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ON A GENERALIZED CRITICAL POINT THEORY ON GAUGE SPACES AND APPLICATIONS TO ELLIPTIC PROBLEMS ON $\mathbb{R}^N$

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Abstract. In this paper, we introduce some aspects of a critical point theory for multivalued functions $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ defined on E a complete gauge space and with closed graph. The existence of a critical point is established in presence of linking. Finally, we present applications of this theory to semilinear elliptic problems on $\mathbb{R}^N$.

1. Introduction

In the last six years, critical point theory for lower semi-continuous functionals defined on complete metric spaces was developped by Degiovanni, Marzocchi, Corvellec [7], [8], Ioffe, Schwartzman [14], [15], and Katriel [16]. Since then, some applications of this theory to partial differential equations were given. They concern mainly problems from which the associated functional defined on a Banach space is not continuously differentiable. The starting point of this work was to see if this theory can be applied to semilinear elliptic problems on $\mathbb{R}^N$. Indeed, the Fréchet space $H^1_{\mathrm{loc}}(\mathbb{R}^N)$ is a very natural metric space (which is not Banach) one can think of. Many difficulties occured; in particular, the associated functional is not in general lower semi-continuous on its domain.

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In this paper, we generalize in some aspects the critical point theory developped by Degiovanni, Marzocchi and Corvellec [7], [8] and by the author [11] for multivalued functionals, see also [21]. We consider complete gauge spaces E and multivalued functions $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ with closed graph. Obviously, this contains the particular case of continuous functions $\phi : E \to \mathbb{R}^{\mathbb{N}}$.

A notion of slope of Φ and the associated notion of critical point are introduced. Also, we define a notion of linking in the spirit of the one introduced in [12]. Our main theorem establishes under suitable assumptions the existence of a critical point of Φ in presence of linking. As corollary, the existence of a critical point of some functions Φ bounded from below is obtained. The proofs of these results rely on a deformation lemma for subsets of graph Φ which is established in Section 7.

Finally, two simple applications to the semilinear elliptic partial differential equation

$$-\Delta u + a(x)u = g(x, u), \quad x \in \mathbb{R}^N \tag{1.1}$$

are presented. The existence of a solution in $H^1_{\text{loc}}(\mathbb{R}^N)$ to (1.1) is obtained. Here, no group invariance conditions are imposed on a and g. Also, no restrictions on the behavior of a and g as $\|x\| \to \infty$ are assumed. Moreover, g is not necessarily superlinear at 0.

This type of problems has been and is still studied by many authors, see for example [1]–[4], [9], [17], [19], [20], [22]–[24] and the references therein. Some of them, as Strauss [22], Berestycki, Lions [4], Bartsch, Willem [2], [3], and Bartsch, Wang [1], seeked radial solutions or $O(N)$-invariant solutions under appropriate group invariance conditions on a and g. Others, as Lions [17], treated the case where a and g have a limit at ∞, that is $a(x) \to \overline{a}$, $g(x, u) \to \overline{g}(u)$ as $\|x\| \to \infty$. The concentration-compactness principle of Lions [17] is an important tool in these results. On the other hand, Bartsch, Wang [1], Ding, Ni [9], and Rabinowitz [20] established existence results for the problem (1.1) without group invariance conditions but with stronger assumptions on a than ours; for instance, in [20], it is assumed that $a(x) \to \infty$ as $\|x\| \to \infty$. In [9] and [20], the results are obtained in using a sequence $\{u_n\}$ with u_n a solution of (1.1) on B_n the ball of radius n in $\mathbb{R}^N$. All those results can not be compared to ours since, even though our assumptions on a are weaker, those on g are different to theirs; also g satisfies a subcritical growth condition as in [5], [6], [13]. However, it is worthwile to mention that in most of the results in the literature on the problem (1.1), it is assumed that $\inf a > 0$ while this is not required here. Finally, we point out that the approach presented in this paper has some similarities with the one used in [11].

2. Notations

In this paper, we consider complete gauge spaces $(\mathbf{E}, \{\mathbf{d}_n\}_{n\in\mathbb{N}})$ (topological spaces with topology induced by a family of gauges (semi-metric) $\{\mathbf{d}_n\}_{n\in\mathbb{N}}$), see [10] for more details. In order to simplify the presentation, we choose $\{\mathbf{d}_n\}$ such that

$$\mathbf{d}_1(u,v) \le \mathbf{d}_2(u,v) \le \dots. \tag{2.1}$$

We write for $y \in \mathbf{E}$, $S \subset \mathbf{E}$, and $r > 0$

$$\begin{aligned} B_n(y,r) &= \{x \in \mathbf{E} \mid \mathbf{d}_n(y,x) < r\}, \\ \mathcal{O}(y,r) &= \{x \in \mathbf{E} \mid \sup_{n\in\mathbb{N}} \mathbf{d}_n(x,y) < r\}, \\ S_r &= \overline{\bigcup_{y\in S} \mathcal{O}(y,r)}. \end{aligned}$$

We denote by $\mathcal{T}_u$ the uniform topology on $\mathbf{E}$ generated by the open sets $\mathcal{O}(y,r)$. Notice that $\mathbf{E}$ is endowed with the gauge space topology $\mathcal{T}_g$ generated by $\{\mathbf{d}_n\}_{n\in\mathbb{N}}$ unless the contrary is mentioned.

Observe that if $(E, \{d_n\}_{n\in\mathbb{N}})$ is a complete gauge space satisfying (2.1), then so is $(E \times \mathbb{R}^{\mathbb{N}}, \{D_n\}_{n\in\mathbb{N}})$ where D_n is the gauge defined by:

$$D_n((u,c),(v,b)) = \sqrt{d_n(u,v)^2 + (\max_{k\le n} |c_k - b_k|)^2}.$$

Also, if $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ is a multivalued map with closed graph, that is such that

$$\operatorname{graph}\Phi = \{y = (u,c) \in E \times \mathbb{R}^{\mathbb{N}} \mid c \in \Phi(u)\}$$

is closed in $E \times \mathbb{R}^{\mathbb{N}}$, then, $(\operatorname{graph}\Phi, \{D_n\}_{n\in\mathbb{N}})$ is a complete gauge space satisfying (2.1).

We say that a subset $C \subset \mathbb{R}^{\mathbb{N}}$ is *bounded from below* (resp. *from above*) if there exists $m = (m_1, m_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ (resp. $M = (M_1, M_2, \dots)$) such that for every $c = (c_1, c_2, \dots) \in C$, $c_n \ge m_n$ (resp. $c_n \le M_n$) for every $n \in \mathbb{N}$; we write $C \subset [m, \infty[\, = \prod_{n\in\mathbf{N}} [m_n, \infty[$ (resp. $C \subset \,]-\infty, M]$). We say that C is *bounded* if it is bounded from above and from below, we write $C \subset [m, M] = \prod_{n\in\mathbb{N}} [m_n, M_n]$. For $c \in \mathbb{R}^{\mathbb{N}}$ and $r \in \mathbb{R}$, we write $c + r$ for $(c_1 + r, c_2 + r, \dots)$.

3. Linking

Let $(\mathbf{E}, \{\mathbf{d}_n\}_{n\in\mathbb{N}})$ be a complete gauge space. We introduce a notion of linking in the spirit of the one presented in [12]. For that, we use the following notation. Let $A_1 \subset A_0 \subset \mathbf{E}$ with $A_0 \ne \emptyset$, we set

$$\begin{aligned} \mathcal{N}(A_0, A_1) = \{\eta : (A_0, \mathcal{T}_u) \times [0,1] \to (\mathbf{E}, \mathcal{T}_u) \mid \eta \text{ is continuous} \\ \eta(x,t) = x \text{ for all } (x,t) \in A_0 \times \{0\} \cup A_1 \times [0,1]\}. \end{aligned}$$

Definition 3.1. Let $A_1 \subset A_0 \subset \mathbf{E}$, $Q_1 \subset Q_0 \subset \mathbf{E}$. We say that (A_0, A_1) *links* (Q_0, Q_1) if $A_1 \cap Q_0 = \emptyset$, $A_0 \cap Q_1 = \emptyset$, $A_0 \cap Q_0 \neq \emptyset$, and if for every $\eta \in \mathcal{N}(A_0, A_1)$, one of the following statements holds:

(1) $\eta(A_0, 1) \cap Q_0 \neq \emptyset$,
(2) $\eta(A_0,]0,1[) \cap Q_1 \neq \emptyset$.

Remark 3.2. Notice that A_1 can be empty and Q_1 nonempty (which is impossible in the usual notions of linking). An analogous definition can be stated with $\mathcal{T}_g$ instead of $\mathcal{T}_u$. In fact, we chose to impose the continuity with respect to $\mathcal{T}_u$ in order to obtain more linking sets.

Here are some examples of linking. Many others could be given.

Proposition 3.3. *Let $(F, \{\|\cdot\|_n\}_{n\in\mathbb{N}})$ be a Fréchet space such that $F = F_1 \oplus F_2$ where $F_1 = \operatorname{span}(e_1, \dots, e_k)$ with $\{e_1, \dots, e_k\}$ linearly independant and $\|e_i\|_n \le R < \infty$ for every $n \in \mathbb{N}$, and $i = 1, \dots, k$. Let $M > 0$ and denote*

$$\begin{aligned} A_0 &= \{\lambda_1 e_1 + \dots + \lambda_k e_k \mid \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \text{ with } \|\lambda\| \le 1\}, \\ A_1 &= \{\lambda_1 e_1 + \ldots + \lambda_k e_k \mid \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \text{ with } \|\lambda\| = 1\}, \\ Q_0 &= \{u \in F_2 \mid \liminf_{n\to\infty} \|u\|_n \le M\}, \\ Q_1 &= \{u \in Q_0 \mid \liminf_{n\to\infty} \|u\|_n = M\}. \end{aligned}$$

Then (A_0, A_1) links (Q_0, Q_1); also, (A_0, A_1) links $(F_2, \emptyset)$.

Proof. Let $\eta \in \mathcal{N}(A_0, A_1)$. Define $H : A_0 \times [0,1] \to F_1$ by $h(u,t) = P_{F_1}(\eta(u,t))$, where P_{F_1} is the projection on F_1. It follows from degree theory the existence of a continuum (in the uniform topology) $\mathcal{C} \subset \{(u,t) \in A_0 \times [0,1] : h(u,t) = 0\}$ such that $\mathcal{C} \cap A_0 \times \{i\} \neq \emptyset$, $i = 0, 1$. This implies that $\eta(\mathcal{C} \cap A_0 \times \{1\}) \cap F_2 \neq \emptyset$. So, (A_0, A_1) links $(F_2, \emptyset)$.

On the other hand, define $L : F \to \mathbb{R} \cup \{\pm\infty\}$ by $L(u) = \liminf_{n\to\infty} \|u\|_n$. Observe that $L \circ \eta : A_0 \times [0,1] \to \mathbb{R}$ is well defined and continuous with respect to the uniform topology. Since $L \circ \eta(\mathcal{C})$ is also a continuum and $L \circ \eta(\mathcal{C} \cap A_0 \times \{0\}) = 0$, we deduce that $\eta(\mathcal{C} \cap A_0 \times \{1\}) \cap Q_0 \neq \emptyset$ or $\eta(\mathcal{C} \cap A_0 \times]0,1[) \cap Q_1 \neq \emptyset$. Hence (A_0, A_1) links (Q_0, Q_1). $\square$

Proposition 3.4. *Let $(E, \{d_n\}_{n\in\mathbb{N}})$ be a complete gauge space and let $A_1 \subset A_0 \subset E$, $Q_1 \subset Q_0 \subset E$ be such that (A_0, A_1) links (Q_0, Q_1). Assume that $\phi : E \to \mathbb{R}^{\mathbb{N}}$ is a continuous map with $\phi|_{A_0}$ continuous with respect to the uniform topology on A_0 and $\mathbb{R}^{\mathbb{N}}$. Consider the gauge space $\mathbf{E} = \{(u,c) \in E \times \mathbb{R}^{\mathbb{N}} \mid c \in [\phi(u), \infty[\}$. Then $(\operatorname{graph}\phi(A_0), \operatorname{graph}\phi(A_1))$ links $(\widetilde{Q_0}, \widetilde{Q_1})$ in $\mathbf{E}$, where $\widetilde{Q_i} = \{(u,c) \in Q_i \times \mathbb{R}^{\mathbb{N}} \mid c \in [\phi(u), \infty[\}$, $i = 0, 1$.*

Proof. Obviously, $\operatorname{graph}\phi(A_0) \cap \widetilde{Q_0} \neq \emptyset$, $\operatorname{graph}\phi(A_0) \cap \widetilde{Q_1} = \emptyset$, and $\operatorname{graph}\phi(A_1) \cap \widetilde{Q_0} = \emptyset$. Let $\eta = (\eta_1, \eta_2) \in \mathcal{N}(\operatorname{graph}\phi(A_0), \operatorname{graph}\phi(A_1))$. Define

$$\widehat{\eta} : A_0 \times [0,1] \to E \quad \text{by} \quad \widehat{\eta}(u,t) = \eta_1((u, \phi(u)), t).$$

Since (A_0, A_1) links (Q_0, Q_1), either $\widehat{\eta}(A_0, 1) \cap Q_0 \neq \emptyset$, and hence

$$\eta(\operatorname{graph}\phi(A_0), 1) \cap \widetilde{Q_0} \neq \emptyset, \quad \text{or} \quad \widehat{\eta}(A_0 \times]0,1[) \cap Q_1 \neq \emptyset,$$

and hence $\eta(\operatorname{graph}\phi(A_0) \times\ ]0,1[) \cap \widetilde{Q_1} \neq \emptyset$. So, $(\operatorname{graph}\phi(A_0), \operatorname{graph}\phi(A_1))$ links $(\widetilde{Q_0}, \widetilde{Q_1})$. □

4. Slope and critical points

Let $(E, \{d_n\}_{n\in\mathbb{N}})$ be a complete gauge space satisfying (2.1). We consider $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ a multivalued function with closed graph and the complete gauge space $(\operatorname{graph}\Phi, \{D_n\}_{n\in\mathbb{N}})$. In order to define the notion of critical point of Φ, we need to introduce the notion of slope.

Definition 4.1. Let $y \in \operatorname{graph}\Phi$. The *slope of* Φ *at* y, denoted $|d\Phi|(y)$, is defined as the supremum of $\sigma \geq 0$ such that there exist $m \in \mathbb{N}$, $\delta > 0$ and a map $H = (H_0, H_1, \dots) : B_m(y, \delta) \times [0, \delta] \to \operatorname{graph}\Phi$ continuous with $\operatorname{graph}\Phi$ endowed with the uniform topology such that for every $(v, b) \in B_m(y, \delta)$ and every $t \in [0, \delta]$,

(i) $D_n(H((v,b),t),(v,b)) \leq t\sqrt{1+\sigma^2}$ for every $n \in \mathbb{N}$,
(ii) $H_n((v,b),t) \leq b_n - \sigma t$ for every $n \geq m$,
(iii) $|H_n((v,b),t) - b_n| \leq \sigma t$ for every $n \in \mathbb{N}$.

Remark 4.2. For $k \in [1, \infty]$, we can define $|k\text{-}d\Phi|(y)$ by replacing (iii) by

(iii)$_k$ $|H_n((v,b),t) - b_n| \leq k\sigma t$ for every $n \in \mathbb{N}$.

So, $|d\Phi|(y) = |1\text{-}d\Phi|(y)$ and $|d\Phi|(y) \leq |k\text{-}d\Phi|(y)$ for every $k \geq 1$.

Remark 4.3. (1) Let (E, d) be a complete metric space, and $\phi_1 : E \to \mathbb{R} \cup \{\infty\}$ a lower semi-continuous functional. Take $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ defined by

$$\Phi(u) = \begin{cases} \{\infty\} & \text{if } \phi_1(u) = \infty, \\ \{(c, c, \dots) \mid c \geq \phi_1(u)\} & \text{otherwise.} \end{cases}$$

Then the weak slope of ϕ_1 at u in the sense of Degiovanni and Marzocchi [8] $|d\phi_1|(u) = |d\Phi|(y)$ with $y = (u, (\phi_1(u), \phi_1(u), \dots))$.

(2) Let (E, d) be a complete metric space, and $\Phi_1 : E \to \mathbb{R} \cup \{\infty\}$ a multivalued map with closed graph. Define $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ by

$$\Phi(u) = \begin{cases} \{(c, c, \dots) \mid c \in \Phi_1(u)\} & \text{if } \Phi_1(u) \cap \mathbb{R} \neq \emptyset, \\ \{\infty\} & \text{otherwise.} \end{cases}$$

Then $|d\Phi_1|(u,c) = |\infty\text{-}d\Phi|(u,(c,c,\dots))$ (see Remark 4.2), where $|d\Phi_1|(u,c)$ is the weak slope introduced in [11].

(3) Let E be a Banach space and $f_n \in C^1(E,\mathbb{R})$ for $n \in \mathbb{N}$. Take $\Phi : E \to \mathbb{R}^{\mathbb{N}}$ defined by $\Phi(u) = (f_1(u), f_2(u), \dots)$. Then

$$|\infty\text{-}d\Phi|(u,(f_1(u),f_2(u),\dots)) \le \liminf_{n\to\infty} \|f_n'(u)\|.$$

It is easy to show the following result.

Lemma 4.4. *The slope $|d\Phi|$ is lower semi-continuous.*

Definition 4.5. We say that $u \in E$ is a *critical point of* Φ *at level* $c \in \mathbb{R}^{\mathbb{N}}$ if $(u,c) \in \operatorname{graph}\Phi$ and $|d\Phi|(u,c) = 0$; c is called a *critical value of* Φ; we write $u \in K_c$. We say that u is a *critical point of* Φ if $u \in K = \bigcup_{c\in\mathbb{R}^{\mathbb{N}}} K_c$.

Definition 4.6. Let $C \subset \mathbb{R}^{\mathbb{N}}$. We say that Φ satisfies the *Palais–Smale condition at* C, noted $(\mathrm{PS})_C$ if every sequence $\{y_k\}$ such that $|d\Phi|(y_k) \to 0$ and $y_k \in \operatorname{graph}\Phi \cap E \times C + [-r_k, r_k]$ with $r_k \to 0$, has a convergent subsequence.

Remark 4.7. This definition seems to be the most suitable in our context. Indeed, let us consider the case where $C = \{c\} \subset \mathbb{R}^{\mathbb{N}}$. The set of sequences $\{y_k = (u_k, c_k)\}$ in $\operatorname{graph}\Phi$ with $c_k \to c$ and $|d\Phi|(y_k) \to 0$ is much larger than the set of sequences $\{y_k\}$ considered in the previous definition. Also, in practice, it is extremely difficult to know exactly what will be the critical value; so, in our case, C will be an interval in $\mathbb{R}^{\mathbb{N}}$.

5. Single-valued continuous maps

For a better understanding of our Main Theorem, we first consider the particular case where $\phi : E \to \mathbb{R}^{\mathbb{N}}$ is a single-valued continuous map. Obviously, it can be considered as a multivalued map $u \mapsto \{\phi(u)\}$ with closed graph. However, the Palais–Smale condition would hardly be satisfied because condition (iii) in Definition 4.1 is very restrictive in this context. So, in practice, it seems more appropriate to consider the associate multivalued map defined by

$$\Phi(u) = [\phi(u), \infty[\,.$$

We will say that u is a *critical point of* ϕ *at level* c if there exists $c \in [\phi(u), \infty[$ such that u is a critical point of Φ at level c. Similarly, we will say that ϕ satisfies $(\mathrm{PS})_C$ if Φ satisfies $(\mathrm{PS})_C$.

Lemma 5.1. *Let $\phi : E \to \mathbb{R}^{\mathbb{N}}$ be a single-valued continuous map. If u is a critical point of ϕ at level c, then*

$$\liminf_{n\to\infty} c_n - \phi_n(u) = 0.$$

In this particular case, we state a corollary of our main theorem which will be presented in the next section.

Theorem 5.2. *Let $\phi : E \to \mathbb{R}^{\mathbb{N}}$ be a continuous single-valued map, and $\overline{c} \in \mathbb{R}^{\mathbb{N}}$. Let $A_1 \subset A_0 \subset E$, $Q_1 \subset Q_0 \subset E$ be such that (A_0, A_1) links (Q_0, Q_1), and*

$$\begin{aligned}(5.1)\qquad \sup_{u\in A_1}\liminf_{n\to\infty}\phi_n(u)-\overline{c}_n<\beta&=\inf_{u\in Q_0}\liminf_{n\to\infty}\phi_n(u)-\overline{c}_n\\ &\le\gamma=\sup_{u\in A_0}\liminf_{n\to\infty}\phi_n(u)-\overline{c}_n\le\inf_{u\in Q_1}\liminf_{n\to\infty}\phi_n(u)-\overline{c}_n,\end{aligned}$$

with $\gamma \in \mathbb{R}$. Assume that $\operatorname{graph}\phi(A_0)$ *is compact with the uniform topology, and ϕ satisfies* $(\mathrm{PS})_C$ *for every bounded $C \subset \mathbb{R}^{\mathbb{N}}$. Then ϕ has a critical point.*

6. Main theorem

We state our main result establishing the existence of a critical point of $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ a multivalued map with closed graph. By convention $\sup(\emptyset) = -\infty$, $\inf(\emptyset) = \infty$.

Main Theorem 6.1. *Let $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ be a multivalued map with closed graph. Let $\overline{c} = (\overline{c}_1, \overline{c}_2, \dots) \in \mathbb{R}^{\mathbb{N}}$, $A_1 \subset A_0 \subset \operatorname{graph}\Phi$, $Q_1 \subset Q_0 \subset \operatorname{graph}\Phi$ be such that (A_0, A_1) links (Q_0, Q_1), and*

$$\begin{aligned}(6.1)\qquad \sup_{(u,c)\in A_1}\liminf_{n\to\infty}(c_n-\overline{c}_n)<\beta&=\inf_{(u,c)\in Q_0}\liminf_{n\to\infty}(c_n-\overline{c}_n)\\ &\le\gamma=\sup_{(u,c)\in A_0}\liminf_{n\to\infty}(c_n-\overline{c}_n)\le\inf_{(u,c)\in Q_1}\liminf_{n\to\infty}(c_n-\overline{c}_n),\end{aligned}$$

with $\gamma \in \mathbb{R}$. Assume that A_0 is compact with the uniform topology, and Φ satisfies $(\mathrm{PS})_C$ *with*

$$C=\{c+r\in\mathbb{R}^{\mathbb{N}}\mid |r|\le\gamma-\beta \text{ and there exists } (u,c)\in A_0 \text{ such that } \liminf_{n\to\infty}c_n-\overline{c}_n\ge\beta\}.$$

Then there exists $c \in C$ a critical value of Φ such that

$$\beta-(\gamma-\beta)\le\liminf_{n\to\infty}(c_n-\overline{c}_n)\le\gamma+(\gamma-\beta).$$

Remark 6.2. The previous theorem still holds if we replace $|d\Phi|$ by $|k\text{-}d\Phi|$ for $k \in]1, \infty]$ and $\gamma - \beta$ by $k(\gamma - \beta)$.

In imposing extra conditions on the values of Φ, we can get more precisions on the critical value.

Theorem 6.3. *Assume the assumptions of the previous theorem are satisfied, and assume that*

$$\begin{aligned}&b\in\Phi(u) \text{ with } b_n<\overline{c}_n+\beta \text{ for some } n\in\mathbb{N}\\ &\qquad\Rightarrow(b_1,\dots,b_{n-1},\widehat{b}_n,b_{n+1},\dots)\in\Phi(u) \text{ for every } \widehat{b}_n\in[b_n,\overline{c}_n+\beta].\end{aligned}$$

Then there exists $c \in C$ a critical value of Φ such that $\liminf_{n\to\infty}(c_n - \overline{c}_n) \geq \beta$.

On the other hand, if we do not impose a compactness assumption on A_0, we get less precisions on the critical value.

Theorem 6.4. *Let $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ be a multivalued map with closed graph. Let $\overline{c} = (\overline{c}_1, \overline{c}_2, \dots) \in \mathbb{R}^{\mathbb{N}}$, $A_1 \subset A_0 \subset \operatorname{graph}\Phi$, $Q_1 \subset Q_0 \subset \operatorname{graph}\Phi$ be such that (A_0, A_1) links (Q_0, Q_1). Assume that* (6.1) *holds, and Φ satisfies* $(\mathrm{PS})_C$ *with $C = \widetilde{C} + [\beta - \gamma, \gamma - \beta]$ where $\widetilde{C}$ is the closure (in $\mathcal{T}_g$) of $\{c \in \mathbb{R}^{\mathbb{N}} \mid \exists (u, c) \in A_0\}$. Then there exists $c \in C$ a critical value of Φ.*

In order to prove these theorems, a deformation lemma which will be established in the next section is needed. Before, we present a corollary.

In our context, we do not talk of minimum but sometimes, one can obtain a critical point of Φ when it is bounded from below. This is shown in the following result which is a direct consequence of Theorem 6.1 applied with $A_0 = \{(\widehat{u}, \widehat{c})\}$, $Q_0 = \operatorname{graph}\Phi$, and $A_1 = Q_1 = \emptyset$.

Corollary 6.5. *Let $\Phi : E \to \mathbb{R}^{\mathbb{N}} \cup \{\infty\}$ be a multivalued map with closed graph such that $\Phi(E)$ is bounded from below, that is $\Phi(E) \subset [m, \infty[\cup \{\infty\}$. Assume that there exists $(\widehat{u}, \widehat{c}) \in \operatorname{graph}\Phi$ such that*

$$\liminf_{n\to\infty}(m_n - \widehat{c}_n) = -r > -\infty.$$

If $(\mathrm{PS})_C$ *is satisfied with $C = [\widehat{c} - r, \widehat{c} + r]$, then Φ has a critical value $c \in C$.*

It is worthwhile to mention that the situation here is different from the classical critical point theory. Indeed, one can find a map Φ bounded from below, satisfying $(\mathrm{PS})_C$ for every C, which does not have critical points.

Example 6.6. Let $E = \mathbb{R}^{\mathbb{N}}$, and $\Phi : E \to \mathbb{R}^{\mathbb{N}}$ defined by

$$\Phi(u) = \{c \in \mathbb{R}^{\mathbb{N}} \mid c_n \geq e^{n + \max\{u_1, \dots, u_n\}},\ n \in \mathbb{N}\}.$$

Obviously, Φ has closed graph and is bounded from below. Moreover, Φ has no critical points. Indeed, for every $(u, c) \in \operatorname{graph}\Phi$, $|d\Phi|(u, c) \geq e^{-1}$. To see this, define $H : \operatorname{graph}\Phi \times [0, 1] \to \operatorname{graph}\Phi$ by

$$\begin{aligned} &H((v, b), t) \\ &\quad = (v - t, \max\{b_1 - te^{-1}, e^{1 - t + v_1}\}, \max\{b_2 - te^{-1}, e^{2 - t + \max\{v_1,, v_2\}}\}, \dots). \end{aligned}$$

Obviously, Φ satisfies $(\mathrm{PS})_{\mathbb{R}^{\mathbb{N}}}$.

7. Deformation lemma

In this section, we obtain a deformation result which will be used in the proof of Theorems 6.1 and 6.3. The proof is analogous to the one presented in [7].

THEOREM 7.1. *Let S be a subset of* $\operatorname{graph}\Phi$ *such that there exist $\sigma > 0$, and $R > 0$ such that*

$$|d\Phi|(y) > \sigma \quad \textit{for every } y \in S_R. \tag{7.1}$$

Then there exist $r > 0$, a map $\eta = (\eta_0, \eta_1, \dots) : S \times [0, r] \to \operatorname{graph}\Phi$ continuous with $\operatorname{graph}\Phi$ *endowed with the uniform topology, and there exists a map $N : S \to \mathbb{N}$ locally bounded such that for every $(u, c) \in S$ and every $t \in [0, r]$,*

(i) $D_n(\eta((u, c), t), (u, c)) \le t\sqrt{1 + \sigma^2}$ *for every $n \in \mathbb{N}$,*
(ii) $|\eta_n((u, c), t) - c_n| \le \sigma t$ *for every $n \in \mathbb{N}$,*
(iii) $\eta_n((u, c), t) \le c_n - \sigma t$ *for every $n \ge N(u, c)$.*

PROOF. For every $y \in S_R$, there exist $\delta_y > 0$, $m_y \in \mathbb{N}$, and a continuous in the uniform topology map $H_y : B_{m_y}(y, \delta_y) \times [0, \delta_y] \to \operatorname{graph}\Phi$ satisfying (i)–(iii) of Definition 4.1 with σ.

By Milnor's Lemma (see [18]), the open cover $\{B_{m_y}(y, \delta_y/2)\}_{y \in S_R}$ of S_R admits a locally finite refinement $\{\mathcal{V}_{j,\lambda} \mid j \in \mathbb{N},\ \lambda \in \Lambda_j\}$ such that $\mathcal{V}_{j,\lambda} \cap \mathcal{V}_{j,\mu} = \emptyset$ if $\lambda \ne \mu$. Let $\{\theta_{j,\lambda} : S_R \to [0, 1] \mid j \in \mathbb{N},\ \lambda \in \Lambda_j\}$ be a continuous partition of unity subordinate to $\{\mathcal{V}_{j,\lambda} \mid j \in \mathbb{N},\ \lambda \in \Lambda_j\}$.

For every (j, λ), choose $y_{j,\lambda} \in S_R$ such that $\mathcal{V}_{j,\lambda} \subset B_{m_{j,\lambda}}(y_{j,\lambda}, \delta_{j,\lambda}/2)$, where we set $m_{j,\lambda} = m_{y_{j,\lambda}}$, $\delta_{j,\lambda} = \delta_{y_{j,\lambda}}$, and $H_{j,\lambda} = H_{y_{j,\lambda}}$. For $y \in S_R$, denote

$$\widetilde{N}(y) = \max\{m_{j,\lambda} \mid y \in \mathcal{V}_{j,\lambda}\}.$$

Let $T : S_R \to]0, \infty[$ be a continuous function such that

$$0 < T(y) < \frac{1}{2\sqrt{1 + \sigma^2}} \min\{\delta_{j,\lambda} \mid y \in \mathcal{V}_{j,\lambda}\}. \tag{7.2}$$

Such a function exists, and $\widetilde{N}$ is locally bounded on S_R since the refinement is locally finite.

Define $\eta^1 = (\eta^1_0, \eta^1_1, \dots) : \{(y, t) \in S_R \times [0, \infty[\mid t \le T(y)\} \to \operatorname{graph}\Phi$ by

$$\eta^1(y, t) = \begin{cases} H_{1,\lambda}(y, \theta_{1,\lambda}(y)t) & \text{if } y \in \overline{\mathcal{V}_{1,\lambda}}, \\ y & \text{if } y \notin \bigcup\limits_{\lambda \in \Lambda_1} \mathcal{V}_{1,\lambda}. \end{cases}$$

This function is continuous in the uniform topology since the inclusion

$$i : (\operatorname{graph}\Phi, \mathcal{T}_u) \to (\operatorname{graph}\Phi, \mathcal{T}_g)$$

is continuous. Also, it satisfies for every $y=(u,c)\in S_R$ and every $t\le T(y)$,

$$\begin{aligned} D_n(\eta^1(y,t),y) &\le t\sqrt{1+\sigma^2}\sum_{\lambda\in\Lambda_1}\theta_{1,\lambda}(y) \quad &&\text{for every } n\in\mathbb{N},\\ |\eta^1_n(y,t)-c_n| &\le \sigma t\sum_{\lambda\in\Lambda_1}\theta_{1,\lambda}(y) &&\text{for every } n\in\mathbb{N},\\ \eta^1_n(y,t) &\le c_n-\sigma t\sum_{\lambda\in\Lambda_1}\theta_{1,\lambda}(y) &&\text{for every } n\ge \widetilde{N}(y). \end{aligned}$$

Moreover, this with (7.2) imply that

$$D_n(\eta^1(y,t),y)\le t\sqrt{1+\sigma^2}<\frac{1}{2}\min\{\delta_{j,\lambda}\mid y\in\mathcal{V}_{j,\lambda}\}\quad\text{for every } n\in\mathbb{N}.$$

Therefore, we can define inductively

$$\eta^j=(\eta^j_0,\eta^j_1,\dots):\{(y,t)\in S_R\times[0,\infty[\,\mid t\le T(y)\}\to\operatorname{graph}\Phi$$

by

$$\eta^j(y,t)=\begin{cases} H_{j,\lambda}(\eta^{j-1}(y,t),\theta_{j,\lambda}(y)t) & \text{if } y\in\overline{\mathcal{V}_{j,\lambda}},\\ \eta^{j-1}(y,t) & \text{if } y\notin\bigcup_{\lambda\in\Lambda_j}\mathcal{V}_{j,\lambda}.\end{cases}$$

This function is continuous in the uniform topology and satisfies for every $y\in S_R$ and every $t\le T(y)$,

$$\begin{aligned} D_n(\eta^j(y,t),y) &\le t\sqrt{1+\sigma^2}\sum_{i=1}^{j}\sum_{\lambda\in\Lambda_i}\theta_{i,\lambda}(y) \quad &&\text{for every } n\in\mathbb{N},\\ |\eta^j_n(y,t)-c_n| &\le \sigma t\sum_{i=1}^{j}\sum_{\lambda\in\Lambda_i}\theta_{i,\lambda}(y) &&\text{for every } n\in\mathbb{N},\\ \eta^j_n(y,t) &\le c_n-\sigma t\sum_{i=1}^{j}\sum_{\lambda\in\Lambda_i}\theta_{i,\lambda}(y) &&\text{for every } n\ge \widetilde{N}(y). \end{aligned}$$

Since the refinement is locally finite, for every $y\in S_R$, there exists a neighbourhood $\mathcal{V}$ of y which intersects a finite number of $\mathcal{V}_{j,\lambda}$. So, there exists $\widehat{k}$ such that

$$\eta^j(x,t)=\eta^{\widehat{k}}(x,t)\quad\text{for every } j\ge\widehat{k}\text{ and every } x\in\mathcal{V}.$$

Therefore, the map $\widehat{\eta}:S_R\times[0,\infty[\,\to\operatorname{graph}\Phi$ defined by

$$\widehat{\eta}(y,t)=\lim_{j\to\infty}\eta^j(y,\min\{t,T(y)\})$$

is continuous in the uniform topology. Also, it verifies for every $y=(u,c)\in S_R$,

$$D_n(\widehat{\eta}(y,t),y)\le t\sqrt{1+\sigma^2}\quad\text{for every } n\in\mathbb{N},\tag{7.3}$$

$$|\widehat{\eta}_n(y,t)-c_n|\le\sigma t\quad\text{for every } n\in\mathbb{N},\tag{7.4}$$

$$\widehat{\eta}_n(y,t)\le c_n-\sigma t\quad\text{for every } n\ge\widetilde{N}(y)\text{ and every } t\le T(y).\tag{7.5}$$

Now, fix $r = R/\sqrt{1+\sigma^2}$. We define inductively $\eta : S \times [0,r] \to \operatorname{graph}\Phi$ by

$$\eta(y,t) = \begin{cases} \widehat{\eta}(y,t) & \text{if } T_0(y) \le t \le T_1(y), \\ \widehat{\eta}(\eta(y,T_1(y)), t - T_1(y)) & \text{if } T_1(y) < t \le T_2(y), \\ \widehat{\eta}(\eta(y,T_2(y)), t - T_2(y)) & \text{if } T_2(y) < t \le T_3(y), \\ \quad\vdots \end{cases}$$

where $T_0(y) = 0$ and

$$T_{h+1}(y) = \min\{r, T_h(y) + T(\eta(y, T_h(y)))\}, \quad h = 0, 1, \ldots. \tag{7.6}$$

Hence, from (7.3), we deduce that for every $h \ge 0$, $k \ge 1$, $t \in \,]T_{h+k-1}(y), T_{h+k}(y)]$, and $y \in S$, we have for every $n \in \mathbb{N}$,

$$\begin{aligned} D_n(\eta(y,t), \eta(y,T_h(y))) &\le D_n(\eta(y,t), \eta(y, T_{h+k-1}(y))) + \ldots + D_n(\eta(y,T_{h+1}(y)), \eta(y,T_h(y))) \\ &\le [(t - T_{h+k-1}(y)) + \ldots + (T_{h+1}(y) - T_h(y))]\sqrt{1+\sigma^2} \\ &\le (t - T_h(y))\sqrt{1+\sigma^2}, \end{aligned} \tag{7.7}$$

and in particular, $D_n(\eta(y,t), y) \le t\sqrt{1+\sigma^2} \le R$; so, $\eta(y,t) \in S_R$. Also, from (7.4), for every $y = (u,c) \in S$, t, and every $n \in \mathbb{N}$, we have

$$|\eta_n(y,t) - c_n| \le \sigma t.$$

We claim that for every $y \in S$, there exists $\widehat{h} \in \mathbb{N}$ such that

$$T_h(y) = r \quad \text{for all } h \ge \widehat{h}. \tag{7.8}$$

If this is not the case, for some $y \in S$,

$$T_h(y) < r \quad \text{for every } h \in \mathbb{N}. \tag{7.9}$$

It follows from (7.7) that $\{\eta(y, T_h(y))\}$ is a Cauchy sequence in S_R. Therefore, there exists $\alpha > 0$ such that

$$T(\eta(y, T_h(y))) \ge \alpha \quad \text{for every } h \ge 0, \tag{7.10}$$

since T is continuous and positive on S_R. Combining (7.6), (7.9) and (7.10), we deduce that for all $h \in \mathbb{N}$,

$$r > T_h(y) = \sum_{i=0}^{h-1} T(\eta(y, T_i(y))) \ge h\alpha,$$

a contradiction. This permits us to conclude that η is well defined and continuous in the uniform topology. Moreover, for every $y \in S$, there exist a neighbourhood $\mathcal{V}$ of y, and $h_y \in \mathbb{N}$ such that

$$T_h(z) = T_{h_y}(z) = r \quad \text{for every } z \in \mathcal{V} \text{ and } h \ge h_y. \tag{7.11}$$

On the other hand, from (7.5), we have for every $y \in S$,

$$\eta_n(y,t) \le c_n - \sigma t \quad \text{for all } n \ge \widetilde{N}(y), \text{ and all } t \le T_1(y);$$

and for all $t \in [T_1(y), T_2(y)]$, and all $n \ge \widetilde{N}(\eta(y, T_1(y)))$,

$$\eta_n(y,t) \le \eta_n(y, T_1(y)) - \sigma(t - T_1(y));$$

so that for all $t \le T_2(y)$, and all $n \ge \max\{\widetilde{N}(y), \widetilde{N}(\eta(y, T_1(y)))\}$,

$$\eta_n(y,t) \le c_n - \sigma t;$$

and hence, for all $t \le r$,

$$\eta_n(y,t) \le c_n - \sigma t \quad \text{for all } n \ge N(y),$$

where $N(y) = \max\{\widetilde{N}(y), \widetilde{N}(\eta(y, T_1(y))), \widetilde{N}(\eta(y, T_2(y))), \dots\}$. From (7.11), we deduce that $N : S \to \mathbb{N}$ is well defined and is locally bounded in S. $\square$

Remark 7.2. An analogous result is also true with $|k\text{-}d\Phi|$.

8. Proof of the Main Theorem

We define $L : E \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R} \cup \{-\infty, \infty\}$ by

$$L(u,c) = \liminf_{n\to\infty} c_n - \overline{c}_n.$$

Proof of the Main Theorem 6.1. If there is no critical value in C, by the Palais–Smale condition there exist $\sigma > 0$, $R > 0$ such that

$$|d\Phi|(y) > \sigma \quad \text{for every } y \in \operatorname{graph}\Phi \cap E \times C + [-2R, 2R].$$

Define $L_0 : A_0 \to [-\infty, \gamma]$ by $L_0 = L|_{A_0}$. Observe that L_0 is continuous in the uniform topology. Fix $\overline{\varepsilon} > 0$ such that

$$\sup_{(u,c)\in A_1} L_0(u,c) < \beta - 2\overline{\varepsilon}.$$

There exists $\varepsilon \in \,]0, \overline{\varepsilon}[$ such that

$$L_0^{-1}\,[\beta - \varepsilon, \infty[\, \subset E \times \{c + r \in \mathbb{R}^{\mathbb{N}} \mid \text{there exists } (u,c) \in A_0 \\ \text{such that } \liminf_{n\to\infty} c_n - \overline{c}_n \ge \beta \text{ and } |r| \le R\},$$

and there exists an Urysohn's function $\lambda_1 : A_0 \to [0,1]$ such that

$$\lambda_1(y) = 0 \text{ for } y \in L_0^{-1}[-\infty, \beta - \varepsilon] \quad \text{and} \quad \lambda_1(y) = 1 \text{ for } y \in L_0^{-1}\,[\beta, \infty[\,.$$

Denote $S = \operatorname{graph}\Phi \cap E \times C + [-R, R]$. Let $r > 0$ and η be given by Theorem 7.1. We define $\eta^1 : A_0 \times [0,1] \to \operatorname{graph}\Phi$ by

$$\eta^1(y,t) = \begin{cases} \eta(y, rt\lambda_1(y)) & \text{if } \lambda_1(y) > 0, \\ y & \text{otherwise,} \end{cases}$$

so that $\eta^1 \in \mathcal{N}(A_0, A_1)$. We have $L(\eta^1(y,t)) < \gamma$ for every $y \in A_0$ and every $t \in]0,1]$. Therefore, $\eta^1(A_0,]0,1]) \cap Q_1 = \emptyset$, and hence, by definition of linking,

$$\eta^1(A_0, 1) \cap Q_0 \neq \emptyset. \tag{8.1}$$

Define $L_1 : A_0 \to [-\infty, \gamma]$ by $L_1 = L \circ \eta^1(\cdot, 1)$. We have $L_1(y) = L_0(y) - \sigma r \lambda_1(y)$. So,

$$L_1(y) \leq \gamma - \sigma r \quad \text{or} \quad L_1(y) < \beta. \tag{8.2}$$

If $\beta > \gamma - \sigma r$, (6.1), (8.1) and (8.2) lead to a contradiction.

On the other hand, if $\gamma - \beta \geq \sigma r$, from the properties of η, we deduce that

$$\eta^1(y, 1) \in S \quad \text{for every } y \in L_1^{-1}\,[\beta - \varepsilon, \infty[.$$

Again, there exists an Urysohn's function $\lambda_2 : A_0 \to [0,1]$ such that $\lambda_2(y) = 0$ for $y \in L_1^{-1}[-\infty, \beta - \varepsilon]$ and $\lambda_2(y) = 1$ for $y \in L_1^{-1}\,[\beta, \infty[$.

We define $\eta^2 : A_0 \times [0,1] \to \operatorname{graph} \Phi$ by

$$\eta^2(y,t) = \begin{cases} \eta(\eta^1(y,1), rt\lambda_2(y)) & \text{if } \lambda_2(y) > 0, \\ \eta^1(y,1) & \text{otherwise,} \end{cases}$$

We have $\eta^2 \in \mathcal{N}(A_0, A_1)$, $\eta^2(A_0,]0,1]) \cap Q_1 = \emptyset$, and

$$\eta^2(A_0, 1) \cap Q_0 \neq \emptyset. \tag{8.3}$$

We define $L_2 : A_0 \to [-\infty, \gamma]$ by $L_2 = L \circ \eta^2(\,\cdot\,, 1)$. As before, $L_2(y) = L_1(y) - \sigma r \lambda_2(y)$, and

$$L_2(y) \leq \gamma - 2\sigma r \quad \text{or} \quad L_2(y) < \beta. \tag{8.4}$$

If $\beta > \gamma - 2\sigma r$, (6.1), (8.3) and (8.4) lead to a contradiction. On the other hand, if $\gamma - \beta \geq 2\sigma r$, we deduce that

$$\eta^2(y, 1) \in S \quad \text{for every } y \in L_2^{-1}\,[\beta - \varepsilon, \infty[.$$

In doing this argument k times with $\gamma - k\sigma r < \beta$, we get a contradiction. □

Proof of Theorem 6.3. As before, let us define $L_0 : A_0 \to [-\infty, \gamma]$ by $L_0 = L|_{A_0}$. Fix $\varepsilon_1 > 0$ such that $L_0(A_1) < \beta - 2\varepsilon_1$. There exists $n_1 \in \mathbb{N}$ such that

$$L_0^{-1}\,[\beta - \varepsilon_1, \infty[\subset E \times \widetilde{C_1} = E \times (\overline{c} + \mathbb{R}^{n_1} \times [\beta - 2\varepsilon_1, \infty[\times [\beta - 2\varepsilon_1, \infty[\times \ldots).$$

Indeed, for every $y \in L_0^{-1}\,[\beta - \varepsilon_1, \infty[$, there exists n_y such that

$$y \in \mathcal{O}_y = \operatorname{graph} \Phi \cap E \times (\overline{c} + \mathbb{R}^{n_y} \times]\beta - 2\varepsilon_1, \infty[\times]\beta - 2\varepsilon_1, \infty[\times \ldots).$$

So $\{\mathcal{O}_y\}$ is an open cover of $L_0^{-1}\,[\beta - \varepsilon_1, \infty[$ in the uniform topology. Since this set is compact, $L_0^{-1}\,[\beta - \varepsilon_1, \infty[\subset \bigcup_{i=1}^k \mathcal{O}_{y_i}$. Set $n_1 = \max\{n_{y_1}, \ldots, n_{y_k}\}$.

Let us denote $C_1 = C \cap \widetilde{C_1}$. If there is no critical value in C_1, by the Palais–Smale condition there exist $\sigma > 0$ and $R > 0$ such that

$$|d\Phi|(y) > \sigma \quad \text{for every } y \in \operatorname{graph}\Phi \cap E \times C_1 + [-2R, 2R].$$

There exists $\varepsilon \in \,]0, \varepsilon_1[$ such that

$$L_0^{-1}\,[\beta - \varepsilon, \infty[\, \subset E \times \widetilde{C_1} \cap \{c + r \in \mathbb{R}^{\mathbb{N}} \mid \text{there exists } (u, c) \in A_0 \text{ such that } \liminf_{n\to\infty} c_n - \overline{c}_n \geq \beta \text{ and } |r| \leq R\}.$$

Denote $S = \operatorname{graph}\Phi \cap E \times C_1 + [-R, R]$, and let $r > 0$ and $\eta = (\eta_0, \eta_1, \ldots)$ be given by Theorem 7.1.

We proceed as in the previous proof except that we define $\eta^1 = (\eta_0^1, \eta_1^1, \ldots) : A_0 \times [0,1] \to \operatorname{graph}\Phi$ by

$$\eta_n^1(y,t) = \begin{cases} \eta_n(\eta^0(y,1), rt\lambda_1(y)) & \text{if } \lambda_1(y) > 0, \\ \eta_n^0(y,1) & \text{otherwise,} \end{cases}$$

for $n = 0, \ldots, n_1 - 1$, and for $n \geq n_1$

$$\eta_n^1(y,t) = \begin{cases} \max\{\eta_n(\eta^0(y,1), rt\lambda_1(y)), \overline{c}_n + \beta - 2\varepsilon_1\} & \text{if } \lambda_1(y) > 0, \\ \theta_1(y)\max\{\eta_n^0(y,1), \overline{c}_n + \beta - 2\varepsilon_1\} & \\ +(1 - \theta_1(y))\eta_n^0(y,1) & \text{otherwise,} \end{cases}$$

where $\eta^0(y,1) = y$, and $\theta_1 : A_0 \to [0,1]$ is an Urysohn's function such that $\theta_1(x) = 1$ on $L_0^{-1}\,[\beta - \varepsilon, \infty[$ and $\theta_1(x) = 0$ on $L_0^{-1}\,]-\infty, \beta - 2\varepsilon]$. This insures that if $\gamma - \beta \geq \sigma r$

$$\eta^1(y,1) \in S \quad \text{for every } y \in L_1^{-1}\,[\beta - \varepsilon, \infty[\,,$$

where $L_1 : A_0 \to [-\infty, \gamma]$ is defined by $L_1 = L \circ \eta^1(\,\cdot\,, 1)$. We have

$$L_1(y) = L_0(y) - \theta_1(y)\min\{\sigma r\lambda_1(y), L_0(y) - (\beta - 2\varepsilon)\}.$$

So,

$$L_1(y) \leq \gamma - \sigma r \quad \text{or} \quad L_1(y) < \beta.$$

Therefore, $\eta^1 \in \mathcal{N}(A_0, A_1)$, $\eta^1(A_0,]0,1]) \cap Q_1 = \emptyset$, and $\eta^1(A_0, 1) \cap Q_0 \neq \emptyset$. As before, if $\beta > \gamma - \sigma r$, we get a contradiction. Otherwise we repeat this argument a finite number of times until we get a contradiction.

Therefore, Φ has a critical point at level $c^1 \in C_1$.

Now, take $\varepsilon_2 \in \,]0, \varepsilon_1[$. One can find

$$\widetilde{C_2} = \overline{c} + \mathbb{R}^{n_1} \times ([\beta - 2\varepsilon_1, \infty[)^{n_2} \times [\beta - 2\varepsilon_2, \infty[\, \times [\beta - 2\varepsilon_2, \infty[\, \times \ldots,$$

such that

$$L_0^{-1}\,[\beta - \varepsilon_2, \infty[\, \subset E \times \widetilde{C_2}.$$

The same argument as before yields to the existence of u_2 a critical point of Φ at level $c^2 \in C_2 = C \cap \widetilde{C_2}$. We continue this process with $\{\varepsilon_k\}$ a decreasing sequence converging to 0 in order to get a sequence of sets

$$C \supset C_1 \supset C_2 \supset \ldots$$

and a sequence $\{u_k\}$ such that u_k is a critical point of Φ at level $c^k \in C_k$. From the Palais–Smale condition, we deduce the existence of u a critical point of Φ at some level c with

$$c \in \bigcap_{n\in\mathbb{N}} C_n,$$

and hence, $\liminf_{n\to\infty}(c_n - \overline{c}_n) \geq \beta$. □

The proof of Theorem 6.4 is similar to the proof of the Main Theorem 6.1.

9. Applications

In this section, we present simple applications of our results to nonlinear partial differential equations on $\mathbb{R}^N$.

Let us consider the problem

$$-\Delta u(x) + a(x)u(x) = g(x, u(x)), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{9.1}$$

We look for $u \in H^1_{\text{loc}}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \nabla u \nabla w + a(x)uw - g(x,u)w\,dx = 0 \quad \text{for every } w \in C_c^\infty(\mathbb{R}^N).$$

We assume

(H1) $a \in C(\mathbb{R}^N,]0,\infty[)$,

(H2) $g : \mathbb{R}^{N+1} \to \mathbb{R}$ is such that $x \mapsto g(x,u)$ is measurable for every $u \in \mathbb{R}$, $u \mapsto g(x,u)$ is continuous for almost every $x \in \mathbb{R}^N$, and for every $\varepsilon > 0$, there exists $a_\varepsilon \in L^{2N/(N+2)}_{\text{loc}}(\mathbb{R}^N)$, such that

$$|g(x,u)| \leq a_\varepsilon(x) + \varepsilon |u|^{N-2/(N+2)},$$

(H3) (1) there exist $\theta < 2$, $a_3 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, $a_4 \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that

$$G(x,u) \leq a_3(x)|u|^\theta + a_4(x) \quad \text{for every } u \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N,$$

or

(2) there exist $\beta > 2 > \theta \geq 0$, $R \in L^{2N/(N-2)}_{\text{loc}}(\mathbb{R}^N)$, $a_3 \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, and $a_4 \in L^1_{\text{loc}}(\mathbb{R}^N)$ such that

$$\beta G(x,u) - g(x,u)u \leq a_3(x)|u|^\theta + a_4(x) \quad \text{for every } |u| \geq R(x) \text{ and a.e. } x \in \mathbb{R}^N,$$

where

$$G(x,u)=\int_0^u g(x,s)\,ds.$$

Let $p=2N/(N-2)$. We define $\Phi : L^p_{\mathrm{loc}}(\mathbb{R}^N)\to\mathbb{R}^{\mathbb{N}}\cup\{\infty\}$ by

$$\Phi(u)=\begin{cases}\{c\in\mathbb{R}^{\mathbb{N}}\mid c_n\geq\phi_n(u)\text{ for all }n\in\mathbb{N}\} & \text{if } u\in H^1_{\mathrm{loc}}(\mathbb{R}^N),\\ \infty & \text{otherwise,}\end{cases}$$

where

$$\phi_n(u)=\int_{B_n}\frac{|\nabla u|^2}{2}+\frac{a(x)u^2}{2}-G(x,u)\,dx,$$

and $B_n=\{x\in\mathbb{R}^N\mid \|x\|\leq n\}$. It is clear that Φ has closed graph.

The following result gives some information on the slope of Φ.

Proposition 9.1. *Assume that* (H1) *and* (H2) *hold. Let* $(u,c)\in\operatorname{graph}\Phi$ *be such that* $|d\Phi|(u,c)<\infty$, *then there exists* $\alpha\in L^q(\mathbb{R}^N)$ (*with* $1/p+1/q=1$) *such that for every* $n\in\mathbb{N}$,

$$\int_{B_n}\alpha w\,dx=\int_{B_n}\nabla u\nabla w+a(x)uw-g(x,u)w\,dx\quad\text{for every } w\in H^1(B_n).$$

Moreover, $\|\alpha\|_{L^q(\mathbb{R}^N)}\leq|d\Phi|(u,c)$.

Proof. If, for every $n\in\mathbb{N}$,

$$\sup_{\substack{w\in C^\infty(B_n)\\ \|w\|_{L^p(B_n)}\leq 1}}\left\{\int_{B_n}\nabla u\nabla w+a(x)uw-g(x,u)w\,dx\right\}\leq|d\Phi|(u,c),\tag{9.2}$$

then, for every $n\in\mathbb{N}$, the functional defined on $C^\infty(B_n)$ by

$$w\mapsto\int_{B_n}\nabla u\nabla w+a(x)uw-g(x,u)w\,dx,$$

admits a continuous extension f_n on $L^p(B_n)$. By Riesz's Theorem, there exists $\alpha_n\in L^q(B_n)$ such that $f_n(w)=\int_{B_n}\alpha_n w\,dx$ for every $w\in L^p(B_n)$. In particular,

$$\int_{B_n}\alpha_n w=\int_{B_n}\nabla u\nabla w+a(x)uw-g(x,u)w\,dx\quad\text{for every } w\in C^\infty(B_n).$$

Observe that $\int_{B_n}(\alpha_m-\alpha_n)w\,dx=0$ for every $w\in C^\infty_c(B_n)$ and every $m\geq n$; so, $\alpha_m=\alpha_n$ a.e. on B_n. Hence, there exists $\alpha\in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying for every $n\in\mathbb{N}$

$$\int_{B_n}\alpha w\,dx=\int_{B_n}\nabla u\nabla w+a(x)uw-g(x,u)w\,dx\quad\text{for every } w\in H^1(B_n).$$

Moreover, $\|\alpha\|_{L^q(B_n)}\leq|d\Phi|(u,c)$ for every $n\in\mathbb{N}$, so,

$$\alpha\in L^q(\mathbb{R}^N)\quad\text{and}\quad\|\alpha\|_{L^q(\mathbb{R}^N)}\leq|d\Phi|(u,c).$$

Now, suppose that (9.2) is false. Let n_0 be the infimum of $n \in \mathbb{N}$ such that

$$|d\Phi|(u,c) < \sup_{\substack{w \in C^\infty(B_n) \\ \|w\|_{L^p(B_n)} \le 1}} \left\{ \int_{B_n} \nabla u \nabla w + a(x)uw - g(x,u)w \, dx \right\}.$$

Fix $\rho \in \mathbb{R}$ such that

$$|d\Phi|(u,c) < \rho \le \sup_{\substack{w \in C^\infty(B_{n_0}) \\ \|w\|_{L^p(B_{n_0})} \le 1}} \left\{ \int_{B_{n_0}} \nabla u \nabla w + a(x)uw - g(x,u)w \, dx \right\},$$

and choose $\varepsilon > 0$ such that

$$|d\Phi|(u,c) < \rho - 6\varepsilon. \tag{9.3}$$

There exists $w_0 \in C_c^\infty(B_{n_0})$ such that $\|w_0\|_{L^p(B_{n_0})} \le 1$, and

$$\rho - \frac{\varepsilon}{2} < \int_{B_{n_0}} \nabla u \nabla w_0 + a(x)uw_0 - g(x,u)w_0 \, dx. \tag{9.4}$$

Let us recall that

$$-|d\Phi|(u,c) \le \int_{B_n} \nabla u \nabla w_0 + a(x)uw_0 - g(x,u)w_0 \, dx \quad \text{for all } n < n_0. \tag{9.5}$$

We can find $\delta_1 > 0$ such that for every $(v,b) \in B_{n_0}((u,c),\delta_1)$, we have

$$\int_{B_{n_0}} \nabla v \nabla w_0 + a(x)vw_0 - g(x,v)w_0 \, dx > \rho - \varepsilon, \tag{9.6}$$

$$\int_{B_n} \nabla v \nabla w_0 + a(x)vw_0 - g(x,v)w_0 \, dx > -|d\Phi|(u,c) - \varepsilon \tag{9.7}$$

for all $n < n_0$. If (9.6) is false, there exists a sequence $\{(v_k,b_k)\}$ such that $D_{n_0}((v_k,b_k),(u,c)) < 1/k$ and

$$\int_{B_{n_0}} \nabla v_k \nabla w_0 + a(x)v_k w_0 - g(x,v_k)w_0 \, dx \le \rho - \varepsilon. \tag{9.8}$$

Since

$$\phi_{n_0}(v_k) = \int_{B_{n_0}} \frac{|\nabla v_k|^2}{2} + \frac{a(x)v_k^2}{2} - G(x,v_k) \, dx \le b_{n_0,k},$$

$b_{k,n_0} \to c_{n_0}$, and $v_k \to u$ in $L^p(B_{n_0})$ as $k \to \infty$, we deduce that $\{v_k\}$ is bounded in $H^1(B_{n_0})$, and hence has a subsequence converging weakly to u. This with (9.4) and (9.8) lead to a contradiction. The inequality (9.7) can be proved by a similar argument. On the other hand, there exists $\delta_2 > 0$ such that for every $(v,b) \in B_{n_0}((u,c),\delta_2)$, $0 \le t \le \delta_2$, and every $n \le n_0$,

$$\int_{B_n} \left[\int_{v - tw_0}^{v} g(x,s) \, ds \right] - tg(x,v)w_0 \, dx \le t\varepsilon. \tag{9.9}$$

Indeed, otherwise, we could find $n_1 \le n_0$, and sequences $\{(v_k, b_k)\}$, $\{t_k\}$ such that $(v_k, b_k) \in B_{n_0}((u,c), 1/k)$, $t_k \in]0, 1/k[$, and

$$\int_{B_{n_1}} \left[\frac{1}{t_k}\int_{v_k - t_k w_0}^{v_k} g(x,s)\,ds\right] - g(x, v_k) w_0\,dx > \varepsilon. \tag{9.10}$$

Since $v_k \to u$ in $L^p(B_{n_0})$, without lost of generality, $v_k(x) \to u(x)$ a.e. on B_{n_0}. It follows from the mean value Theorem that

$$\frac{1}{t_k}\int_{v_k(x) - t_k w_0(x)}^{v_k(x)} g(x,s)\,ds - g(x, v_k(x)) w_0(x) \to 0 \quad \text{a.e. } x \in B_{n_0}.$$

The Lebesgue dominated convergence Theorem leads to a contradiction.

Finally, we set $\delta = \min\{\delta_1, \delta_2, 2\varepsilon/\zeta\}$ with $\zeta = \int_{B_{n_0}} (|\nabla w_0|^2 + a(x) w_0^2)\,dx$, and we define

$$H : B_{n_0}((u,c),\delta) \times [0,\delta] \to \operatorname{graph}\Phi$$

by

$$H((v,b),t) = (v - t w_0, b - t(M - 3\varepsilon)),$$

where $M \in \mathbb{R}^{\mathbb{N}}$ is given by

$$M_n = \begin{cases} -|d\Phi|(u,c) & \text{if } n < n_0, \\ \rho & \text{otherwise.} \end{cases}$$

Using (9.6), (9.7) and (9.9), we obtain

$$\begin{aligned} \phi_n(v - t w_0) &= \phi_n(v) + \frac{t^2}{2}\int_{B_n} |\nabla w_0|^2 + a(x) w_0^2\,dx \\ &\quad - t\left[\int_{B_n} \nabla v \nabla w_0 + a(x) v w_0 - g(x,v) w_0\,dx\right] \\ &\quad + \int_{B_n} \left[\int_{v - t w_0}^{v} g(x,s)\,ds\right] - t g(x,v) w_0\,dx \\ &\le \phi_n(v) - t\left[\int_{B_n} \nabla v \nabla w_0 + a(x) v w_0 - g(x,v) w_0\,dx\right] + t\varepsilon + t\varepsilon \\ &\le b_n - t(M_n - 3\varepsilon). \end{aligned}$$

So H is well defined. Obviously, H is continuous when $\operatorname{graph}\Phi$ is endowed with $\mathcal{T}_u$.

Observe that $\|v - t w_0 - v\|_{L^p(B_n)} \le t$ and $|M_n - 3\varepsilon| \le \rho - 3\varepsilon$ for every $n \in \mathbb{N}$ by (9.3). Hence, $|d\Phi|(u,c) \ge \rho - 3\varepsilon$, a contradiction. □

Now, we establish the Palais–Smale condition.

Proposition 9.2. *Assume that* (H1)–(H3) *hold. Then for every bounded subset $C \subset \mathbb{R}^{\mathbb{N}}$, Φ satisfies* $(\mathrm{PS})_C$.

Proof. Let $\{(u_k, c_k)\}$ be a sequence such that $|d\Phi|(u_k, c_k) \to 0$ and

$$(u_k, c_k) \in \operatorname{graph}\Phi \cap L^p_{\mathrm{loc}}(\mathbb{R}^N) \times C + [-r_k, r_k] \quad \text{with } r_k \to 0.$$

Since C is bounded, we can find $M \in \mathbb{R}^{\mathbb{N}}$ such that $c_k \in [-M, M]$ for every $k \in \mathbb{N}$.

From the previous proposition, for every $k \in \mathbb{N}$, there exists $\alpha_k \in L^q(\mathbb{R}^N)$ such that

$$\|\alpha_k\|_{L^q(\mathbb{R}^N)} \leq |d\Phi|(u_k, c_k),$$

and

$$\int_{B_n} \alpha_k w\, dx = \int_{B_n} \nabla u_k \nabla w + a(x) u_k w - g(x, u_k) w\, dx$$

for all $w \in H^1(B_n)$ and all $n \in \mathbb{N}$. If (H3)(1) is satisfied, fix $\gamma = 0$ and $R \equiv 0$; and fix $\gamma = 1/\beta$ if (H3)(2) holds. We deduce that, for every $n \in \mathbb{N}$, there exist $\gamma_1, \gamma_2 \geq 0$ such that for every $k \in \mathbb{N}$,

$$\begin{aligned}
M_n \geq c_{k,n} &\geq \phi_n(u_k) - \gamma \int_{B_n} \alpha_k u_k\, dx + \gamma \int_{B_n} \alpha_k u_k\, dx \\
&\geq \left(\frac{1}{2} - \gamma\right) \int_{B_n} |\nabla u_k|^2 + a(x) u_k^2\, dx - \gamma \|\alpha_k\|_{L^q(B_n)} \|u_k\|_{L^p(B_n)} \\
&\quad + \int_{B_n} \gamma g(x, u_k) u_k - G(x, u_k)\, dx \\
&\geq \left(\frac{1}{2} - \gamma\right) \int_{B_n} |\nabla u_k|^2 + a(x) u_k^2\, dx - \gamma |d\Phi|(u_k, c_k) \|u_k\|_{L^p(B_n)} \\
&\quad + \left[\int_{B_n \cap \{x \mid u_k(x) \leq R(x)\}} + \int_{B_n \cap \{x \mid u_k(x) > R(x)\}}\right] \gamma g(x, u_k) u_k - G(x, u_k)\, dx \\
&\geq \left(\frac{1}{2} - \gamma\right) \int_{B_n} |\nabla u_k|^2 + a(x) u_k^2\, dx - \gamma_1 \|u_k\|_{L^p(B_n)}^{\widetilde{\theta}} - \gamma_2,
\end{aligned}$$

with $\widetilde{\theta} = \max\{1, \theta\}$. Hence, for every $n \in \mathbb{N}$, $\{u_k\}$ is bounded in $H^1(B_n)$. So, without lost of generality, we can assume that $\{u_k\}$ is pointwise convergent almost everywhere to some $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$.

For $\varepsilon > 0$, define $g_\varepsilon : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ by

$$g_\varepsilon(x, s) = \begin{cases} g(x, s) & \text{if } |g(x, s)| \leq a_\varepsilon(x), \\ a_\varepsilon(x) & \text{if } g(x, s) > a_\varepsilon(x), \\ -a_\varepsilon(x) & \text{if } g(x, s) < -a_\varepsilon(x), \end{cases}$$

where a_ε is given in (H2). Observe that

$$\begin{aligned}
|g(x, u(x)) - g(x, u_k(x))| &\leq |g(x, u(x)) - g_\varepsilon(x, u(x))| \\
&\quad + |g_\varepsilon(x, u(x)) - g_\varepsilon(x, u_k(x))| + |g_\varepsilon(x, u_k(x)) - g(x, u_k(x))|.
\end{aligned}$$

Since

$$|g_\varepsilon(x, u(x)) - g_\varepsilon(x, u_k(x))| \leq 2a_\varepsilon(x),$$

we deduce that for every $n \in \mathbb{N}$,

$$\|g_\varepsilon(\,\cdot\,, u) - g_\varepsilon(\,\cdot\,, u_k)\|_{L^{2N/(N+2)}(B_n)} \to 0$$

from Lebesgue's Theorem. On the other hand, for every $n \in \mathbb{N}$,

$$\|g(\,\cdot\,,u)-g_\varepsilon(\,\cdot\,,u)\|_{L^{2N/(N+2)}(B_n)} \le \varepsilon\|u\|^{p-1}_{L^p(B_n)},$$
$$\|g(\,\cdot\,,u_k)-g_\varepsilon(\,\cdot\,,u_k)\|_{L^{2N/(N+2)}(B_n)} \le \varepsilon\|u_k\|^{p-1}_{L^p(B_n)}.$$

Therefore, it follows that for every $n \in \mathbb{N}$,

$$\|g(\,\cdot\,,u)-g(\,\cdot\,,u_k)\|_{L^{2N/(N+2)}(B_n)} \to 0 \quad \text{as } k \to 0.$$

Now, using standard arguments, we deduce the existence of a subsequence of $\{u_k\}$ converging in $H^1(B_n)$ and hence, in $L^p(B_n)$. So, there exists a convergent subsequence $\{u_{k_l}\}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$. Finally, $[-M,M]$ is compact in $\mathbb{R}^{\mathbb{N}}$. Therefore, $\{(u_{k_l},c_{k_l})\}$ has a convergent subsequence in graph Φ. □

We start with the following example. Consider the problem

$$-\Delta u + a(x)u = a(x)u^{1/3} + g(x,u), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{9.11}$$

Theorem 9.3. *Assume* (H1), (H2)*, and there exists $k \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ such that $k - G(\,\cdot\,,1) \in L^1(\mathbb{R}^N)$ and*

$$G(x,u) \le k(x) \quad \textit{for all } u \in \mathbb{R} \textit{ and } x \in \mathbb{R}^N.$$

Then the problem (9.11) *has a solution. Moreover, if either $g(x,0) \not\equiv 0$ or $\|k-G(\,\cdot\,,1)\|_{L^1(\mathbb{R}^N)} < \|(4k+a)/8\|_{L^1(B_n)}$ for some n, then the solution is non trivial.*

Proof. Fix $\overline{c}_n = -\|k + a/4\|_{L^1(B_n)}$. For every $u \in H^1_{\mathrm{loc}}(\mathbb{R}^N)$,

$$\phi_n(u) = \int_{B_n} \frac{|\nabla u|^2}{2} + a(x)\left(\frac{u^2}{2} - \frac{3u^{4/3}}{4}\right) - G(x,u)\,dx \ge \overline{c}_n.$$

On the other hand, take $u_0 \equiv 1$. We have

$$\phi_n(u_0) = \overline{c}_n + \int_{B_n} k(x) - G(x,1)\,dx \le \overline{c}_n + \|k - G(\,\cdot\,,1)\|_{L^1(\mathbb{R}^N)} = \overline{c}_n + r.$$

Corollary 6.5 and Proposition 9.2 imply that Φ has a critical point at level c with $\liminf_{n\to\infty} c_n - \overline{c}_n \le 2r$. In particular, if $2r < \|k + a/4\|_{L^1(B_n)}$ for some n,

$$2r < \liminf_{n\to\infty} \phi_n(0) - \overline{c}_n \le \liminf_{n\to\infty} b_n - \overline{c}_n \text{ for all } b \in \mathbb{R}^{\mathbb{N}} \text{ such that } (0,b) \in \text{graph}\,\Phi.$$

So, the critical point is non trivial. □

In the next result, we establish the existence of a solution to problem (9.1).

From (H1), we deduce that for every bounded domain $\Omega \in \mathbb{R}^N$, $H^1(\Omega)$ can be endowed with the norm

$$\|u\|_{H^1(\Omega)} = \int_\Omega \frac{|\nabla u|^2}{2} + \frac{a(x)u^2}{2}\,dx,$$

which is equivalent to the usual one. Let $p = 2^*$. For every $n \in \mathbb{N}$, let $a_n > 0$ be the constant obtained by the continuous imbedding $H^1(B_n) \to L^p(B_n)$; that is

$$a_n \|u\|^2_{L^p(B_n)} \le \|u\|^2_{H^1(B_n)}.$$

THEOREM 9.4. *Assume* (H1)–(H3) *and*

(H4) *there exist* $l \in L^1(\mathbb{R}^N)$, *and a sequence of positive numbers* $\{k_n\}$ *such that*

$$\inf_{n\in\mathbb{N}} \frac{a_n}{k_n} > 0 \quad \text{and} \quad G(x,u) \le l(x) + k_n|u|^p \quad \text{for all } u \in \mathbb{R} \text{ and a.e. } x \in B_n.$$

If $\|l\|_{L^1(\mathbb{R}^N)}$ *is sufficiently small, the problem* (9.1) *has a solution.*

PROOF. Denote

$$\alpha = \inf_{n\in\mathbb{N}} \frac{4a_n^p}{p^p k_n^2} > 0.$$

If l is such that $\|l\|_{L^1(\mathbb{R}^N)} < (p-2)\alpha^{1/(p-2)}$, fix $\varepsilon > 0$ such that

$$\xi = \frac{(p-2)\alpha^{1/(p-2)}}{(1+\varepsilon)^2} > \|l\|_{L^1(\mathbb{R}^N)}.$$

For $n \in \mathbb{N}$, take

$$r_n = \left(\frac{2a_n}{pk_n}\right)^{1/(p-2)},$$

and set

$$\delta = \left(\frac{\varepsilon}{1+\varepsilon}\right) \inf_{n\in\mathbb{N}} r_n.$$

Denote

$$\begin{aligned} Q_0 &= \operatorname{graph}\Phi \cap (\{u \in L^p_{\mathrm{loc}}(\mathbb{R}^N) \mid \|u\|_{L^p(B_n)} \le r_n \text{ for all } n \in \mathbb{N}\} \times \mathbb{R}^{\mathbb{N}}), \\ Q_1 &= Q_0 \cap (\{u \in L^p_{\mathrm{loc}}(\mathbb{R}^N) \mid \inf_{n\in\mathbb{N}} (r_n - \|u\|_{L^p(B_n)}) = 0\} \times \mathbb{R}^{\mathbb{N}}). \end{aligned}$$

For $(u,c) \in Q_1$, there exists $n \in \mathbb{N}$ such that $\|u\|_{L^p(B_n)} \ge r_n - \delta$. Thus,

$$\begin{aligned} c_n \ge \phi_n(u) &= \int_{B_n} \frac{|\nabla u|^2}{2} + \frac{a(x)u^2}{2} - G(x,u)\,dx \\ &\ge a_n\|u\|^2_{L^p(B_n)} - k_n\|u\|^p_{L^p(B_n)} - \|l\|_{L^1(B_n)} \ge \xi - \|l\|_{L^1(B_n)}, \end{aligned}$$

and, for $m \ge n$, $c_m \ge \phi_m(u) \ge \xi - \|l\|_{L^1(B_m)}$. So, $\liminf_{m\to\infty} c_m \ge \xi - \|l\|_{L^1(\mathbb{R}^N)}$.

Similarly, for every $(u,c) \in Q_0$, $\liminf_{m\to\infty} c_m \ge -\|l\|_{L^1(\mathbb{R}^N)}$.

It is easy to see that $(\{(0,(0,0,\dots))\},\emptyset)$ links (Q_0,Q_1). From Theorem 6.1 and Proposition 9.2, we deduce that ϕ has a critical point, and hence the problem (9.1) has a solution by Proposition 9.1. □

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