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NONLINEAR RIEMANN–HILBERT PROBLEMS FOR DOUBLY CONNECTED DOMAINS AND CLOSED BOUNDARY DATA

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Dedicated to Katrin Wendland on the occasion of her 30th birthday

Abstract. In this paper, for nonlinear Riemann–Hilbert problems in doubly connected domains with smooth as well as Lipschitz continuous boundary data, existence of at least two topologically different solutions is established. The main tools are the topological degree of quasi-ruled Fredholm mappings, Montel's theorem, a priori estimates and the employment of classical modular function theory.

1. Introduction

We use properties of the class of complex analytic functions on the ring domain G_2 and at the same time the properties of the lifted problem on the universal covering Ξ which is defined by the reflexion principle on the Riemannian manifold corresponding to the closed boundary conditions. On the universal covering we use a very handy numbering and practical parametrization. The existence proof rests on the use of nonlinear systems of singular integral equations which can be treated as quasiruled Fredholm maps. In addition, we can show a priori bounds for the solutions of the Riemann–Hilbert problem and of the nonlinear singular integral equations. Hence, a degree of quasiruled Fredholm mappings can be used which turns out to be nonzero since the nonlinear operator

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can homotopically be deformed to a simple linear problem that is easily solvable due to our useful parametrization. The corresponding homotopy is constructed explicitly.

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2. The nonlinear Riemann–Hilbert problem and its lifting to the universal covering

Consider the nonlinear RHP_2 in $G_2 := \{z \in \mathbb{C} \mid 0 < q < |z| < 1\}$, i.e.

$$\frac{\partial Z}{\partial \overline{z}} = 0 \quad \text{in } G_2, \tag{1}$$

subject to the nonlinear boundary conditions

$$Z(qe^{i\tau}) \in \gamma_{\tau,1} \quad \text{and} \quad Z(e^{i\tau}) \in \gamma_{\tau,2}, \tag{2}$$

where $\tau \in [0, 2\pi)$ is the angular coordinate on the circles ∂G_2; and $\gamma_{\tau,j}$ $(j = 1, 2)$ are two families of closed, non–selfintersecting Lipschitz curves in the complex $\mathbb{C}_Z$-plane depending smoothly on the parameter τ.

The boundary conditions (2) can also be written as

$$F_1\big(\tau, U(qe^{i\tau}), V(qe^{i\tau})\big) = 0, \quad F_2\big(\tau, U(e^{i\tau}), V(e^{i\tau})\big) = 0, \tag{3}$$

if the family of curves $\gamma_{\tau,j}$ is given implicitly:

$$\gamma_{\tau,j} = \{Z = U + iV \in \mathbb{C}_Z \mid F_j(\tau, U, V) = 0\}, \quad j = 1, 2. \tag{4}$$

In the special case when the curves $\gamma_{\tau,j}$ $(j = 1, 2)$ are independent of τ and the curves γ_1 and γ_2 form the boundary of a bounded doubly–connected domain G_2^* one obtains the classical problem of conformal mapping of G_2 onto G_2^*. As is well known, this problem is only solvable under an additional condition (see [7]). In the special case that the two separated curves γ_1, γ_2 form the boundary of two separated bounded components it is clear that no solutions exist at all due to the open mapping properties of nonconstant holomorphic functions. These two examples show us that – even in these special cases – the solvability of the RHP_2 depends crucially on the geometric configuration of the curves $\gamma_{\tau,j}$.

In this paper we present sufficient conditions on the boundary curves $\gamma_{\tau,j}$ which guarantee existence of at least two different solutions. We emphasize that the case of a multiply connected domain is completely different from the simply connected one (see [8], [9], [11]) and, hence, requires completely new arguments.

CONDITIONS 2.1.

(a) *The curves of the two families* $\{\gamma_{\tau,j} \mid 0 \le \tau \le 2\pi,\ j = 1,2\}$ *are closed, non–selfintersecting Lipschitz curves in the complex Z-plane $\mathbb{C}_Z$ for every $\tau \in [0,2\pi)$, depending smoothly on the parameter $\tau \in [0,2\pi)$.*

(b) *There exist real numbers $Z_\ell \in \mathbb{R} \subset \mathbb{C}_Z$ $(\ell = 1,2,3)$ with*

$$Z_1 < Z_2 < Z_3 < Z_4 = \infty$$

such that, for every $\tau \in [0,2\pi)$,

$$\begin{aligned} &Z_1 \in \operatorname{int}\gamma_{\tau,1} \cap \operatorname{ext}\gamma_{\tau,2}, \quad Z_2 \in \operatorname{int}\gamma_{\tau,1} \cap \operatorname{int}\gamma_{\tau,2},\\ &Z_3 \in \operatorname{ext}\gamma_{\tau,1} \cap \operatorname{int}\gamma_{\tau,2}, \quad Z_4 \in \operatorname{ext}\gamma_{\tau,1} \cap \operatorname{ext}\gamma_{\tau,2}, \end{aligned}$$

where int $\gamma_{\tau,j}$ denotes the interior domain in $\mathbb{C}_Z$ bounded by $\gamma_{\tau,j}$ and ext $\gamma_{\tau,j}$ denotes the exterior domain, respectively.

In addition, for every $\tau \in [0,2\pi]$, we require,

$$\begin{aligned} (-\infty, Z_1) \cap \gamma_{\tau,2} &= \emptyset, \qquad (Z_1, Z_2) \cap \gamma_{\tau,1} = \emptyset,\\ (Z_2, Z_3) \cap \gamma_{\tau,2} &= \emptyset, \qquad (Z_3, \infty) \cap \gamma_{\tau,1} = \emptyset. \end{aligned}$$

The situation under Conditions 2.1 for any fixed value of τ is indicated in Figure 1.

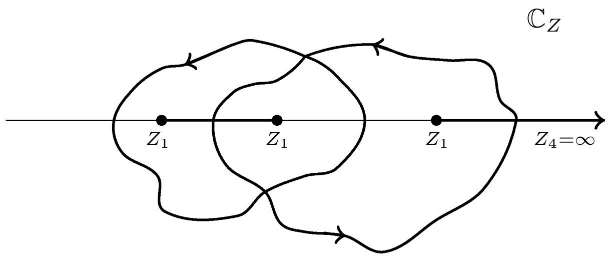

FIGURE 1

Without loss of generality, we may set $Z_2 = 0$.

Compactness arguments imply that there exist two positive constants $0 < \varrho_0 < R_0$ with the property

$$\bigcup_{\tau\in[0,2\pi)} \gamma_{\tau,j} \subset \{\varrho_0 \le |Z| \le R_0\} \quad \text{for } j = 1,2. \tag{5}$$

In addition to Conditions 2.1 for the boundary conditions we specify the class of solutions by the requirement for their winding numbers,

$$\frac{1}{2\pi}\int_0^{2\pi} d\arg Z(qe^{i\tau}) = \frac{1}{2\pi}\int_0^{2\pi} d\arg Z(e^{i\tau}) = 0. \tag{6}$$

Definition 2.1. The solution $Z(z)$ of the RHP_2 is called “liftable” if the mapping

$$Z(z) : G_2 \to \mathcal{S}^2 \setminus \{Z_1, Z_2, Z_3, Z_4 = \infty\}$$

is homotopic to the trivial one which here means that $Z(z)$ satisfies condition (6), where $\mathcal{S}^2$ denotes the Riemannian shere.

Now we are ready to formulate our main result.

Theorem 2.1. *Let $\{\gamma_{\tau,j} \mid 0 \le \tau < 2\pi, j = 1, 2\}$ satisfy Conditions 2.1. Then there exist at least two different “liftable” solutions of the nonlinear RHP_2.*

In order to prove existence of solutions to (1)–(6) we reduce the nonlinear RHP_2 to nonlinear RHPs on the universal covering Ξ of $\mathcal{S}^2 \setminus \{Z_1, Z_2, Z_3, \infty\}$ on the Riemannian sphere $\mathcal{S}^2$. To this end, we denote by $\Gamma_1 := [Z_1, Z_2], \Gamma_2 := [Z_2, Z_3], \Gamma_3 := [Z_3, \infty]$ and $\Gamma_4 := [\infty, Z_1]$ and let $f(Z)$ be the conformal mapping of the upper half-plane onto $T \subset \Xi$ where ∂T consists of four circular arcs $\alpha_j = f(\Gamma_j)$ which are orthogonal to the unit circle $\partial\Xi$ at their endpoints $f(Z_j)$. Let S_j be the circular quadrangle which is obtained from T by reflexion with respect to the arc α_j. It is clear, that the corners of ∂S_j lie on $\partial\Xi$. The four sides of $S_j,\ j = 1, \dots, 4$ consist of circular arcs which are orthogonal to $\partial\Xi$. According to the reflexion principle, S_j is the conformal image of the lower half-plane. Clearly, $f(Z) = f(1/\overline{Z})$ for the conformal mapping. Moreover, the corners of the quadrangle S_j are images of the points Z_1, Z_2, Z_3, Z_4. By $\beta_j^{(1)}, \beta_j^{(2)}, \beta_j^{(3)}, \beta_j^{(4)}, j = 1, \dots, 4$, we denote the images of the segments Γ_j, respectively. Next, let $T_{j,k}$ be the circular quadrangle with the sides $\alpha_{j,k}^{\ell}, \ell = 1, \dots, 4$, which is obtained from S_j by reflexion across the side $\beta_j^{(k)}, j \neq k$. It is clear that we get the original quadrangle if we reflect S_j backwards across the side $\beta_j^{(j)}$. According to the reflexion principle, each of those $T_{j,k}$ is the image of the upper half-plane. Of course, the mapping function $f(Z)$ is analytically extended across Γ_j onto the half-plane of the next Riemann sheet. Next, we introduce the circular quadrangle $S_{j,k,\ell}$ by reflecting $T_{j,k}$ across the side $\alpha_{j,k}^{\ell}$. Proceeding in this way, as a result we obtain that Ξ is divided into a countable number of circular quadrangles of the form

$$S_{j_1,k_1,\dots,j_{r-1},k_{r-1},j_\tau} \quad \text{and} \quad T_{j_1,k_1,\dots,j_\tau,k_\tau},$$

which are obtained from T by odd and even numbers of reflexions, respectively.

Now we are in the position to consider the images of the boundary curves $\gamma_{\tau,1}$ and $\gamma_{\tau,2}$ on Ξ.

Let $\pi : \Xi \to \mathcal{S}^2 \setminus \{Z_1, Z_2, Z_3, \infty\}$ denote the canonical projection associated with our construction. Now, according to the condition (6) there exists a holomorphic function $\zeta(z)$ mapping $G_2 \to \Xi$ such that $\pi\zeta(z) = Z(z)$ will define the desired solution provided it exists [5], [10], [6].

From the boundary condition (2) it follows that

$$\zeta(qe^{i\tau}) = \pi^{-1}\big(Z(qe^{i\tau})\big) \in \pi^{-1}(\gamma_{\tau,1}), \quad \zeta(e^{i\tau}) = \pi^{-1}\big(Z(e^{i\tau})\big) \in \pi^{-1}(\gamma_{\tau,2}). \tag{7}$$

Therefore it remains to describe the preimages of $\gamma_{\tau,j}$ $(j = 1, 2)$ lying in the universal covering Ξ. Note that, from our construction, it follows that any of these preimages consists of a countable number of curves $\widetilde{\gamma}_{\tau,j,m}$ with $m \in \mathbb{Z}$. Let the point $Z^{(j)}(t)$ for $-\infty < t < +\infty$ trace through the closed curve $\gamma_{\tau,j}$ in the counter–clockwise direction and suppose $Z^{(j)}(t) = Z^{(j)}(t+1)$ with Im $Z^{(j)}(t_0) > 0$. Let $\xi^{(j)}(t)$ be one of the preimages of $Z^{(j)}(t)$ in Ξ of the mapping π which continuously depends on $t \in \mathbb{R}$. The curve $\xi^{(j)}(t)$ coincides with $\widetilde{\gamma}_{\tau,j,m}$ for some $m \in \mathbb{Z}$. From Conditions 2.1 it follows that if

$$\xi^{(j)}(t_0) \in T_{j_1,k_1,\dots,j_\tau,k_\tau} \quad \text{for } j = 1, 2,$$

then

$$\xi^{(1)}(t_0+1) \in T_{j_1,k_1,\dots,j_\tau,k_\tau,4,2} \quad \text{and} \quad \xi^{(2)}(t_0+1) \in T_{j_1,k_1,\dots,j_\tau,k_\tau,1,3},$$

respectively. Let $h^{(j)} : \Xi \to \Xi$ be the mapping corresponding to an even number of consecutive reflexions of the upper half-plane of $\mathbb{C}_Z$. Due to our construction of Ξ, it is enough to define $h^{(j)}$, $j = 1, 2$, in one of the domains:

$$h^{(1)} : T_{j_1,k_1,\dots,j_\tau,k_\tau} \to T_{j_1,k_1,\dots,j_\tau,k_\tau,4,2}, \tag{8}$$

$$h^{(2)} : T_{j_1,k_1,\dots,j_\tau,k_\tau} \to T_{j_1,k_1,\dots,j_\tau,k_\tau,1,3}. \tag{9}$$

It is clear that

$$\xi^{(j)}(t+1) = h^{(j)} \circ \xi^{(j)}(t).$$

According to Brouwer's fixed point theorem, the mapping $h^{(j)} : \Xi \to \Xi$ has two distinct fixed points on $\partial\Xi$. Since the curve $\xi^{(j)}(t)$ is invariant under the mapping $h^{(j)}$, these points are the two endpoints for the curve $\xi^{(j)}(t)$. Now we are in the position to reduce the original problem RHP_2 to a different problem of the same type, which is more suitable for our analysis. Let $Z(z)$ be a solution of the RHP_2 (1)–(6), then there exists a function $\zeta(z)$ which is holomorphic in G_2 and continuous on $\overline{G}_2$ with the following properties:

$$\zeta(z) : G_2 \to \Xi \quad \text{and} \quad \pi\big(\zeta(z)\big) = Z(z), \tag{$\widetilde{\mathrm{RHP}}_2$}$$

(see [10]) and there exist two integers $m_1, m_2 \in \mathbb{Z}$ such that

$$\zeta(qe^{i\tau}) \in \widetilde{\gamma}_{\tau,1,m_1} \quad \text{and} \quad \zeta(e^{i\tau}) \in \widetilde{\gamma}_{\tau,2,m_2}. \tag{10}$$

Now we fix the chain of all $T_{j_1,k_1,\dots,j_\tau,k_\tau}$ and $S_{j_1,k_1,\dots,j_\tau,k_\tau,j_{\tau+1}}$ through which the curves $\widetilde{\gamma}_{\tau,1,m_1}$ and $\widetilde{\gamma}_{\tau,2,m_2}$ are trespassing, respectively. Then exactly one of the following cases is possible (see also Figure 2):

(1) These chains have an intersection coinciding with $T_{j_1,k_1,\dots,j_\tau,k_\tau}$;

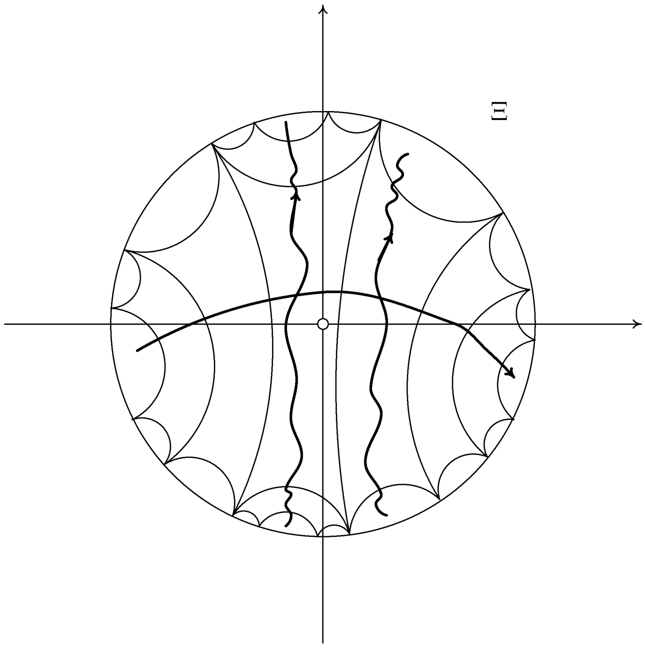

Figure 2. The chains on Ξ

(2) These chains have an intersection coinciding with $S_{j_1,k_1,\dots,j_\tau,k_\tau,j_{\tau+1}}$;

(3) These chains do not intersect at all.

Let $\widetilde{\gamma}_{\tau,j,m_\ell}$ and $\widetilde{\gamma}_{\tau,j,m_r}$ be two families of curves corresponding to the case (1) each. Then it is not difficult to see that they can be transformed into each other by the mappings $h^{(j)} : \Xi \to \Xi$ $(j = 1, 2)$. Hence, they are equivalent and it suffices to consider the following two representative cases only:

Let $\widetilde{\gamma}_{\tau,1,1}$ be the family which trespasses through the circular quadrangles

$$\dots, S_{424}, T_{42}, S_4, T, S_2, T_{24}, S_{242}, \dots, \tag{11}$$

and let $\widetilde{\gamma}_{\tau,2,1}$ be the family which trespasses through

$$\dots, S_{313}, T_{31}, S_3, T, S_1, T_{13}, S_{131}, \dots \tag{12}$$

and let $\widetilde{\gamma}_{\tau,2,2}$ be the family which trespasses through the chains

$$\dots, S_{213}, T_{21}, S_2, T_{23}, S_{231}, \dots . \tag{13}$$

Each of the two families:

Case 1. $[\widetilde{\gamma}_{\tau,1,1}, \widetilde{\gamma}_{\tau,2,1}]$,

Case 2. $[\widetilde{\gamma}_{\tau,1,2}, \widetilde{\gamma}_{\tau,2,2}]$,

represents one of the cases, respectively. For each of these cases we shall prove an existence theorem for solutions of the nonlinear $\widetilde{\mathrm{RHP}}_2$. As a consequence, we shall show that the corresponding sets of solutions of the original problem RHP_2 lead to two different sets of solutions.

We now restrict ourselves to Case 1. The Case 2 can be treated analogously. Note that it is not difficult to see that there is no solution in the Case 3.

We denote by D_1 the domain which is the union of all circular quadrangles which contain $\widetilde{\gamma}_{\tau,1,1}$ and the arcs dividing the neighbouring quadrangles in order to obtain a connected domain including all sets in (11). By D_2 we denote the analogous domain for the family $\widetilde{\gamma}_{\tau,2,2}$. Hence, we consider now the following $\widetilde{\mathrm{RHP}}_2$:

Find a holomorphic function $\zeta(z)$ in G_2 and continuous in $\overline{G}_2$ such that

(1) $\zeta(qe^{i\tau}) \in \widetilde{\gamma}_{\tau,1,1}$ and $\zeta(e^{i\tau}) \in \widetilde{\gamma}_{\tau,2,1}$,

(2) $\zeta(G_2) \subset D_1 \cup D_2$.

To prove the existence of the $\widetilde{\mathrm{RHP}}_2$ we reduce it to an equivalent system of nonlinear singular integral equations on ∂G_2.

To this end, similar to the ideas in [3], we use smooth approximations $\gamma^{\varepsilon}_{\tau,j}$ satisfying Condition 2.1 uniformly with respect to ε, and use Montel's theorem. From now on we deal with the familiy of smooth curves $\gamma^{\varepsilon}_{\tau,j}$ and with $\widetilde{\gamma}^{\varepsilon}_{\tau,j}$, correspondingly.

For every fixed $\varepsilon > 0$, let the curves $\gamma^{\varepsilon}_{\tau,j}$ in the complex Z-plane be implicitly represented by the real-valued functions

$$F^{\varepsilon}_j(\tau, U, V) = 0, \quad j = 1, 2$$

as in (3) and satisfy the following conditions:

(1) F^{ε}_1 is a smooth function of all its arguments for $0 \le \tau < 2\pi$ and $Z = U + iV \in \mathcal{S}^2 \setminus (\Gamma_1 \cup \Gamma_3)$ and F^{ε}_2 is a smooth function of all its arguments for $0 \le \tau < 2\pi$ and $Z \in \mathcal{S}^2 \setminus (\Gamma_2 \cup \Gamma_4)$.

(2) $\mathrm{grad}_{U,V} F^{\varepsilon}_1(\tau, U, V) \ne 0 \quad$ for $U + iV = Z \in \mathcal{S}^2 \setminus (\Gamma_1 \cup \Gamma_3)$,
$\mathrm{grad}_{U,V} F^{\varepsilon}_2(\tau, U, V) \ne 0 \quad$ for $U + iV = Z \in \mathcal{S}^2 \setminus (\Gamma_2 \cup \Gamma_4)$,
uniformly with respect to $\varepsilon > 0$.

In order to formulate our problem on Ξ, let us define

$$f^{\varepsilon}_j(\tau, \zeta) := F^{\varepsilon}_j\big(\tau, \mathrm{Re}\,\pi(\zeta), \mathrm{Im}\,\pi(\zeta)\big) \quad \text{for } \zeta \in \Xi, \tag{14}$$

where $\pi : \Xi \to \mathcal{S}^2 \setminus \{Z_1, Z_2, Z_3, \infty\}$ is the canonical projection. Let $H^s(\mathcal{S}^1)$ be the Sobolev–Slobodetskiĭ space of order $s \ge 0$ on the unit circle $\mathcal{S}^1$ and let $\mathcal{X}^s$ be the Banach space of functions holomorphic in G_2 and continuous on $\overline{G}_2$ with $H^s(\mathcal{S}^1)$-bounded traces on $\overline{G}_2$ equipped with the norm

$$\| \zeta \|_s := \{\| \zeta(qe^{i\bullet}) \|^2_{H^s(\mathcal{S}^1)} + \| \zeta(e^{i\bullet}) \|^2_{H^s(\mathcal{S}^1)}\}^{1/2} \tag{15}$$

considered over the scalar field of the reals $\mathbb{R}$. Then $\widetilde{\mathrm{RHP}}_2$ can be reduced to the following problem: *Find a holomorphic function $\zeta(z)$ in G_2 and continuous* *on* $\overline{G}_2$ *such that*

$$f_1^\varepsilon(\tau,\zeta(qe^{i\tau})) = 0 \quad \text{and} \quad f_2^\varepsilon(\tau,\zeta(e^{i\tau})) = 0. \tag{16}$$

Note that the families of curves $f_1^\varepsilon(\tau,\zeta) = 0$ and $f_2^\varepsilon(\tau,\zeta) = 0$ are non-closed and are "going to $\partial\Xi$".

To prove the existence of a solution to (16), we convert this problem to a system of nonlinear singular integral equations on $\Gamma = \partial G_2$ based on the following theorems from [3] which are proved there.

Theorem 2.2. *For every pair* $(u_1, v_1) \in [H^s(\mathcal{S}^1) \times H^s(\mathcal{S}^1)]$ *with* $s \geq 0$, *there exists a unique holomorphic function* $w(z) = u(z) + iv(z)$ *in* $G_2 = \{z \in \mathbb{C} \mid q < |z| < 1\}$ *such that*

$$\operatorname{Re} w(qe^{i\tau}) = u_1(\tau), \quad \operatorname{Im} w(e^{i\tau}) = v_2(\tau). \tag{17}$$

Moreover, the correspondence $(u_1, v_2) \mapsto w(z) = u(z) + iv(z)$ *is an isomorphism between the space* $H^s(\mathcal{S}^1)$ *with* $s \geq 0$ *and the Banach space* $\mathcal{X}^s$.

Let $\mathbf{S}$ *denote the map* $\mathbf{S} : H^s(\mathcal{S}^1) \to \mathcal{X}^s$ *given by* $(u_1, v_2) \mapsto w(z)$, *and write*

$$\mathbf{S}_1(u_1, v_2) := \operatorname{Im} w(qe^{i\tau}) \quad \text{and} \quad \mathbf{S}_2(u_1, v_2) := \operatorname{Re} w(e^{i\tau}). \tag{18}$$

The mapping $\mathbf{S}_1, \mathbf{S}_2$ *is well defined due to Theorem* 2.2.

Theorem 2.3. *The linear operators* $\mathbf{S}_1$ *and* $\mathbf{S}_2$ *admit representations of the form*

$$\begin{aligned} \mathbf{S}_1(u_1, v_2) &= \mathbf{H}_0 u_1(\tau) + \mathbf{A}u_1(\tau) + \mathbf{B}v_2(\tau), \\ \mathbf{S}_2(u_1, v_2) &= \mathbf{H}_0 v_2(\tau) + \mathbf{C}u_1(\tau) + \mathbf{D}v_2(\tau), \end{aligned} \tag{19}$$

where $\mathbf{H}_0$ *ist the Hilbert transform and* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ *are linear integral operators with analytical kernels.*

3. The nonlinear singular integral equations

Thus, the nonlinear RHP_2 is reduced to a system of nonlinear integral equations for the yet unknown traces of $\zeta(z) = u(z) + iv(z)$ on $\mathcal{S}^1$ of the form

$$\mathcal{A}_\varepsilon(u_1, v_2) := \begin{pmatrix} f_1^\varepsilon(\tau, u_1(\tau), \mathbf{H}_0 u_1 + \mathbf{A}u_1 + \mathbf{B}v_2) \\ f_2^\varepsilon(\tau, \mathbf{H}_0 v_2 + \mathbf{C}u_1 + \mathbf{D}v_2, v_2(\tau)) \end{pmatrix} = 0. \tag{20}$$

We emphasize that the solution of this system of nonlinear integral equations obeys the structure of the universal covering and will provide us with the periodic solutions u_1, v_2 which define via

$$\begin{aligned} Z = \pi\zeta &= u_1 + i\mathbf{S}_1(u_1, v_2) \quad \text{on } |z| = q, \quad \text{and} \\ Z = \pi\zeta &= \mathbf{S}_2(u_1, v_2) + iv_2 \quad \text{on } |z| = 1, \end{aligned}$$

the complete Cauchy data of Z on ∂G_2, so Z is defined via the Cauchy integral over ∂G_2.

In order to solve (20), we use the degree theory of quasiruled Fredholm maps (see [1], [4], [9] and the references therein). We now consider this equation on Ξ. Here we solve the integral equations (20) discarding the fact that Ξ carries the structure of an universal covering.

For the application of the degree theory, we shall need a priori bounds for the solution, and, moreover, that the admissible functions will not admit values on $\partial\Xi$. With a suitable homotopy we then find a simplified system of integral equations that has a solution which implies that the degree of the mapping associated with (20) is nonzero which in turn guarantees the desired existence of a solution.

We first notice that the nonlinear integral equations (20) permit this approach.

Theorem 3.1. *For every* $\varepsilon > 0$, *the operator* $\mathcal{A}_\varepsilon(u_1, v_2) : H^s(\mathcal{S}^1) \times H^s(\mathcal{S}^1) \to H^s(\mathcal{S}^1) \times H^s(\mathcal{S}^1)$ *for* $s > 1$ *defined by*

$$\mathcal{A}_\varepsilon(u_1, v_2) := \begin{pmatrix} f_1^\varepsilon\big(\tau, u_1(\tau), \mathbf{H}_0 u_1(\tau) + \mathbf{A}u_1(\tau) + \mathbf{B}v_2(\tau)\big) \\ f_2^\varepsilon\big(\tau, \mathbf{H}_0 v_2(\tau) + \mathbf{C}u_1(\tau) + \mathbf{D}v_2(\tau), v_2(\tau)\big)\big) \end{pmatrix} \tag{21}$$

is quasilinear Fredholm, where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ *are the linear operators defined in Theorem* 2.2.

For the proof see [3].

In order to show the desired a priori estimates, we observe that every solution of the system of boundary integral equations (20) will define some solution of the RHP$_2$. Hence, we can exploit the following results.

Theorem 3.2. *Let* $F_j^\varepsilon \in C^\infty$ *and satisfy Conditions* 2.1. *Then every holomorphic solution* Z *of the nonlinear Riemann–Hilbert problem* (1)–(6) *satisfies the a priori estimates*

$$0 < \varrho_0 \le |Z(z)| \le R_0 \quad \text{and} \quad \| Z \|_{\mathcal{X}^1} \le C_1 \tag{22}$$

uniformly with respect to $\varepsilon > 0$ *with the two constants* $\varrho_0 < R_0$ *from Conditions* 2.1 *and a constant* C_1, *that do not depend on* Z *nor* ε.

The proof of this theorem is based on the following theorem due to Šnirel'man (see [9]).

Theorem 3.3. *Suppose that* $Z(z)$ *is a holomorphic solution of the nonlinear Riemann–Hilbert problem in the unit disc with a boundary condition* $F \in C^\infty$ *and where* $Z(z)$ *is uniformly bounded by* K_0. *Then* Z *satisfies the a priori estimate*

$$\| Z \|_{\mathcal{X}^s} \le C_1 \tag{23}$$

in the Sobolev–Slobodetskiĭ norms (15) *with* $s \geq 1$ *where* C_1 *is independent of* Z (*but depends on* K_0 *and* s).

PROOF OF THEOREM 3.2. The proof of (23) in [9] is based on local estimates for the solutions Z. The global estimate (23) is then obtained by a partition of the unity. Therefore, we can obtain local estimates separately for each of the boundary components of ∂G_2 repeating Snirelman's arguments word by word (see also [2]).

Because of Conditions 2.1 we have the property (5) available which implies with the maximum principle for $Z(z)$ in G_2 via (2) that

$$0 < \varrho_0 \leq |Z(z)| \leq R_0. \tag{24}$$

For the local estimates, in [9, Lemma 4.1], the constants C_h in [9, (4.11)] and C'_h in [9, (4.11')] are bounds of the first derivatives of the boundary functions which in our case can be estimated by the first derivatives of f_j^ε given in (14). Note that our approximations can be chosen in such a way (e.g. by using Friedrichs mollifiers) that

$$\left|\frac{\partial}{\partial \tau} f_j^\varepsilon\right|, \left|\frac{\partial}{\partial u} f_j^\varepsilon\right|, \left|\frac{\partial}{\partial v} f_j^\varepsilon\right| \leq c \cdot L$$

hold uniformly where L is the maximum of the Lipschitz constants of f^j. Hence, the constant C_1 in (22) is independent of ε and depends only on L, ϱ_0, R_0 and π.□

From (22) and continuous differentiability of the argument function, one gets the a priori bounds

$$\left|\frac{1}{2\pi}\int_{\tau=0}^{2\pi} d\arg Z(qe^{i\tau})\right| < K < \infty \quad \text{and} \quad \left|\frac{1}{2\pi}\int_{\tau=0}^{2\pi} d\arg Z(e^{i\tau})\right| < K \tag{25}$$

for all solutions with an integer constant K depending only on ϱ_0, R_0, C_1 but not on the solution Z.

We now emphasize that all these constants only depend on the functions F_j and Conditions 2.1 and, hence, are available a priori. Therefore, the transformation $\widetilde{z} = z^K$ of the independent variable in G_2 will imply that the transformed solution $\widetilde{Z}$ of the transformed problem in $\widetilde{G}_2$ has zero winding indices on both boundary components of $\partial\widetilde{G}_2$.

From (25) we deduce, due to our construction, that we have a uniform estimate on Ξ:

$$|\zeta^\varepsilon(z)| \leq K_0 < 1. \tag{26}$$

Using Theorem 3.3 again, we also obtain the a priori bounds

$$\| \zeta^\varepsilon \|_{\mathcal{X}^1} \leq C_1 \quad \text{and} \quad \| \zeta^\varepsilon \|_{\mathcal{X}^s} \leq \widetilde{C}_1(\varepsilon, s) \quad \text{with } s > 1 \tag{27}$$

for every periodic solution $\zeta^\varepsilon = u^\varepsilon + iv^\varepsilon$ of the integral equations (20). Hence, for every fixed $\varepsilon > 0$, a degree of the mapping $\mathcal{A}_\varepsilon$ given by (20) is well defined

[1], [4], [9]. It remains to compute the degree of $\mathcal{A}_\varepsilon$. To this end, we construct a homotopy and a familiy of nonlinear integral equations that connects $\mathcal{A}_\varepsilon$ with a solvable system $\mathcal{A}^1$.

Construction of the homotopy. We now associate the solution of the nonlinear integral equations (20) with the nonlinear RHP$'_2$ for solutions ζ in $\widetilde{G}_2$ satisfying the boundary conditions (14) and, in what follows, we shall skip ε in the notation of the solutions $\zeta = u + iv$. Since these solutions satisfy the a priori estimate (26), for $|\zeta(z)| > K_0$ we may formulate any ficticious boundary conditions, e.g. with curves $f_j^\varepsilon(\tau, u, v) = 0$ going off to infinity as in [1]. As indicated in Figure 3 we further choose two strips $|u| \le c_u$ and $|v| \le c_v$ which contain all curves $\widetilde{\gamma}_{\tau,1,1}$ and $\widetilde{\gamma}_{\tau,2,1}$, respectively, together with their ficticious extensions.

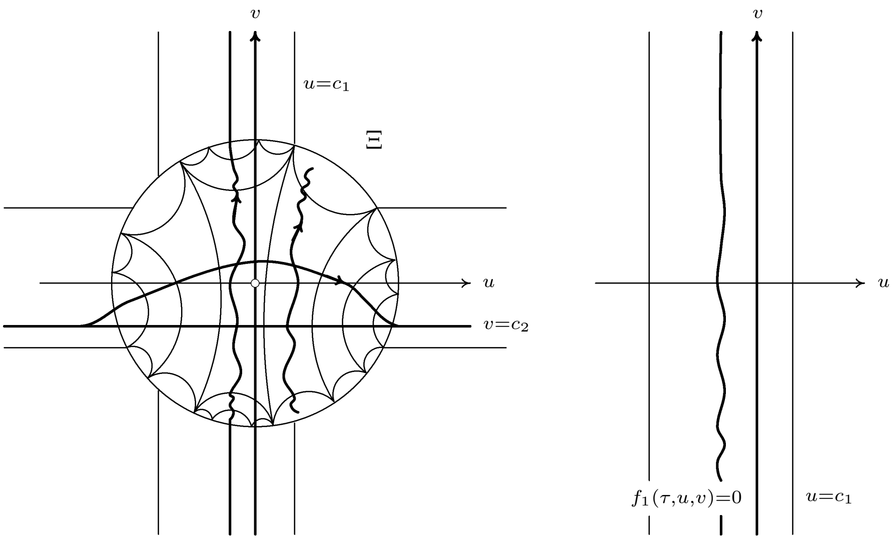

FIGURE 3

Now we choose a straight line $u = c_1$ such that all ficticiously extended curves $\Gamma_0 : f_1^\varepsilon(\tau, u, v) = 0$ lie on the left side of $\Gamma_1 : u = c_1$ and solve the auxiliary Dirichlet problem

$$\Delta_{u,v}\Psi_1 = 0 \text{ for } \{(u,v) \mid u_0 \le u \le c_1, f_1^\varepsilon(\tau, u_0, v) = 0\},$$
$$\text{with } \Psi_1|_{\Gamma_0} = 0 \text{ and } \Psi_1|_{\Gamma_1} = 1,\ \Psi_1 \le 1,$$

and set $f_1^{\varepsilon,t}(\tau, u, v) := \Psi_1(\tau, u, v) - t$ for $0 \le t \le 1$. Then for every constant value of $t \in [0, 1]$, the equation $f_1^{\varepsilon,t}(\tau, u, v) = 0$ describes a level curve of Ψ_1

which is non-selfintersecting and lies in the strip $|u| \le c_u$. Moreover,

$$f_1^{\varepsilon,0}(\tau,u,v) = 0 \quad \text{on } \Gamma_0, \text{ i.e. } f_1^{\varepsilon}(\tau,u,v) = 0,$$
$$f_1^{\varepsilon,1}(\tau,u,v) = 0 \quad \text{on } \Gamma_1, \text{ i.e. } u = c_1.$$

In the same manner we construct $f_2^{\varepsilon,t}(\tau,u,v)$ and obtain the second family of boundary condition curves in $|v| \le c_v$. Clearly, the family of integral equations

$$\mathcal{A}_\varepsilon^t(u_1,v_2) = \begin{pmatrix} f_1^{\varepsilon,t}(\tau,u_1(\tau),\mathbf{H}_0u_1(\tau)+\mathbf{A}u_1(\tau)+\mathbf{B}v_2(\tau)) \\ f_2^{\varepsilon,t}(\tau,\mathbf{H}_0v_2(\tau)+\mathbf{C}u_1(\tau)+\mathbf{D}v_2(\tau),v_2(\tau)) \end{pmatrix} = 0$$

is now homotopic to

$$\mathcal{A}^1(u_1,v_2) = \begin{cases} u_1 - c_1 = 0 & \text{for } |\widetilde{z}| = q^K, \\ v_2 - c_2 = 0 & \text{for } |\widetilde{z}| = 1, \end{cases}$$

which has exactly one solution.

Consequently, the degree of the quasi Fredholm ruled mapping $\mathcal{A}^1$ is 1, so is the degree of $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^0$. Hence, the integral equations (20) admit at least one solution $\zeta_T^\varepsilon(z)$ in case 1 for every $\varepsilon > 0$.

In the same manner we obtain at least one solution $\zeta_S^\varepsilon(z)$ in case 2.

As a result we obtain $Z_T^\varepsilon(z) = \pi\zeta_T^\varepsilon(z)$ and $Z_S^\varepsilon(z) = \pi\zeta_S^\varepsilon(z)$. These are solutions of the original problems (1)–(6) for the boundary curves F_j^ε. For $\varepsilon \to 0$, it now remains to use convergence arguments. Taking into account the uniform estimate (27) in $\mathcal{X}^1$ and using Montel's theorem, we obtain desired solutions $Z_T(z)$ and $Z_S(z)$ of our original problem (see also [3]).

It remains to show that the two solutions obtained in case 1 and in case 2 are different. For this property it suffices to show that they are not homotopic. This will be proved by contradiction. Let $Z_T(z) = \pi\zeta_T(z)$ and $Z_S(z) = \pi\zeta_S(z)$ be two solutions obtained from the first and the second case, respectively. Then

$$Z_T \text{ and } Z_S : G_2 \to \mathcal{S}^2 \setminus \{Z_1,\ldots,Z_4\},$$

and

1. both solutions define holomorphic mappings;
2. due to boundary conditions we also have

$$Z_T(qe^{i\tau}) \notin \Gamma_2 \cup \Gamma_4 \quad \text{and} \quad Z_T(e^{i\tau}) \notin \Gamma_1 \cup \Gamma_3,$$
$$Z_S(qe^{i\tau}) \notin \Gamma_2 \cup \Gamma_4 \quad \text{and} \quad Z_S(e^{i\tau}) \notin \Gamma_1 \cup \Gamma_3.$$

Let us now suppose that Z_T and Z_S are homotopic and consider

$$g_T := Z_{T|[q,1]}, \quad g_S := Z_{S|[q,1]}.$$

Then there exists a continuous function

$$g(t,x) \quad \text{with } g : [0,1] \times [q,1] \to \mathcal{S}^2 \setminus \{Z_1,\ldots,Z_4\}$$

and

$$g(0,x) = g_T(x), \quad g(1,x) = g_S(x);$$

moreover, $g(t,q) \notin \Gamma_2 \cup \Gamma_4$ and $g(t,1) \notin \Gamma_1 \cup \Gamma_3$ for all $t \in [0,1]$.

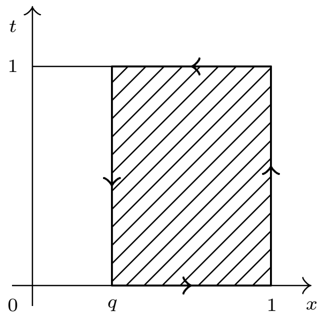

Figure 4

Then $Z_T(qe^{i\tau})$ and $Z_S(qe^{i\tau})$ can also homotopically be deformed into each other. In view of our construction, this will require that the corresponding curves $\widetilde{\gamma}_{\tau,1,1}$ or $\widetilde{\gamma}_{\tau,1,2}$ on Ξ can homotopically be deformed into each other which implies that some of these curves must have a nonempty intersection with the image of Γ_2 or Γ_4, or some of the curves $\gamma^t_{\tau,1}$ will intersect with Γ_2, or with Γ_4 which contradicts Conditions 2.1.

In other words, one can easily see that to the curves $g(t,x)|_{\partial([q,1]\times[0,1])}$ there corresponds the unity of the fundamental group $\pi_1(\mathcal{S}^2 \setminus \{Z_1, Z_2, Z_3, Z_4\})$ since $[q,1]\times[0,1]$ is contractible. On the other hand, a direct computation shows that the curve $g(t,x)|_{\partial([q,1]\times[0,1])}$ corresponds to some element $(g_1g_2)^{n_1}(g_2g_3)^{n_2}$ with $n_1, n_2 \in \mathbb{Z}$ of the fundamental group $\pi_1(\mathcal{S}^2 \setminus \{Z_1, Z_2, Z_3, Z_4\})$ which obviously is always different from the unity e contradicting the previous statement. Here, we denote by g_j the generators of the free group π_1 [10]. This completes the proof of Theorem 2.1. $\square$

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