing multioperators. We describe the main properties of the degree including the general fixed point principle and derive from it the nonlinear alternative and Leray–Schauder type fixed point theorem. As application of the developed abstract theory we consider the solvability problems for semilinear inclusions, inclusions with m-accretive operators and time-dependent subdifferentials, nonlinear evolution inclusions and Volterra type problems for integral inclusions in a Banach space.

Preliminaries. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, $P(Y)$ denote the collection of all nonempty subsets of Y. A multivalued map (multimap)

$$\mathcal{F} : X \to P(Y)$$

is said to be *lower semicontinuous at a point* $x \in X$ provided for every open set $V \subset Y$ such that $\mathcal{F}(x) \cap V \neq \emptyset$ there exists such $\delta > 0$ that $\mathcal{F}(x') \cap V \neq \emptyset$ for all $x' \in X$, $\rho_X(x, x') < \delta$. If this property is fulfilled for every point $x \in X$ then $\mathcal{F}$ is called lower semicontinuous (l.s.c.) (see, e.g. [4]–[6] for further details).

Let E be a Banach space; $K(E)$ denote the collection of all nonempty compact subsets of E. A multifunction $G : [0, d] \to K(E)$ is said to be

(i) *measurable* if $G^{-1}(V) = \{t \in [0, d] : G(t) \subset V\}$ is Lebesgue measurable for every open set $V \subseteq E$ (see e.g. [8], [5], [6] for equivalent definitions and details),

(ii) *p-integrable* ($p \geq 1$) provided it has a Bochner p-summable selection $g \in L^p([0, d]; E)$, i.e. $g(t) \in G(t)$ for a.e. $t \in [0, a]$; (see e.g. [8], [4] for equivalent defintions and details). For p-integrable multifunction G the set of all p-summable selections of G will be denoted as S^p_G.

Recall also the following notions (see, e.g. [1]). Let E be a Banach space, $B(E)$ denote the collection of all bounded subsets of E and $(A, \geq)$ be a partially ordered set. A map

$$\beta : B(E) \to A$$

is called *a measure of noncompactness* (MNC) in E if, for every $\Omega \in B(E)$,

$$\beta(\overline{\mathrm{co}}\, \Omega) = \beta(\Omega).$$

A MNC β is called:

(i) *monotone* if $\Omega_0, \Omega_1 \in B(E)$, $\Omega_0 \subseteq \Omega_1$ implies $\beta(\Omega_0) \leq \beta(\Omega_1)$,

(ii) *nonsingular* if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E, \Omega \in B(E)$,

(iii) *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As the example of MNC possessing all these properties we may consider the Hausdorff MNC

$$\chi(\Omega) = \inf \{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

At last, let us consider the following notion. The nonempty subset $M \subset L^1([0, d]; E)$ is said to be *decomposable* provided for every $f, g \in M$ and each Lebesgue measurable subset m in $[0, d]$,

$$f\chi_m + g\chi_{[0,d]\setminus m} \in M,$$

where χ_m is the characteristic function of the set m.

Lemma 1 (see [11]). *Let X be a compact metric space, E a Banach space, $Z = L^1([0, d]; E)$. Then every l.s.c. multimap $\mathcal{F} : X \to P(Z)$ with closed decomposable values has a continuous selection, i.e. there exists a continuous map $f : X \to Z$ such that $f(x) \in \mathcal{F}(x)$ for all $x \in X$.*

2. Topological degree for a class of condensing multimaps

Let E be a Banach space. We will consider a multimap $F : [0,d]\times E \to K(E)$ satisfying the following assumptions.

- (F1) F is almost lower semicontinuous (a.l.s.c.) in the sense that for given $\varepsilon > 0$ and $\emptyset \neq C \subset E$ compact, there exists a compact $I_\varepsilon \subset [0,d]$ with $\operatorname{meas}([0,d] \setminus I_\varepsilon) < \varepsilon$ such that restriction of F on $I_\varepsilon \times C$ is l.s.c. and $\overline{\operatorname{span}}\, F(I_\varepsilon \times C)$ is separable,
- (F2) there exisits $I \subset [0,d]$ of full measure such that for each $D \in B(E)$ the set $F(I \times D)$ is bounded,
- (F3) there exists a function $k(\,\cdot\,) \in L^1_+[0,d]$ such that for every bounded set $D \subset E$ we have that

$$\chi(F(t,D)) \leq k(t)\cdot \chi(D) \quad \text{for a.e. } t \in [0,d].$$

From (F1) it follows easily (cf. e.g. [10]) that for every function $x(\,\cdot\,) \in C([0,d];E)$ the multifunction $F(t,x(t))$ is measurable with range in a separable Banach space and thus p-integrable for any p, $1 \leq p \leq \infty$, as it follows from the property (F2). Hence, we may define the superposition multioperator $\mathcal{P}_F : C([0,d];E) \to P(L^p([0,d];E))$ in the following way:

$$\mathcal{P}_F(x) = S^p_{F(\,\cdot\,,x(\,\cdot\,))}.$$

Also it is clear that, for every $x \in C([0,d];E)$, the set $\mathcal{P}_F(x)$ is closed and decomposable. Further, following [16], [10] and [12], we may prove the next statement.

Lemma 2. *The multimap $\mathcal{P}_F$ is l.s.c.*

As an easy consequence of Lemmas 1 and 2 we have the following statement:

Corollary 1. *Let $X \subset C([0,d];E)$ compact. Then, for every $p \geq 1$, the superposition multioperator $\mathcal{P}_F : X \to P(L^p([0,d];E))$ has a continuous selection.*

Consider now a continuous map $S : L^p([0,d];E) \to C([0,d];E)$ satisfying the following assumptions:

- (S1) there is a constant $N > 0$ such that, for every $f,g \in L^p([0,d];E)$,

$$\|S(f)(t) - S(g)(t)\| \leq N \int_0^t \|f(s) - g(s)\|\, ds \quad \text{for every } t \in [0,d],$$

- (S2) for every compact set $K \subset E$ the set $S(M_K)$, where

$$M_K = \{f \in L^p([0,d];E) : f(t) \in K \text{ for a.e. } t \in [0,d]\},$$

  is relatively compact in $C([0,d];E)$.

Denote $\ell = C([0,d];E)$ and let $U \subset \ell$ be an open bounded subset. Under above assumptions consider a composition

$$\mathcal{F} = S \circ \mathcal{P}_F : \overline{U} \to P(\ell) \tag{1}$$

and assume that it is fixed point tree on the boudary ∂U:

$$x \notin \mathcal{F}(x) \quad \text{for all } x \in \partial U.$$

Our aim is to present the topological degree for the multimap $\mathcal{F}$. The fact that this multimap is condensing with respect to a special MNC in ℓ will be crucial in our constructions.

Consider the partially ordered set $(\mathbb{R}\times\mathbb{R}, \geq)$ with the order $\geq$ induced by the cone $\mathbb{R}^2$ of nonnegative pairs and define the following MNC $\psi : B(\ell) \to (\mathbb{R}\times\mathbb{R}, \geq)$:

$$\psi(\Omega) = (\nu(\Omega), \mathrm{mod}_C(\Omega))$$

where

$$\nu(\Omega) = \max_{\Delta\in D(\Omega)} \sup_{t\in[0,d]} \{e^{-Lt}\chi(\Delta(t))\},$$

$D(\Omega)$ denotes the collection of all denumerable subsets of Ω; $\Delta(t) = \{y(t) :\ y \in \Delta\}$, $L > 0$ and

$$\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0} \sup_{x\in\Omega} \max_{\substack{|t_1-t_2|<\delta \\ t_1,t_2\in[0,d]}} \|x(t_1) - x(t_2)\|$$

is the modulo of equicontinuity of the set Ω. It is easy to see that the MNC ψ is monotone and nonsingular.

Lemma 3. *For L large enough, the multioperator $\mathcal{F}$ is ψ-condensing, i.e. the relation*

$$\psi(\mathcal{F}(\Omega)) \geq \psi(\Omega) \tag{2}$$

for any bounded $\Omega \subset \overline{U}$ implies the relative compactness of Ω.

For the proof we need the following result proved in [9], [13] which is given in the form convenient for the sequel.

Lemma 4. *Let $\{f_n\} \subset L^p([0,d];E)$ be an p-integrably bounded sequence such that*

$$\chi(\{f_n(t)\}) \leq \mu(t) \quad \text{for a.e. } t \in [0,d]$$

where $\mu(\,\cdot\,) \in L^1_+[0,d]$. Then for every $\varepsilon > 0$ there exist a compact $K_\varepsilon \subset E$, a measurable set $e_\varepsilon \subset [0,d]$, and a sequence of functions $\{g_n^\varepsilon\} \subset L^p([0,d];E)$ such that:

(i) $\mathrm{meas}\, e_\varepsilon < \varepsilon$,
(ii) $\{g_n^\varepsilon(t)\} \subset K_\varepsilon$ *for a.e.* $t \in [0,d]$,
(iii) $\|f_n(t) - g_n^\varepsilon(t)\| < 2\mu(t) + \varepsilon$ *for a.e.* $t \in [0,d] \setminus e_\varepsilon$.

Proof of Lemma 3. Choose $L > 0$ large enough to provide

$$q = 2N \cdot \sup_{t\in[0,d]} e^{-Lt} \int_0^t k(s)e^{Ls}\,ds < 1$$

where N is a constant from the condition (S1) and $k(\,\cdot\,)$ is the function from the condition (F3). Let us demonstrate that the multimap $\mathcal{F}$ is (q,ν)-condensing, i.e.

$$\nu(\mathcal{F}(\Omega)) \le q\nu(\Omega) \tag{3}$$

for every bounded $\Omega \subset \overline{U}$. In fact, let $\{z_n\}$ be any sequence of elements from $\mathcal{F}(\Omega)$. Then there exists a sequence $\{x_n\}$ in Ω, a sequence of elements $\{f_n\}$ such that $f_n \in \mathcal{P}_F(x_n), n \ge 1$, and

$$z_n = S(f_n) \quad \text{for all } n \ge 1. \tag{4}$$

Notice that from the condition (F2) it follows that the sequence $\{f_n\}$ is bounded and we have also that, for a.e. $t \in [0,d]$,

$$\chi(\{f_n(t)\}) \le \chi(F(t,\{x_n(t)\})) \le k(t)\cdot\chi(\{x_n(t)\}) \le k(t)\cdot e^{Lt}\nu(\Omega).$$

By virtue of Lemma 4, for the sequence $\{f_n(t)\}$ there exist a a compact set $K_\varepsilon \subset E$, a measurable set $e_\varepsilon \subset [0,d]$, and a sequence of functions $\{g_n^\varepsilon\} \subset L^p([0,d];E)$ satisfying the properties (i)–(iii) for $\mu(t) = k(t)\cdot e^{Lt}\nu(\Omega)$. Further, from the hypothesis (S2) it follows that the set $\{Sg_n^\varepsilon\}$ is relatively compact in ℓ. But applying the properties (S1) and (iii) we obtain that

$$\begin{aligned} e^{-Lt}\|Sf_n(t) - Sg_n^\varepsilon(t)\| &\le Ne^{-Lt}\int_0^t \|f_n(s) - g_n^\varepsilon(s)\|\,ds \\ &\le 2Ne^{-Lt}\nu(\Omega)\int_0^t k(s)\cdot e^{Ls}\,ds + C\varepsilon \le q\cdot\nu(\Omega) + C\varepsilon, \end{aligned} \tag{5}$$

where C is a certain constant. Now the estimate (3) follows from the arbitrariness of ε.

Further, let us mention that the equality $\nu(\Omega) = 0$ implies the relative compactness of the set $\mathcal{F}(\Omega)$ (notice that the MNC ν is not regular). In fact, from the estimate (5) it follows that in this case for any sequence $\{z_n\}$ defined by the equality (4) we may construct a compact net consisting of functions $\{Sg_n^\varepsilon\}$ which would be arbitrary close to $\{z_n\}$.

Now from relations (2) and (3) it follows that

$$\nu(\Omega) \le \nu(\mathcal{F}(\Omega)) \le q\cdot\nu(\Omega)$$

and hence $\nu(\Omega) = 0$ and so $\mathcal{F}(\Omega)$ is relatively compact. But then $\operatorname{mod}_C(\mathcal{F}(\Omega)) = 0$ and the relation (2) implies that also $\operatorname{mod}_C(\Omega) = 0$, and so $\psi(\Omega) = 0$. From the known Arzela–Ascoli criterion it follows that the MNC ψ is regular and hence Ω is relatively compact. □

Now let X be a closed subset of a Banach space E and $\mathcal{G} : X \to P(E)$ be a multimap. Recall (see e.g. [4], [5]) that a closed convex set $T \subseteq E$ is said to be fundamental for $\mathcal{G}$ if:

(a) $\mathcal{G}(X \cap T) \subseteq T$,
(b) $x_0 \in \overline{\text{co}}\,(\mathcal{G}(x_0) \cup T)$ implies $x_0 \in T$.

We emphasize that this definition does not exclude the case $T = \emptyset$ or $X \cap T = \emptyset$, which necessarily implies that the fixed points set $\text{Fix}\,\mathcal{G} := \{x \in X : x \in \mathcal{G}(x)\}$ is empty.

Let us note that the whole space E and $\overline{\text{co}}\,\mathcal{G}(X)$ are examples of fundamental sets.

The following properties of fundamental sets can be easily verified.

Proposition 1. *The fixed points set* $\text{Fix}\,\mathcal{G}$ *is included in every fundamental set of* $\mathcal{G}$.

Proposition 2. *If T is a fundamental set for a multimap $\mathcal{G}$ and $P \subset T$, then the set*
$$\widetilde{T} = \overline{\text{co}}\,(\mathcal{G}(X \cap T) \cup P)$$
is also fundamental.

Proposition 3. *If $\{T_\tau\}$ is an arbitrary system of fundamental sets of $\mathcal{G}$ then the set*
$$\widehat{T} = \bigcap_\tau T_\tau$$
is also fundamental.

Lemma 5. *The multimap $\mathcal{F}$ given in* (1) *has a compact fundamental set T_∞ such that $\overline{U} \cap T_\infty \neq \emptyset$.*

Proof. Take an arbitrary point $p \in \overline{U}$ and consider the collection $\{T_\sigma\}$ of all fundamental sets of $\mathcal{F}$ containing p. Notice that this collection is nonempty since it contains ℓ. Now the set
$$T_\infty = \bigcap_\sigma T_\sigma$$
is the desirable one. In fact, from the minimality of T_∞ it follows that
$$T_\infty = \overline{\text{co}}\,(\mathcal{F}(\overline{U} \cap T_\infty)) \cup p)$$
and therefore
$$\psi(\overline{U} \cap T_\infty) \leq \psi(T_\infty) = \psi(\mathcal{F}(\overline{U} \cap T_\infty)),$$
due to the monotonicity and nonsingularity properties of the MNC ψ. Applying Lemma 3 we obtain that the set $\overline{U} \cap T_\infty$ is compact and thus $\nu(\overline{U} \cap T_\infty) = 0$. This implies that (by the argument following the estimate (5) in the proof of Lemma 3) the set $\mathcal{F}(\overline{U} \cap T_\infty)$ is relatively compact. Thus T_∞ is compact, too. □

From Corollary 1 it follows that there exists a continuous selection $\gamma : \overline{U} \cap T_\infty \to L^p([0,d];E)$ of a superposition multioperator $\mathcal{P}_F$. Consider a compact continuous map $S \circ \gamma : \overline{U} \cap T_\infty \to T_\infty$. It is clear that it is fixed point free on the relative boundary ∂U_{T_∞}.

Definition. The topological degree $\operatorname{Deg}(\mathcal{F}, \overline{U})$ of a multimap $\mathcal{F} = S \circ \mathcal{P}_F$ is defined as

$$\operatorname{Deg}(\mathcal{F}, \overline{U}) := \deg_{T_\infty}(S \circ \gamma, \partial U_{T_\infty})$$

where $\deg_{T_\infty}$ denotes the relative topological degree of a compact continuous map (see e.g. [3]).

Let us justify the correctness of the above definition.

Lemma 6. *The degree* $\operatorname{Deg}(\mathcal{F}, \overline{U})$ *does not depend on the choice of a selection* γ.

Proof. In fact, let γ and δ be two continuous selections of a superposition multioperator $\mathcal{P}_F$. Then the maps $S \circ \gamma$ and $S \circ \delta$ are homotopic on the relative boundary ∂U_{T_∞}: the family $h : \partial U_{T_\infty} \times [0,1] \to T_\infty$,

$$h(x,\lambda) = S \circ (\kappa_{[0,\lambda d]} \cdot \gamma(x) + \kappa_{[\lambda d, d]} \cdot \delta(x)),$$

where κ denotes the characteristic function of the set, is obviously continuous, $x \neq h(x,\lambda)$ for all $(x,\lambda) \in \partial U_{T_\infty} \times [0,1]$ and $h(\cdot,0) = S \circ \delta$, $h(\cdot,1) = S \circ \gamma$. Therefore

$$\deg_{T_\infty}(S \circ \delta, \partial U_{T_\infty}) = \deg_{T_\infty}(S \circ \gamma, \partial U_{T_\infty}). \qquad \square$$

Lemma 7. *The degree* $\operatorname{Deg}(\mathcal{F}, \overline{U})$ *does not depend on the choice of the fundamental set* T_∞.

Proof. Let T'_∞, T''_∞ be two compact fundamental sets of $\mathcal{F}$. Notice that in case $T'_\infty \cap T''_\infty = \emptyset$ the multimap $\mathcal{F}$ is fixed point free, and therefore

$$\deg_{T'_\infty}(S \circ \gamma', \partial U_{T'_\infty}) = \deg_{T''_\infty}(S \circ \gamma'', \partial U_{T''_\infty}) = 0.$$

Otherwise consider a compact fundamental set $\widetilde{T} = T'_\infty \cap T''_\infty$. Let $\rho : \ell \to \widetilde{T}$ be any retraction. The family $g : \partial U_{T'_\infty} \times [0,1] \to T'_\infty$,

$$g(x,\lambda) = (1-\lambda) \cdot S \circ \gamma'(x) + \lambda \cdot \rho \circ S \circ \gamma'(x)$$

is fixed point free: if $x = g(x,\lambda)$ then

$$x \in \overline{\operatorname{co}}\,(S \circ \gamma'(x) \cup \widetilde{T}) \subseteq \overline{\operatorname{co}}\,(\mathcal{F}(x) \cup \widetilde{T})$$

and hence $x \in \widetilde{T}$ but then also $S \circ \gamma'(x) \in \widetilde{T}$ and $\rho \circ S \circ \gamma'(x) = S \circ \gamma'(x)$ and so $x = S \circ \gamma'(x)$ giving a contradiction.

Now applying the homotopy invariance property of the relative topological degree and the principle of the map restriction (see [3]) we obtain that

$$\deg_{T'_\infty}(S\circ\gamma',\partial U_{T'_\infty}) = \deg_{T'_\infty}(\rho\circ S\circ\gamma',\partial U_{T'_\infty}) = \deg_{\widetilde{T}}(S\circ\gamma',\partial U_{\widetilde{T}}).$$

Analogously,

$$\deg_{T''_\infty}(S\circ\gamma'',\partial U_{T''_\infty}) = \deg_{\widetilde{T}}(S\circ\gamma'',\partial U_{\widetilde{T}})$$

and an aplication of Lemma 6 gives

$$\deg_{\widetilde{T}}(S\circ\gamma',\partial U_{\widetilde{T}}) = \deg_{\widetilde{T}}(S\circ\gamma'',\partial U_{\widetilde{T}}). \qquad \square$$

From the definition we may deduce usual properties of the topological degree of the multimap $S\circ\mathcal{P}_F$.

Property 1 (Normalization). *If $S\circ\mathcal{P}_F(x)\equiv A$, then*

$$\operatorname{Deg}(S\circ\mathcal{P}_F,\overline{U}) = \begin{cases} 1 & \text{if } A\subset U,\\ 0 & \text{if } A\subset \ell\setminus\overline{U}.\end{cases}$$

Property 2 (Additive dependence on the domain). *Let $\{U_j\}_{j\in J}$ be a disjoint system of open sets, $U_j\subseteq U$ such that*

$$\operatorname{Fix}(S\circ\mathcal{P}_F)\cap\Big(\overline{U}\setminus\bigcup_j U_j\Big)=\emptyset.$$

Then the degrees $\operatorname{Deg}(S\circ\mathcal{P}_F,\overline{U}_j)$ nonvanish only for a finite number of indexes j and

$$\operatorname{Deg}(S\circ\mathcal{P}_F,\overline{U}) = \sum_j \operatorname{Deg}(S\circ\mathcal{P}_F,\overline{U}_j).$$

Property 3 (Homotopy invariance). *Let the family $G:[0,d]\times E\times[0,1]\to K(E)$ satisfy the following assumptions.*

- (G1) *G is a.l.s.c. in the sense that for given $\varepsilon>0$ and $\emptyset\neq C\subset E$ compact, there exists a compact $I_\varepsilon\subset[0,d]$ with $\operatorname{meas}([0,d]\setminus I_\varepsilon)<\varepsilon$ such that restriction of G on $I_\varepsilon\times C\times[0,1]$ is l.s.c. and $\overline{\operatorname{span}}\,G(I_\varepsilon\times C\times[0,1])$ is separable,*
- (G2) *there exists $I\subset[0,d]$ of full measure such that for each $D\in B(E)$ the set $G(I\times D\times[0,1])$ is bounded,*
- (G3) *there exists a function $k(\cdot)\in L^1_+[0,d]$ such that for every bounded set $D\subset E$ we have that*

$$\chi(G(t,D,[0,1]))\le k(t)\cdot\chi(D)\qquad \text{for a.e. } t\in[0,d].$$

For a continuous family $R : L^p([0,d];E) \times [0,1] \to C([0,d];E)$ *we assume the following hypothesis:*

(R1) *there is a constant* $N > 0$ *such that for every* $f, g \in L^p([0,d];E)$ *and* $\lambda \in [0,1]$

$$\|R(f,\lambda)(t) - R(g,\lambda)(t)\| \leq N \int_0^t \|f(s) - g(s)\|\, ds \quad \text{for every } t \in [0,d],$$

(R2) *for every compact set* $K \subset E$ *the set* $R(M_K \times [0,1])$ *is relatively compact in* $C([0,d];E)$.

If $x \notin R(\,\cdot\,,\lambda) \circ \mathcal{P}_{G(\,\cdot\,,\,\cdot\,,\lambda)}(x)$ *for all* $x \in \partial U$ *and* $\lambda \in [0,1]$ *then*

$$\operatorname{Deg}(R(\,\cdot\,,0) \circ \mathcal{P}_{G(\,\cdot\,,\,\cdot\,,0)}, \overline{U}) = \operatorname{Deg}(R(\,\cdot\,,1) \circ \mathcal{P}_{G(\,\cdot\,,\,\cdot\,,1)}, \overline{U}).$$

We obtain also the following fixed point property.

Proposition 4. *If* $\operatorname{Deg}(S \circ \mathcal{P}_F, \overline{U}) \neq 0$ *then* $\operatorname{Fix} S \circ \mathcal{P}_F \neq \emptyset$.

From the above general fixed point principle one can derive other fixed point theorems for the maps under consideration. As an example we prove the Nonlinear Alternative and Leray–Schauder type fixed point theorem.

Theorem 1. *Let* $\overline{B}_r \subset \ell$ *be a closed ball with the center at the origin. Then for a multimap* $S \circ \mathcal{P}_F : \overline{B}_R \to P(\ell)$ *under assumptions* (F1)–(F3) *and* (S1), (S2) *either there exists* x_0, $\|x_0\| = r$ *and* λ, $0 < \lambda < 1$, *such that*

$$x_0 \in \lambda \cdot S \circ \mathcal{P}_F(x_0) \tag{6}$$

or

$$\operatorname{Fix} S \circ \mathcal{P}_F \neq \emptyset.$$

Proof. Assume that $S \circ \mathcal{P}_F$ is fixed point free on ∂B_r (otherwise we are done). Then the degree $\operatorname{Deg}(S \circ \mathcal{P}_F, \overline{B}_r)$ is defined. It is easy to see that the family $R : L^p([0,d];E) \times [0,1] \to C([0,d];E)$,

$$R(f,\lambda) = \lambda \cdot S(f)$$

satisfies the properties (R1), (R2). Supposing that the assumption (6) is not valid and applying homotopy and normalization properties we obtain that

$$\operatorname{Deg}(S \circ \mathcal{P}_F, \overline{B}_r) = \operatorname{Deg}(R(\,\cdot\,,1) \circ \mathcal{P}_F, \overline{B}_r) = \operatorname{Deg}(R(\,\cdot\,,0) \circ \mathcal{P}_F, \overline{B}_r) = 1. \quad \square$$

Corollary 2. *For a multimap* $S \circ \mathcal{P}_F : \ell \to P(\ell)$ *either the set*

$$\{x : x \in \lambda \cdot S \circ \mathcal{P}_F(x) \text{ for some } \lambda \in (0,1)\}$$

is unbounded or $\operatorname{Fix} S \circ \mathcal{P}_F \neq \emptyset$.

We may prove now the following abstract existence result.

Theorem 2. *Suppose that a multimap* $F : [0,d] \times E \to K(E)$ *satisfy hypothesis* (F1), (F3) *and*

(F2') $\|F(t,x)\| := \sup\{\|y\| : y \in F(t,x)\} \le K(1+\|x\|)$ *for a.e.* $t \in [0,d]$ *and* $x \in E$

holds (*where* $K > 0$). *Let* $S : L^p([0,d];E) \to \ell$ *satisfy* (S1), (S2). *Then* $\operatorname{Fix} S \circ \mathcal{P}_F \ne \emptyset$.

Proof. For some $\lambda \in (0,1)$, let $x \in \ell$, $x \in \lambda \cdot S \circ \mathcal{P}_F(x)$. Take $f \in \mathcal{P}_F(x)$ such that $x = \lambda \cdot S(f)$. Define $y = \lambda \cdot S(0)$. Then, for every $t \in [0,d]$, we see that

$$\begin{aligned}\|x(t) - y(t)\| &\le \lambda N \cdot \int_0^t \|f(s)\|\, ds \le \lambda NK \int_0^t (1 + \|x(s)\|)\, ds \\ &\le \lambda NKd + \lambda NK \int_0^t \|x(s)\|\, ds\end{aligned}$$

and hence

$$\|x(t)\| \le \|y\| + \lambda NKd + \lambda NK \int_0^t \|x(s)\|\, ds.$$

Applying the Gronwall's inequality we get

$$\|x(t)\| \le (\lambda \cdot \|S(0)\| + \lambda NKd)e^{\lambda NKt} \le (\|S(0)\| + NKd)e^{NKd}$$

and Corollary 2 may be applied to conclude the proof. □

3. Applications

As application of the above developed abstract theory we will consider the Cauchy problem for differential inclusions of the form

$$\begin{cases} x'(t) \in A(t,x(t)) + F(t,x(t)), & t \in [0,d], \\ x(0) = x_0, \end{cases} \tag{7}$$

where multivalued nonlinearity F satisfies conditions (F1), (F2'), (F3). As an operator $S : L^p([0,d],E) \to C([0,d],E)$ we take the solution operator of the quasi-linear problem

$$\begin{cases} x'(t) \in A(t,x(t)) + f(t), & t \in [0,d], \\ x(0) = x_0, \end{cases} \tag{8}$$

Then it is clear that the solutions to (7) coincide with the fixed point set $\operatorname{Fix} S \circ \mathcal{P}_F$ of the multimap $S \circ \mathcal{P}_F$.

Describe some concrete situations.

(a) *Semilinear inclusions.* $A(t,x(t)) = Ax(t)$ where $A : D(A) \subseteq E \to E$ is a densely defined linear operator generating a (noncompact) semigroup $\exp\{At\}$. In this case the (mild) solution operator (p=1) can be written in the explicit form:

$$S(f)(t) = \exp\{At\}x_0 + \int_0^t \exp\{A(t-s)\}f(s)\, ds.$$

The condition (S1) can be easily verified and the condition (S2) follows from the property of the Cauchy operator proved in [9] (see also [13]). Applying Theorem 2 we obtain the existence of a mild solution for the Cauchy problem (7) (see [12]).

(b) *Multivalued Volterra problems.* The previous example is a particular case of more general problems of existence of solutions to the following inclusion:

$$x(t) \in L(t) + \int_0^t k(t, s, F(s, x(s))ds \quad \text{for } t \in [0, d].$$

Here $L : [0, d] \to E$ is a continuous function and the kernel $k : \Delta \times E \to E$, where $\Delta = \{(t, s) \in [0, d] \times [0, d], s \le t\}$ satisfies the following conditions (comp. [9]):

(V1) k is continuous in the first variable,
(V2) the function $s \mapsto k(t, s, g(s))$ is integrable for each $g \in L^1([0, d], E)$,
(V3) $\|k(t, s, y) - k(t, s, z)\| \le M\|y - z\|$ for $(t, s) \in \Delta, x, y \in E$,
(V4) for every compact $K \subset E$ there is a function $\mu \in L^1([0, d])$ such that for $t \in [0, d]$ and $z \in K$ we have $\|k(t, s, z)\| \le \mu(s)$ for a.e. $s \in [0, d]$.

Here we take

$$S(f)(t) = L(t) + \int_0^t k(t, s, f(s))\, ds.$$

Then the assumptions (V1)–(V4) imply that the map

$$S : L^1([0, d]; E) \to C([0, d], E)$$

satisfies the assumptions (S1) and (S2). Thus Theorem 2 shows the existence of a solution to the inclusion

$$x(t) \in L(t) + \int_0^t k(t, s, F(s, x(s)))\, ds.$$

(c) *Inclusions with m-accretive operators.* $A(t, x(t)) = Ax(t)$ where $A : D(A) \subseteq E \to E$ is an m-accretive operator. Assume that the topological dual E^* is uniformly convex and that A generates a (noncompact) equicontinuous semigroup. For the mild solution operator S (with $p = 1$) of the problem (8), we obtain in this case condition (S1) as a weak form of the Benilan integral inequalities; condition (S2) is proven in [17, p. 60]. Thus an application of Theorem 2 shows the existence of a solution to (7).

(d) *Inclusions with time-dependent subdifferentials.* Let $E = H$ be a Hilbert space and let $\varphi : [0, d] \times H \to \mathbb{R} \cup \{\infty\}$ be a function such that for each $t \in [0, d]$, $\varphi(t, \cdot\,)$ is proper, convex and lower semicontinuous. We suppose that $A(t, x(t)) = \partial\varphi(t, x(t))$ for a.e. $t \in [0, d]$ and $t = 0$, where $\partial\varphi$ denotes the subdifferential of a function. Assume also that φ satisfies the Yotsutani conditions (see [18]), but we do not suppose that $\varphi(t, \cdot\,)$ is of a compact type. Then it was shown in [18] that for each $x_0 \in \operatorname{Dom} \varphi(0, \cdot\,)$ and for each $f \in L^1([0, d], H)$ problem (8) has a unique strong solution $S(f)$. Finally, the assumptions (S1) and (S2) can be justified for the solution mapping S by estimates given in [18] (see also [9]).

(e) *Nonlinear evolution inclusions.* Let (E, H, E^*) be an evolution triple of spaces. Assume that $A : [0, d] \times E \to E^*$ is an operator measurable in t and monotone and hemicontinuous in x satisfying the assumptions given in Zeidler [19, p. 770]. Let $F : [0, d] \times H \to K(H)$ satisfy the assumptions (F1), (F2'), (F3) and let $x_0 \in H$. Then for each $f \in L^q([0, d], H)$, $q > 1$, there exists a unique solution $S(f) \in W_p^1([0, d]; E, H)$, $1/p + 1/q = 1$ of problem (8). Since $W_p^1([0, d]; E, H)$ can be embedded continuously into $C([0, d], H)$ we thus obtain a mapping $S : L^q([0, d], H) \to C([0, d], H)$ and it can be shown that this mapping satisfies the assumptions (S1) and (S2). Notice that the estimation in (S1) is based upon the integration by parts formula for maps in $W_p^1([0, d]; E, H)$, see [19], whereas (S2) can be proven by arguments similar to those givem in [17, p. 60]. Thus we see that the Cauchy problem (7) for the evolution inclusion has a solution in this case.

References

[1] R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskiĭ, *Measures of Noncompactness and Condensing Operators*, Birkhäuser-Verlag, Boston–Basel–Berlin, 1992.

[2] Ph. Benilan, *Equations d'évolution dans un espace de Banach quelconque et application*, Thèse, Orsay (1972).

[3] Yu. G. Borisovich, *The relative rotation of compact vector fields in linear spaces*, Trudy Sem. Funktsion. Anal. Voronezh Gos. Univ. **12** (1969), 3–27. (Russian)

[4] Yu. G. Borisovich, B. D. Gelman, A. D. Myshkis and V. V. Obukhovskiĭ, *Topological methods in the fixed-point theory of multivalued maps*, Russian Math. Surveys **35** (1980), 65–143.

[5] ———, *Multivalued mappings*, J. Sov. Math. **24** (1984), 719–791.

[6] ———, *Introduction to the Theory of Multivalued Maps*, Voronezh Gos. Univ., Voronezh, 1986. (Russian)

[7] A. Bressan and V. Staicu, *On nonconvex perturbations of maximal monotone differential inclusion*, Set-Valued Anal. **2** (1994), 415–437.

[8] C. Castaing and M. Valadier, *Convex Analysis and Measurable Multifunctions,* Lecture Notes in Math., vol. 580, Springer–Verlag, Berlin–Heidelberg–New York, 1977.

[9] J. F. Couchouron and M. Kamenskiĭ, *A unified topological point of view for integro-differential inclusions*, Differential Inclusions and Optimal Control, Lecture Notes in Nonlinear Anal. (J. Andres, L. Górniewicz and P. Nistri, eds.), vol. 2, 1998, pp. 123–137.

[10] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin–New York, 1992.

[11] A. Fryszkowski, *Continuous selections for a class of non-convex multivalued maps*, Studia Math. **76** (1983), 163–174.

[12] M. Kamenskiĭ, V. Obukhovskiĭ and P. Zecca, *On semilinear differential inclusions with lower semicontinuous nonlinearities*, Ann. Mat. Pura Appl. (4) **175** (1999).

[13] ———, *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*, Walter de Gruyter (to appear).

[14] E. Mitidieri and I. I. Vrabie, *Differential inclusions governed by non-convex perturbations of m-accretive operators*, Differential Integral Equations **2** (1989), 525–531.

[15] A. A. Tolstonogov, *Differential Inclusions in a Banach Space*, Nauka, Novosibirsk, 1986. (Russian)

[16] A. A. Tolstonogov and I. A. Finogenko, *On solutions of a differential inclusion with lower semicontinuous nonconvex right-hand side in a Banach space*, Math. USSR Sbornik **53** (1986), 203–231.

[17] I. I. Vrabie, *Compactness Methods for Nonlinear Evolutions*, 2nd ed., Longman, 1995.

[18] S. Yotsutani, *Evolution equations associated with subdifferentials*, J. Math. Soc. Japan **31** (1978), 623–643.

[19] E. Zeidler, *Nonlinear Functional Analysis and its Applications* II/B, Springer-Verlag, 1990.

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