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MORSE THEORY APPLIED TO A T^2-EQUIVARIANT PROBLEM

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Abstract. The following T^2-equivariant problem of periodic type is considered:

$$(\mathrm{P})\qquad \begin{cases} u \in C^2(\mathbb{R}^2,\mathbb{R}), & \\ -\varepsilon\Delta u(x,y) + F'(u(x,y)) = 0 & \text{in } \mathbb{R}^2, \\ u(x,y) = u(x+T,y) = u(x,y+S) & \text{for all } (x,y)\in\mathbb{R}^2, \\ \nabla u(x,y) = \nabla u(x+T,y) = \nabla u(x,y+S) & \text{for all } (x,y)\in\mathbb{R}^2. \end{cases}$$

Using a suitable version of Morse theory for equivariant problems, it is proved that an arbitrarily great number of orbits of solutions to (P) is founded, choosing $\varepsilon > 0$ suitably small. Each orbit is homeomorphic to S^1 or to T^2.

1. Introduction

In this paper Morse theory is applied in order to look for results concerning an equivariant problem, that is in the presence of a group G that acts in such a way to "product", for each solution u to the problem, a whole orbit $G \cdot u$ of solutions. In other words such problem, once expressed in variational form, requires the study of a G-invariant functional. There is a wide literature regarding this kind of problems. We just quote Bott, Benci–Pacella, Mawhin–Willem, Mercuri–Palmieri, among people that studied this argument using Morse theory.

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In the second section some preliminary notions are given, regarding classical Morse theory and a generalization of it, together with a brief introduction to group actions.

Third section deals with the following problem of periodic type

$$
\text{(P)}\qquad \begin{cases} u \in C^2(\mathbb{R}^2,\mathbb{R}), \\ -\varepsilon\Delta u(x,y) + F'(u(x,y)) = 0 & \text{in } \mathbb{R}^2, \\ u(x,y) = u(x+T,y) = u(x,y+S) & \text{for all } (x,y)\in\mathbb{R}^2, \\ \nabla u(x,y) = \nabla u(x+T,y) = \nabla u(x,y+S) & \text{for all } (x,y)\in\mathbb{R}^2, \end{cases}
$$

where $\varepsilon > 0$ is a fixed real number, $T, S > 0$ are two fixed periods and $F : \mathbb{R} \to \mathbb{R}$ is a coercive function which has exactly k maximum points ($k \geq 1$).

Denoting by Ω the set $[0,T]\times[0,S]$ and by $H^1_{S,T}$ the closure of the set

$$
\{u \in C^1(\Omega) \mid u(0,y) = u(T,y),\ u(x,0) = u(x,S) \text{ for all } (x,y)\in\Omega\}
$$

under the norm of $H^1(\Omega)$, solutions to (P) are critical points of the functional

$$
A_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \int_\Omega F(u(x,y))\,dx\,dy
$$

defined on the space $H^1_{S,T}$ on which the compact Lie group $G = \mathbb{R}^2/(\mathbb{Z}T \times \mathbb{Z}S)$ (isomorphic to the torus $T^2 = S^1 \times S^1$) acts. It is proved that A_ε is a G-invariant functional, so that if u is a critical point of A_ε (i.e. a solution to (P)), then the orbit of u is a connected manifold of critical points which is homeomorphic to S^1 or to T^2. Consequently such critical points are degenerate (as they are not isolated), and classical Morse theory cannot be applied to this situation. However, using a Morse theory for equivariant problems the following result is obtained.

Theorem 1.1. *There exists a decreasing sequence $(\mu_i)_{i\in\mathbb{N}}$ which tends to* 0 *such that if $\varepsilon \in \,]\mu_i, \mu_{i-1}[$, then there are at least k_i orbits of solutions. Namely, there are at least k_i critical orbits having Morse index in $\{h, h+1\}$, for each $h = 0, 2, \ldots, 2(i-1)$.*

2. Preliminaries

2.1. Recalls about classical Morse theory. Let f be a C^1-functional defined in an open subset U of a Hilbert space V. Some basic definitions are now recalled

Definition 2.1. A *critical point* of f is an element $x \in U$ such that

$$
df(x) = 0.
$$

The set of critical points of f will be denoted by K_f. A *critical value* of f is a real number c such that

$$\{x \in K_f \mid f(x) = c\} \neq \emptyset.$$

If c is not a critical value, it is called *regular value* for f. Moreover, f verifies the Palais–Smale condition (or briefly, f verifies (PS)) if any sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} f(x_n) = c \quad \text{and} \quad \lim_{n\to\infty} df(x_n) = 0$$

has a subsequence which converges to $x \in U$.

If f is a C^2-functional and $x \in K_f$, the Morse index of x is the maximal dimension of a subspace of V on which $d^2 f(x)$ is negative definite and it is denoted by $m(x)$. The nullity of x is the dimension of the kernel of $d^2 f(x)$ (i.e. the subspace consisting of all y such that $d^2 f(x)(y, z) = 0$ for all $z \in V$). The large Morse index of x is the sum of the Morse index and the nullity and it is denoted by $m^*(x)$. A critical point x is called nondegenerate if its nullity is 0, while in the other case it is called degenerate.

Remark 2.2. If a critical point is nondegenerate, then it is isolated.

Definition 2.3. A functional $f : U \to \mathbb{R}$ is said a *Morse functional* if

(a) f is a C^1-functional and it is of class C^2 in a neighbourhood of its critical points,
(b) all the critical points of f are nondegenerate,
(c) f verifies (PS),
(d) f can be extended to a functional of class C^1 in a neighbourhood of $\overline{U}$.

The following definition introduces the first term of the Morse relation

Definition 2.4. If f is a Morse functional, the *Morse polynomial* of a subset K of K_f is the following formal series in the variable λ

$$m_\lambda(K, f) = \sum_{x\in K} \lambda^{m(x)}$$

with the convention that $\lambda^\infty = 0$.

The following Morse relation is the most famous theorem of classical Morse theory.

Theorem 2.5. *Let $a, b \in \mathbb{R}$, $a < b$ be two regular values for f (eventually $b = \infty$). Denoting by*

$$f^b = \{x \in U \mid f(x) \le b\},$$
$$f_a^b = \{x \in U \mid a \le f(x) \le b\},$$

if f is a Morse functional in a neighbourhood A of f_a^b, then

$$m_\lambda(K_f \cap f_a^b, f) = P_\lambda(f^b, f^a; \mathbb{Z}_2) + (1+\lambda)Q(\lambda),$$

where $Q(\lambda)$ is a formal series in the variable λ with coefficients in $\mathbb{N} \cup \{\infty\}$, while $P_\lambda(f^b, f^a; \mathbb{Z}_2)$ is the Poincare polynomial of (f^b, f^a) with $\mathbb{Z}_2$ as coefficient field.

From now on, $\mathbb{Z}_2$ will always be supposed as coefficient field, even if it will be not specified.

2.2. Morse theory for equivariant problems. In this subsection f will be a C^1 function defined on a Hilbert space V.

Definition 2.6. Let $K \subset (K_f \cap f^{-1}(c))$, where $c \in \mathbb{R}$. The *topological Morse index* of K is the formal series

$$j_\lambda(K) = \sum_{q \in \mathbb{N}} \dim H_q(f^c, f^c \setminus K)\lambda^q.$$

Remark 2.7. If x_0 is a nondegenerate critical point of f, then $j_\lambda(\{x_0\}) = \lambda^{m(x_0)}$, so that the topological Morse index is a generalization of the Morse polynomial.

Let $K \subset (K_f \cap f^{-1}(c))$ be a connected manifold. For each $x \in K$, V admits the orthogonal decomposition $V = V_x \oplus W_x$, where $V_x = T_xK$ and $W_x = (T_xK)^\perp$. If f is a functional of class C^2 in a neighbourhood of K, then each $x \in K$ is certainly degenerate as V_x is included in the kernel of $d^2f(x)$, i.e.

$$d^2f(x)(y,z) = 0 \quad \text{for all } y \in V_x \text{ and } z \in V.$$

So it makes sense to introduce the following

Definition 2.8. A connected manifold $K \subset K_f$ is called *nondegenerate critical manifold* if $d^2f(x)$ is nondegenerate in the directions which are orthogonal to K for each $x \in K$.

In other words K is a nondegenerate manifold if, according to the previous notations, $d^2f(x)|_{W_x \times W_x}$ has nullity 0 for each $x \in K$.

Remark 2.9. It can be proved that if K is a nondegenerate manifold, than it is isolated and $m(x)$ doesn't depend by $x \in K$, so that it is natural to put $m(x) = m(K)$.

It is useful to recall the following known results (see for example [3], [6], [13], [14] for more details).

PROPOSITION 2.10. *If K is a nondegenerate critical manifold such that $m(K)$ is finite, then*

$$j_\lambda(K) = P_\lambda(K)\lambda^{m(K)},$$

where $P_\lambda(K)$ is the Poincare polynomial of K.

THEOREM 2.11. *Let f be a C^2 functional that verifies* (PS) *and let a, b (eventually $b = \infty$) be two regular values of f. If $K_f \cap f^{-1}(]a,b[)$ is made only by isolated critical sets, then the following Morse relation holds*

$$\sum_{K\subset(K_f\cap f^{-1}(]a,b[))} j_\lambda(K) = P_\lambda(f^b, f^a) + (1+\lambda)Q(\lambda), \tag{2.1}$$

where $Q(\lambda)$ is a formal series in the variable λ with coefficients in $\mathbb{N} \cup \{\infty\}$.

2.3. Preliminaries about group actions. Let G be a compact Lie group and X a topological space, an action of G on X is a function $\Phi = (g, x) \in G \times X \mapsto gx \in X$ that verifies the following properties:

- (G_1) $1x = x$ for all $x \in X$,
- (G_2) $g_1(g_2x) = (g_1g_2)x$ for all g_1, $g_2 \in G$ and $x \in X$.

A space with an action of a group G is called a G-space.

DEFINITION 2.12. The *orbit* of x is the set $O \cdot u = \{gx \mid g \in G\}$, and it can be seen that two orbits are either disjointed or equal.

If $O \cdot u = \{x\}$, then x is said to be a *fixed point under the action* of G and the set of the fixed points of X is denoted by $\mathrm{Fix}(G)$.

A function $f : X \to \mathbb{R}$ is said *G-invariant* if $f(gx) = f(x)$ for all $x \in X$ and $g \in G$.

A function $F : X \to Y$ between two G-spaces is *G-equivariant* if

$$F(gx) = gF(x) \quad \text{for all } x \in X \text{ and } g \in G.$$

If V is a Banach space, an action G on V is said *linear representation* (on V) if the function

$$T_g = x \in V \mapsto gx \in V$$

is linear and continuous for each $g \in G$.

EXAMPLE 2.13. If G is a linear representation on a Banach space V, then G acts also on its dual space V' in the following way:

$$(gf)(x) = f(g^{-1}x) \quad \text{for all } g \in G,\ f \in V' \text{ and } x \in X.$$

According to the previous definitions and notations the following results are easily proved

Theorem 2.14. *Let V be a Banach space and $f : V \to \mathbb{R}$ a C^1-functional, if G is a linear representation on V and f is G-invariant, then df is G-equivariant. Consequently, if x is a critical point of f, then its whole orbit $O \cdot u$ is made by critical points.*

Corollary 2.15. *If the function f of the previous theorem is C^2, then*

$$d^2 f(gx)(y,z) = d^2 f(x)(g^{-1}y, g^{-1}z) \quad \textit{for all } x,y,z \in V \textit{ and } g \in G. \tag{2.2}$$

Definition 2.16. If V is a Hilbert space, the action of a group G on V is said to be orthogonal if

$$\langle gx, gy \rangle = \langle x, y \rangle \quad \text{for all } x, y \in V \tag{2.3}$$

where $\langle \cdot\,, \cdot \rangle$ denotes the inner product of V.

Remark 2.17. If V is a real Hilbert space, G acts orthogonally on V and $f : V \to \mathbb{R}$ is a G-invariant C^2-functional, then by (2.2) and (2.3) it is easy to see that

$$m^*(y) = m^*(x) \quad \text{and} \quad m(y) = m(x) \quad \text{for all } x \in K_f \text{ and } y \in O \cdot u \subset K_f.$$

3. The problem

Let us consider the following problem of periodic type

$$\text{(P)} \qquad \begin{cases} u \in C^2(\mathbb{R}^2, \mathbb{R}), \\ -\varepsilon \Delta u(x,y) + F'(u(x,y)) = 0 & \text{in } \mathbb{R}^2, \\ u(x,y) = u(x+T,y) = u(x,y+S) & \text{for all } (x,y) \in \mathbb{R}^2, \\ \nabla u(x,y) = \nabla u(x+T,y) = \nabla u(x,y+S) & \text{for all } (x,y) \in \mathbb{R}^2, \end{cases}$$

where $\varepsilon > 0$ is a fixed real number, $T, S > 0$ are two fixed periods and $F \in C^2(\mathbb{R}, \mathbb{R})$ is a function which satisfies the following assumptions:

(i) $\lim_{t\to\pm\infty} F(t) = \infty$,
(ii) there exist $a, b \ge 0$ such that $|F'(t)| \le a|t|^{p-1} + b$ for all $t \in \mathbb{R}$, where $p \in \,]2, 2^*[$ and $2^* = 2n/(n-2)$ if $n \ge 3$, while $p > 2$ if $n = 2$,
(iii) F has exactly k maximum points ($k \in \mathbb{N}\setminus\{0\}$) and has no critical points except its maximum and minimum points,
(iv) $F" \neq 0$ in the critical points of F.

Introducing the set $\Omega = [0,T] \times [0,S]$ and denoting by $H^1_{S,T}$ the closure of the set

$$\{u \in C^1(\Omega) \mid u(0,y) = u(T,y),\ u(x,0) = u(x,S) \text{ for all } (x,y) \in \Omega\}$$

under the norm of $H^1(\Omega)$, solutions to (P) are critical points of the functional A_ε defined on $H^1_{S,T}$ by

$$A_\varepsilon(u) = \frac{\varepsilon}{2}\int_\Omega |\nabla u|^2 + \int_\Omega F(u(x,y))\,dx\,dy.$$

By standard arguments, assumption (ii) assures that A_ε is a C^1-functional and

$$dA_\varepsilon(u)(v) = \varepsilon\int_\Omega \nabla u\cdot\nabla v + \int_\Omega F'(u(x,y))v(x,y)\,dx\,dy$$

for each $u, v \in H^1_{S,T}$. Moreover, it is not difficult to verify that

PROPOSITION 3.1. *The group $G = \mathbb{R}^2/(\mathbb{Z}T\times\mathbb{Z}S)$ (isomorphic to the torus $T^2 = S^1\times S^1$) is a linear representation on $H^1_{S,T}$, acting as follows*

$$(3.1)\qquad (G_{t,s}u)(x,y) = u(x+t, y+s) \quad \text{for all } (t,s)\in G \text{ and } u \in H^1_{S,T}.$$

REMARK 3.2. The fixed points of $H^1_{S,T}$ under the action of G are the constant functions and A_ε is a G-invariant functional.

PROPOSITION 3.3. *The orbit of each $u \in K_{A_\varepsilon}\setminus \mathrm{Fix}(G)$ is a critical manifold (i.e. it is made by critical points for A_ε) which is homeomorphic to S^1 or to T^2.*

PROOF. As a consequence of Theorem 2.14, the orbit of each $u \in K_{A_\varepsilon}\setminus \mathrm{Fix}(G)$ is a critical set.

Now, if u is a critical point of A_ε which is constant with respect to y but not with respect to x, then

$$(G_{t,s}u)(x,y) = u(x+t,y+s) = u(x+t,y) = (G_{t,0}u)(x,y) \quad \text{for all } (t,s)\in G$$

so that the orbit of u is a manifold homeomorphic to S^1, and the same thing happens if u depends only by y.

Moreover, the orbit of u may be of S^1-type even if u depends on both variables. This happens when the level curves of u are parallel lines, that is when there exists $p \in \mathbb{Q}S/T = \{qS/T \mid q \in \mathbb{Q}\}$ such that u is constant on the lines having equation $y = px + r$, for each $r \in \mathbb{R}$. In this case

$$G_{t,pt}u(x,y) = u(x+t, y+pt) = u(x,y) \quad \text{for all } t \in \mathbb{R}/\mathbb{Z}T$$

so that

$$G_{t,s} = G_{(t,pt)+(0,s-pt)} = G_{t,pt}G_{0,s-pt} = G_{0,s-pt} \quad \text{for all } (t,s)\in G$$

hence the orbit is still homeomorphic to S^1.

It must be remarked that $pT/S \in \mathbb{Q}$ is required in order to assure the compatibility between the existence of such level curves and the periodicity of u. In fact if such a p exists and u is periodic of period T with respect to x and S with respect to y, then

$$u(x,y) = u\left(x + \frac{S}{p}, p\left(x + \frac{S}{p}\right) + y - px\right) = u\left(x + \frac{S}{p}, y + S\right) = u\left(x + \frac{S}{p}, y\right)$$

for all $(x,y) \in \mathbb{R}^2$; hence u is periodic with respect to x both with period T and S/p, so, as u is continuous and non constant, pT/S must belong to $\mathbb{Q}$.

In the other cases the two parameters of the group act independently and the orbit of u is a homeomorphic to a T^2-type manifold. □

Now a few lemmata concerning some properties of the critical points of A_ε will be stated. For the proofs the reader is referred to [22].

LEMMA 3.4. *Let γ and δ denote the smallest and the greatest of the $k+1$ minimum points of F respectively, then*

$$u(x) \in [\gamma, \delta] \quad \text{a.e. in } \Omega, \text{ for all } u \in K_{A_\varepsilon}.$$

LEMMA 3.5. *Let $u \in H^1_{S,T}$ be a critical point of A_ε, then u is a classical solution to* (P)*, i.e.:*

$$\begin{cases} u \in C^2(\mathbb{R}^2, \mathbb{R}) \\ -\varepsilon\Delta u(x,y) + F'(u(x,y)) = 0 & \text{in } \mathbb{R}^2, \\ u(x,y) = u(x+T,y) = u(x,y+S) & \text{for all } (x,y) \in \mathbb{R}^2, \\ \nabla u(x,y) = \nabla u(x+T,y) = \nabla u(x,y+S) & \text{for all } (x,y) \in \mathbb{R}^2. \end{cases}$$

LEMMA 3.6. *A_ε is a coercive functional which satisfies* (PS) *condition.*

REMARK 3.7. From previous lemmata it follows that, up to replacing F with F_0 such that $F = F_0$ on $[\gamma, \delta]$ (i.e. without changing the critical points of the problem), there is no loss of generality in supposing that

(ii') there exists $\overline{a}, \overline{b} \geq 0$ such that $|F''(t)| \leq \overline{a}|t|^{p-2} + \overline{b}$ for each $t \in \mathbb{R}$, where, as in assumption (ii), $p \in \,]2, 2^*[$ and $2^* = 2n/(n-2)$ if $n \geq 3$, while $p > 2$ if $n = 2$,

where it is evident that assumption (ii') is stronger than (ii). So A_ε becomes a C^2-functional and

$$d^2 A_\varepsilon(u)(v,w) = \varepsilon \int_\Omega \nabla v \cdot \nabla w + \int_\Omega F''(u(x))v(x)w(x)\,dx \tag{3.2}$$

for all $u, v, w \in H^1_{S,T}$.

REMARK 3.8. Let us consider the eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$ of $-\Delta$ on Ω with the required periodicity conditions. They are effectively known, and are all the numbers of the form $4\pi^2(m^2/T^2 + h^2/S^2)$ with $m, h \in \mathbb{N}$. Only λ_0 is simple among them, while all the other eigenvalues have at least a double (and in any case even) multiplicity, so that

$$\lambda_0 = 0 < \lambda_1 = \lambda_2 \leq \lambda_3 = \lambda_4 \leq \dots \tag{3.3}$$

In the following, if $a \in \mathbb{R}$, then the symbol u_a will denote the function assuming constant value a. Moreover, $c_1, \dots, c_{k+1}$ will denote the minimum points of F and $d_1, \dots, d_k$ its maximum points, the last ones ordered in such a way that $-F"(d_1) \leq \dots \leq -F"(d_k)$.

Next lemma computes the Morse index of the elements of $K_{A_\varepsilon} \cap \mathrm{Fix}(G)$.

Lemma 3.9.

(1) $m(u_{c_j}) = m^*(u_{c_j}) = 0$ *if* $j = 1, \dots, k+1$.
(2) *If* $j = 1, \dots, k$, *then there exists a decreasing sequence* $(\mu_i^j)_{i\in\mathbb{N}}$ *which tends to* 0 *such that* u_{d_j} *is nondegenerate if and only if* $\varepsilon \neq \mu_i^j$ *for all* $i \in \mathbb{N}$.
(3) *If* $\varepsilon \in \,]\mu_i^j, \mu_{i-1}^j[$, *then* $m(u_{d_j}) = 2i + 1$.
If $\varepsilon \in \,]\mu_0^j, \infty[$, *then* $m(u_{d_j}) = 1$.

Proof. (1) By assumption (iv) and Remark 3.7 $d^2A_\varepsilon(u_{c_j})$ is positive definite and the assertion follows from Definition 2.1.

(2) Let j be fixed in $\{1, \dots, k\}$ and let μ_i^j denote the number $-F"(d_j)/\lambda_{2i+1}$, so that $(\mu_i^j)_{i\in\mathbb{N}}$ clearly is a decreasing sequence which tends to 0. Now u_{d_j} is degenerate if and only if

$$\exists v \neq 0 \in H^1_{S,T} \text{ s.t. } d^2A_\varepsilon(u_{d_j})(v, w) = 0 \text{ for all } w \in H^1_{S,T}$$

i.e., by (3.2),

$$\varepsilon \int_\Omega (\nabla v/\nabla w) + F"(d_j) \int_\Omega v(x)w(x)\,dx = 0 \quad \text{for all } w \in H^1_{S,T}.$$

This means that v solves the problem $-\varepsilon\Delta v + F"(d_j)v = 0$ in Ω with periodicity conditions, and, by (3.3), $-F"(d_j)/\varepsilon$ is one of the eigenvalues λ_{2i+1}. So u_{d_j} is nondegenerate if and only if $\varepsilon \neq \mu_i^j$ for each $i \in \mathbb{N}$.

(3) Letting $(e_h)_{h\in\mathbb{N}}$ be the orthonormal basis for $L^2(\Omega)$ such that e_h is the eigenfunction relative to λ_h, that is $-\Delta e_h = \lambda_h e_h$, it follows that

$$d^2A_\varepsilon(u_{d_j})(e_h, e_h) = \varepsilon \int_\Omega |\nabla e_h|^2 + F"(d_j) \int_\Omega e_h^2(x)\,dx = \varepsilon\lambda_h + F"(d_j).$$

If $h = 0$, then $\lambda_0 = 0$ and

$$d^2A_\varepsilon(u_{d_j})(e_0, e_0) = F"(d_j) < 0$$

thus surely $m(u_{d_j}) \geq 1$.

If $i \geq 1$ and $\varepsilon < \mu^j_{i-1} = -F"(d_j)/\lambda_{2i-1}$, then $\varepsilon\lambda_h + F"(d_j) < 0$ for each $h < 2i+1$. This fact, together with $d^2A_\varepsilon(u_{d_j})(e_s, e_t) = 0$ whenever $s \neq t$, implies that $d^2A_\varepsilon(u_{d_j})$ is negative definite on span $\{e_0, \dots, e_{2i}\}$ and thus

$$m(u_{d_j}) \geq 2i + 1. \tag{3.4}$$

Finally, in order to establish the desired equality, it can be observed that if $\varepsilon > \mu^j_i$, then

$$d^2A_\varepsilon(u_{d_j})(e_h, e_h) > 0 \quad \text{for all } h \geq 2i + 1.$$

This easily implies that $d^2A_\varepsilon(u_{d_j})$ is positive definite on $\overline{\operatorname{span}\{e_{2i+1}, e_{2i+2}, \dots\}}$, so the assert follows from (3.4) and the fact that $H^1_{S,T} = \overline{\operatorname{span}\{e_i, i \in \mathbb{N}\}}$. □

In order to avoid degeneracy, from now on it will be assumed that

$$\varepsilon \neq \mu^j_i \quad \text{for all } j = 1, \dots, k+1 \text{ and } i \in \mathbb{N}$$

and that the orbits of the elements of $K_{A_\varepsilon} \setminus \operatorname{Fix}(G)$ are nondegenerate critical manifolds, according to definition 2.8.

THEOREM 3.10. *Denoted by $(\mu^j_i)_{i\in\mathbb{N}}$ the sequences of the previous lemma, if $\varepsilon \in]\mu^j_{i_j}, \mu^j_{i_j-1}[$ for each $j = 1, \dots, k$ (where $i_j \geq 1$), then there are at least $i_1 + i_2 + \dots + i_k$ critical orbits having Morse index strictly less than $2i_k$. Namely there exists at least one critical orbit having Morse index in $\{2i - 2, 2i - 1\}$ for each $i \in \{1, \dots, i_j\}$ and for each $j \in \{1, \dots, k\}$.*

PROOF. Remark 3.7 and Lemma 3.6 assure that A_ε satisfies the assumptions of theorem 2.11 and is bounded from below, thus (2.1) holds, i.e.

$$\sum_{K\subset K_{A_\varepsilon}} j_\lambda(K) = P_\lambda(H^1_{S,T}) + (1+\lambda)Q(\lambda) \tag{3.5}$$

where

$$\sum_{K\subset K_{A_\varepsilon}} j_\lambda(K) = \sum_{K\subset K_{A_\varepsilon}\cap \operatorname{Fix}(G)} j_\lambda(K) + \sum_{K\subset K_{A_\varepsilon}\setminus \operatorname{Fix}(G)} j_\lambda(K). \tag{3.6}$$

In these hypothesis, as the previous lemma showed, all the elements of $K_{A_\varepsilon} \cap \operatorname{Fix}(G)$ are nondegenerate critical points, so according to Remark 2.7

$$\begin{aligned} \sum_{K\subset K_{A_\varepsilon}\cap \operatorname{Fix}(G)} j_\lambda(K) &= \lambda^{m(u_{c_1})} + \lambda^{m(u_{c_2})} + \dots \lambda^{m(u_{c_{k+1}})} \\ &\quad + \lambda^{m(u_{d_1})} + \lambda^{m(u_{d_2})} + \dots + \lambda^{m(u_{d_k})} \\ &= k + 1 + \lambda^{2i_1+1} + \dots \lambda^{2i_k+1} \end{aligned} \tag{3.7}$$

where $i_1, \dots, i_k \geq 1$ by construction.

Assuming that the critical orbits of $K_{A_\varepsilon} \setminus \mathrm{Fix}(G)$ are nondegenerate manifolds, Proposition 2.10 can be applied and, recalling that $P_\lambda(S^1) = 1 + \lambda$ and $P_\lambda(T^2) = (1+\lambda)^2$, by Proposition 3.3:

$$\sum_{K \subset K_{A_\varepsilon} \setminus \mathrm{Fix}(G)} j_\lambda(K) = (1+\lambda) \sum_{h \in \mathbb{N}} a_h \lambda^h + (1+\lambda)^2 \sum_{h \in \mathbb{N}} b_h \lambda^h \tag{3.8}$$

where a_h and b_h, respectively, are the numbers of orbits of S^1-type and of T^2-type having Morse index h, for each $h \in \mathbb{N}$. Moreover, $H^1_{S,T}$ being contractible,

$$P_\lambda(H^1_{S,T}) = 1. \tag{3.9}$$

By (3.6)–(3.9), equation (3.5) becomes

$$k+1+\lambda^{2i_1+1}+\ldots+\lambda^{2i_k+1}+(1+\lambda)\sum_{h\in\mathbb{N}} a_h\lambda^h+(1+\lambda)^2\sum_{h\in\mathbb{N}} b_h\lambda^h = 1+(1+\lambda)Q(\lambda).$$

Hence, denoting by $\sum_{h\in\mathbb{N}} c_h \lambda^h$ the formal series $Q(\lambda)$, then dividing for $1+\lambda$,

$$\sum_{h\in\mathbb{N}} (a_h + b_h + b_{h-1})\lambda^h = \sum_{h\in\mathbb{N}} c_h\lambda^h - 1 + \lambda + \ldots - \lambda^{2i_1} - 1 + \lambda + \ldots - \lambda^{2i_k}.$$

Now, imposing that the coefficients of the left and right end sides agree,

$$\begin{aligned}
a_0 + b_0 &= c_0 - k \\
a_h + b_h + b_{h-1} &= c_h + (-1)^{h-1}k && \text{if } h = 1, \ldots, 2i_1, \\
a_h + b_h + b_{h-1} &= c_h + (-1)^{h-1}(k-1) && \text{if } h = 2i_1+1, \ldots, 2i_2, \\
&\ldots \\
a_h + b_h + b_{h-1} &= c_h + (-1)^{h-1} && \text{if } h = 2i_{k-1}+1, \ldots, 2i_k,
\end{aligned}$$

so that

$$\begin{aligned}
a_1 + b_1 + b_0 &= c_1 + k \geq k, \\
&\ldots \\
a_{2i_1-1} + b_{2i_1-1} + b_{2i_1-2} &= c_{2i_1-1} + k \geq k, \\
a_{2i_1+1} + b_{2i_1+1} + b_{2i_1} &= c_{2i_1+1} + k - 1 \geq k-1, \\
&\ldots
\end{aligned}$$

therefore there are at least k critical orbits having Morse index in $\{0,1\}$, and other k for each set $\{2,3\}, \ldots, \{2i_1-2, 2i_1-1\}$. Moreover, there are at least $k-1$ critical orbits having index in $\{2i_1, 2i_1+1\}, \ldots$ and so on. So finally the number of critical orbits of A_ε is (at least)

$$ki_1 + (k-1)(i_2 - i_1) + \ldots + (i_k - i_{k-1}) = i_1 + i_2 + \ldots + i_k. \qquad \square$$

Corollary 3.11. *If F has only one maximum point (that is $k = 1$), then there exists a decreasing sequence $(\eta_i)_{i\in\mathbb{N}}$ which tends to 0, such that if $\varepsilon \in]\eta_i, \eta_{i-1}[$, then there are at least i orbits of solutions to the problem* (P). *Namely there exists at least one orbit having Morse index in $\{h, h+1\}$, for each $h = 0, 2, \ldots, 2(i-1)$.*

Proof. It is an immediate consequence of the previous theorem, where $\eta_i = -F"(d)/\lambda_{2i+1}$ and d is the (only) maximum point of F. □

Proof of Theorem 1.1. Putting $\mu_i = \mu_i^1$ and recalling that $-F"(d_1) \leq -F"(d_j)$ for each $j = 1, \ldots, k$, it results that if $\varepsilon < \mu_i = \mu_i^1$, then $\varepsilon < \mu_i^j$ for each j, thus

$$i_j \geq i \quad \text{if } j = 1, \ldots, k$$

and by Theorem 3.10, the assert immediately follows. □

Remark 3.12. In analogous way it is possible to study the problem

$$\begin{cases} u \in C^2(\mathbb{R}^n), \\ -\varepsilon\Delta u(x_1, \ldots, x_n) + F'(u(x_1, \ldots, x_n)) = 0 & \text{in } \mathbb{R}^n, \\ u(x_1, \ldots, x_n) = u(x_1 + S_1, \ldots, x_n) \\ \quad = \ldots = u(x_1, \ldots, x_n + S_n) & \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}^n, \\ \nabla u(x_1, \ldots, x_n) = \nabla u(x_1 + S_1, \ldots, x_n) \\ \quad = \ldots = \nabla u(x_1, \ldots, x_n + S_n) & \text{for all } (x_1, \ldots, x_n) \in \mathbb{R}^n. \end{cases}$$

where $n > 2$, F is a function satisfying assumptions (i)–(iv) and $S_1, \ldots, S_n > 0$ are fixed numbers.

The functional relative to this problem is invariant with respect to a group which is homeomorphic to T^n. Moreover a result that is similar to the statement of the previous theorem is obtained, namely, using the same notations, for $n = 3, 4$ there are at least $(i_1 + \ldots + i_k)/2$ critical orbits having index strictly less than $2i_k$, for $n = 5, 6$ there are at least $(i_1 + \ldots + i_k)/3$ of them, and for a generic n there exist at least $(i_1 + \ldots + i_k)/[(n+1)/2]$ of such orbits.

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