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A FIXED POINT THEOREM FOR MULTIVALUED MAPPINGS WITH NONACYCLIC VALUES

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Dedicated to Professor Lech Górniewicz on his 60th birthday

Abstract. The aim of this paper is to prove that every Borsuk continuous set-valued map of the closed ball in the 3-dimensional Euclidean space, taking values which are one point sets or knots, has a fixed point. This result is a special case of the Górniewicz Conjecture.

1. Introduction

We first recall some results which generalize the Brouwer Fixed Point Theorem for set-valued mappings. Let B^n denote the closed unit ball in $\mathbb{R}^n$, $C(B^n)$ – the family of all nonempty compact subsets of B^n, $*$ – the one point space, $f : B^n \to C(B^n)$ – a map. A point x is called a fixed point of f if $x \in f(x)$. A set $X \in C(B^n)$ is called acyclic if $\check{H}^*(X;Q) = \check{H}^*(*;Q)$. Here $\check{H}^*(\ ;Q)$ denotes the Čech cohomology functor with rational coefficients. The following assumptions on the type of continuity of f and on $f(x)$ for all $x \in B^n$ guarantee that f has a fixed point:

1. (S. Eilenberg, D. Montgomery) f – upper semicontinuous, $f(x)$ – acyclic ([5]).
2. (B. O'Neill) f – Hausdorff continuous, $f(x)$ has 1 or m acyclic components ([13]).

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3. (L. Górniewicz) f – Borsuk continuous[1], $f(x)$ – acyclic or $\check{H}^*(f(x);Q) = \check{H}^*(S^{n-1};Q)$ ([7], [6]).
4. (A. Dawidowicz) f – Borsuk continuous, $f(x)$ – connected, $n = 2$ ([2], [3]).

The basic idea of the proof of (3) and (4) is to apply (1) to a map $\widetilde{f}$ with acyclic values: $\widetilde{f}(x) = f(x) \cup$ (bounded components of $\mathbb{R}^n \setminus f(x)$).

The Górniewicz Conjecture is the extension of (4) for all $n \geq 2$. The following special case was studied in [10].

CONJECTURE 1. *Every Borsuk continuous map* $f : B^n \to C(B^n)$ *with values homeomorphic to* $*$ *or* S^k *has a fixed point* (k *is fixed,* $1 \leq k \leq n-1$).

Note that the class of set-valued mappings, which is considered in the Conjecture 1, generalizes the class of bimaps studied by H. Schirmer in [14] and [15]. By [10, Theorem 1] the Conjecture 1 for $k \neq 4$ is a consequence of the following

CONJECTURE 2. *Let* $M \subset \mathbb{R}^n$ *be a closed connected* PL-*manifold,* $\dim M = n-1$. *Let* $p : E \to M$ *be a locally trivial bundle* ξ *with the fiber* S^k; $1 \leq k \leq n-1$. *If* $E \subset M \times \mathbb{R}^n$ *and the square*

$$\begin{array}{ccc} E & \subset & M \times \mathbb{R}^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ M & = & M \end{array}$$

commutes then $\dim H_k(E; Z_2) > \dim H_k(M; Z_2)$.

Our purpose is to prove both conjectures for $(k, n) = (1, 3)$.

Added in the proof: The Conjecture 2 does not hold for $(k, n) = (1, 4)$. Consider the Hopf fibration $h : S^3 \to S^2$, the map $g : S^1 \times S^2 \to C(S^3)$, $g(x, y) = h^{-1}(y)$, $E = \{(x, y, z) \in S^1 \times S^2 \times S^3 : z \in g(x, y)\} \cong S^1 \times S^3$, $M = S^1 \times S^2$, $p(x, y, z) = (x, y)$. Then $\dim H_1(E; Z_2) = \dim H_1(M; Z_2) = 1$.

2. Preliminaries

The Borsuk distance of continuity [1] in $C(B^n)$ is defined by the formula

$$d_B(X, Y) = \max\{\rho(X, Y), \rho(Y, X)\},$$

where $\rho(X, Y) = \inf\{\max\{d(x, h(x)) : x \in X\} : h \in C(X; Y)\}$ and $C(X; Y)$ is the set of all continuous maps from X to Y. Set-valued maps continuous with respect to d_B are called Borsuk continuous mappings. In the sequel the metric d_B will not appear explicite.

We shall apply the following Borsuk–Ulam type result.

[1] See Preliminaries.

Theorem 1 (Nakaoka [12]). *Let N be a closed n-dimensional manifold with a free involution T and let $g : N \to P$ be a continuous map to an m-dimensional manifold P. Let $c \in H^1(N/T; Z_2)$ be the first Stiefel–Whitney class of the bundle $\pi : N \to N/T$. Assume that $c^m \neq 0$ and $g_* : \widetilde{H}_*(N; Z_2) \to \widetilde{H}_*(P; Z_2)$ is trivial. Then the covering dimension of $A(g) = \{y \in N : g(y) = g(Ty)\}$ is at least $n-m$.*

Let us recall some facts on Stiefel–Whitney classes. The general references here are [8], [11]. Let $p : E \to M$ be a locally trivial bundle ξ with the fiber S^k and the structural group $O(k+1)$. The antipodal map of S^k induces a fiber preserving fixed point free involution $T : E \to E$, $(p \circ T = p;\ T \circ T = id)$. We will denote by $c \in H^1(E/T; Z_2)$ the first Stiefel–Whitney class of the bundle $\pi : E \to E/T$. A projection $q : E/T \to M$ is defined by $q \circ \pi = p$.

Fact 1 ([8, 16.2.5]). *The group $H^*(E/T; Z_2)$ is an $H^*(M; Z_2)$-module freely generated by $\{1, c, c^2, ..., c^k\}$. The multiplication is defined by the formula:*

$$H^*(M; Z_2) \times H^*(E/T; Z_2) \ni (\alpha, \beta) \to \alpha\beta = q^*(\alpha) \cup \beta.$$

Moreover,

$$c^{k+1} = \sum_{j=1}^{k+1} w_j(\xi) c^{k+1-j}$$

where $w_j(\xi) \in H^j(M; Z_2)$ is the j-th Stiefel–Whitney class of ξ.

Fact 2. *If $\vec{\xi}$ is a vector bundle corresponding[2] to ξ then[3]*

- $w(\vec{\xi}) \overset{\text{def}}{=} w(\xi) = 1 + \sum_{j=1}^{k+1} w_j(\xi)$,
- $w(\vec{\xi} \oplus \vec{\eta}) = w(\vec{\xi}) \cup w(\vec{\eta})$ ([11, §4]),
- *if θ is a trivial bundle then $w(\theta) = 1$,* ([11, §4]).

Fact 3 ([11, §8]). *If $\overline{p} : \overline{E} \to M$ is a disc bundle (with the fiber B^{k+1}) corresponding to ξ and $u \in H^{k+1}(\overline{E}, E; Z_2)$ is the Thom class of ξ then*

$$u \to u|_{\overline{E}} \to w_{k+1}(\xi)$$

under the homomorphism

$$H^{k+1}(\overline{E}, E; Z_2) \xrightarrow{\ i^*\ } H^{k+1}(\overline{E}; Z_2) \xrightarrow{(\overline{p}^*)^{-1}} H^{k+1}(M; Z_2).$$

Moreover, $H^{k+1}(\overline{E}, E; Z_2) = Z_2 = \{0, u\}$.

Fact 3 is well known. We here include a proof of it for the convenience of the reader. If $\Phi : H^*(M; Z_2) \to H^{*+k+1}(\overline{E}, E; Z_2)$ is the Thom isomorphism [11, 8.2], $\Phi(x) = \overline{p}^*(x) \cup u$, then $w_{k+1}(\xi) = \Phi^{-1} Sq^{k+1} \Phi(1) = \Phi^{-1} Sq^{k+1}(u) =$

[2] In the sense that the bundle of unit spheres of the vector bundle (with respect to some norm in each fiber) is equivalent to the given sphere bundle.

[3] Another (axiomatic) definition of Stiefel–Whitney classes of vector bundles is given in [11].

$\Phi^{-1}(u \cup u) = \Phi^{-1}(u|_{\overline{E}} \cup u) = (\overline{p}^*)^{-1} u|_{\overline{E}}$. Here Sq^{k+1} denotes the $(k+1)$-Steenrod square [11, §8]. The second assertion of the Fact 3 follows from the Thom isomorphism and the connectedness of M.

3. Two lemmas

In order to apply the Stiefel–Whitney classes, it is now necessary to require that $O(k+1)$ is the structural group of the bundle ξ. This assumption compared with the setting of the Conjecture 2 is more restrictive. Since the group $\mathrm{Homeo}(S^1)$ of all homeomorphisms $S^1 \to S^1$ reduces to $O(2)$ (see [10, Fact 2] and the proof of [16, 11.45]), we shall overcome this difficulty for $k = 1$.

Lemma 1. *$\dim H_k(E; Z_2) > \dim H_k(M; Z_2)$ if and only if $w_{k+1}(\xi) = 0$.*

Proof. The homomorphism $p_{*k} : H_k(E; Z_2) \to H_k(M; Z_2)$ is an epimorphism, (see [10, Fact 1]). This clearly forces that the inequality $\dim H_k(E; Z_2) > \dim H_k(M; Z_2)$ does not hold if and only if p_{*k} is a monomorphism. Since we deal with finite-dimensional vector spaces and the functor *Hom* is exact on this category, p_{*k} is a monomorphism if and only if

$$\mathrm{Hom}(p_{*k}; \mathrm{id}) : \mathrm{Hom}(H_k(M; Z_2); Z_2) \to \mathrm{Hom}(H_k(E; Z_2); Z_2)$$

is an epimorphism, which is equivalent to the statement that

$$p^* : H^k(M; Z_2) \to H^k(E; Z_2)$$

is an epimorphism too. The commutative diagram

$$\begin{array}{ccccccc}
H^k\overline{E} & \xrightarrow{j^*} & H^k E & \xrightarrow{\delta} & H^{k+1}(\overline{E}, E) & \xrightarrow{i^*} & H^{k+1}\overline{E} \\
{\scriptstyle \overline{p}^*}\uparrow{\scriptstyle \cong} & & \uparrow{\scriptstyle p^*} & & & & \\
H^k M & = & H^k M & & & &
\end{array}$$

with the 1st row exact (and Z_2-cohomology coefficients) yields that p^*-epimorphism $\Leftrightarrow$ j^*-epimorphism $\Leftrightarrow$ $\delta = 0 \Leftrightarrow i^*$-monomorphism. Fact 3 now shows that i^*-monomorphism $\Leftrightarrow u|_{\overline{E}} \neq 0 \Leftrightarrow w_{k+1}(\xi) \neq 0$, which completes the proof. □

Let $\widetilde{K}$ denote the reduced topological K-theory functor.

Lemma 2. *If M_g is a closed orientable surface of genus g then*

$$\widetilde{K}(M_g) = (Z_2)^{2g+1}.$$

Proof. (All results of K-theory which will be needed here, can be found in [8] and [9].)

We begin by recalling that $\widetilde{K}(S^1) = Z_2$ and $\widetilde{K}(S^2) = Z_2$. Now suppose that $g \geq 1$. Let SX denote the reduced suspension of the space X and $\widetilde{K}^{-1}(X) =$

$\widetilde{K}(SX)$. Let Y be a closed subset of X. Consider the following exact sequence of abelian groups (see [8, 9.2.8], [9, II.3.29]):

$$\widetilde{K}^{-1}(X) \xrightarrow{\gamma} \widetilde{K}^{-1}(Y) \xrightarrow{\delta} \widetilde{K}(X/Y) \xrightarrow{\alpha} \widetilde{K}(X) \xrightarrow{\beta} \widetilde{K}(Y).$$

Take $X = M_g$ and $Y = \bigvee_{i=1}^{2g} Y_i$, $Y_i \cong S^1$ for $i = 1, \dots, 2g$. If the surface M_g is represented as a polygon (with $4g$ angles and standard identifications) then Y is represented as its boundary. Of course, $X/Y \cong S^2$. Homomorphisms γ and β have their right inverses. Indeed, let $r_i : X \to Y_i$ be a retraction such that $r_i(Y_j) = *$ for $j \neq i$. Then

$$\widetilde{K}(Y) \cong \bigoplus_{i=1}^{2g} \widetilde{K}(Y_i) \xrightarrow{(r_i^!)} \widetilde{K}(X)$$

is a right inverse of β, (fortunately, $\widetilde{K}(*) = 0$). Replacing $\widetilde{K}$ by $\widetilde{K}^{-1}$ we obtain a right inverse of γ. Consequently, γ and β are epimorphisms. We obtain an exact sequence

$$0 \to \widetilde{K}(S^2) \xrightarrow{\alpha} \widetilde{K}(M_g) \xrightarrow{\beta} \bigoplus_{i=1}^{2g} \widetilde{K}(S^1) \to 0,$$

which splits. Thus

$$\widetilde{K}(M_g) \cong \widetilde{K}(S^2) \oplus \bigoplus_{i=1}^{2g} \widetilde{K}(S^1) = (Z_2)^{2g+1}. \qquad \square$$

4. The main result

Theorem 2. *Every Borsuk continuous map $f : B^3 \to C(B^3)$ with values homeomorphic to $*$ or S^1 has a fixed point.*

Proof. It suffices to prove the Conjecture 2 for $(k, n) = (1, 3)$. Let $M \subset \mathbb{R}^3$ be a closed 2-dimensional PL-manifold. Then M is orientable (see [4, VIII.3.9]). By the classification of closed surfaces, $M = M_g$ for some $g \geq 0$. Let $p : E \to M$ be a locally trivial bundle ξ with the fiber S^1. Since the group $\mathrm{Homeo}(S^1)$ reduces to $O(2)$, we can find a bundle ξ_1 equivalent to ξ with the structural group $O(2)$. In fact, it suffices to consider the case $\xi_1 = \xi$. (This sufficiency can be easily verified after reading this proof). Of course M has a differential structure of C^∞-manifold, which makes E, T and E/T smooth. Note that $\dim E = 3$. To obtain a contradiction, suppose that $\dim H_1(E; Z_2) \leq \dim H_1(M; Z_2)$. By Lemma 1, $w_2 \neq 0$. According to the assumption of the Conjecture 2, the following diagram

$$\begin{array}{ccccc} E & \overset{i}{\subset} & M \times \mathbb{R}^3 & \xrightarrow{\pi_2} & \mathbb{R}^3 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_1} & & \\ M & = & M & & \end{array}$$

commutes. Now we assume that $c^3 \neq 0$. From the Nakaoka Theorem (Theorem 1) with $N = E$, $P = \mathbb{R}^3$, $g = \pi_2 \circ i$, we obtain at least one point $x \in E$ such that $\pi_2 \circ i(x) = \pi_2 \circ i(Tx)$. Since $\pi_1 \circ i(x) = p(x) = p(Tx) = \pi_1 \circ i(Tx)$, it follows that $i(x) = i(Tx)$ and $x = Tx$, which contradicts fact that T is fixed point free. It remains to verify that $c^3 \neq 0$.

By Fact 1, $c^2 = w_1c + w_2$. Hence $c^3 = (w_1c + w_2)c = w_1c^2 + w_2c = w_1(w_1c + w_2) + w_2c = ([w_1]^2 + w_2)c + w_1w_2$.

Since $\dim M = 2$, $H^3(M; Z_2) = 0$ and $w_1w_2 = 0$. By Lemma 2, $2\widetilde{K}(M) = 0$, so $\vec{\xi} \oplus \vec{\xi}$ represents zero in $\widetilde{K}(M)$. This gives $\vec{\xi} \oplus \vec{\xi} \oplus \vec{\theta} = \vec{\Theta}$ for some trivial vector bundles $\vec{\theta}$, $\vec{\Theta}$. It follows that $1 = w(\xi) \cup w(\xi) = (1 + w_1 + w_2)^2 = 1 + [w_1]^2 + [w_2]^2 = 1 + [w_1]^2$. Therefore $[w_1]^2 = 0$ and $c^3 = w_2c \neq 0$, (recall that $w_2 \neq 0$ and apply Fact 1). This finishes the proof. □

Corollary 1. *Let $f : B^3 \to C(B^3)$ be a Borsuk continuous map with values homeomorphic to $*$ or S^1. Let $F_i : B^3 \to C(B^3)$ be an upper semicontinuous map with Z_2-acyclic values for $i = 1, \ldots, n$. Then the mapping $F_n \circ \ldots \circ F_1 \circ f$ has a fixed point,* [10, *Statements* 5, 6].

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