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AN AXIOMATIC APPROACH TO A COINCIDENCE INDEX FOR NONCOMPACT FUNCTION PAIRS

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Abstract. We prove that there is a coincidence index for the inclusion $F(x) \in \Phi(x)$ when Φ is convex-valued and satisfies certain compactness assumptions on countable sets. For F we assume only that it provides a coincidence index for single-valued finite-dimensional maps (e.g. F is a Vietoris map). For the special case $F = \mathrm{id}$, the obtained fixed point index is defined if Φ is countably condensing; the assumptions in this case are even weaker than in [36].

Introduction

There are two essentially different approaches to the fixed point theory of multivalued maps: one is applicable for maps with convex values, and the other is applicable for maps with acyclic values. While the assumptions for the two approaches are similar from the viewpoint of applications, the methods applied are essentially different: the first approach uses certain approximations by single-valued maps (see e.g. [26]) while the second approach reduces the problem to a certain coincidence equation which then is attacked by methods from algebraic

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topology [10], [13], [14], [23]. For surveys which also explain the historical development, we refer to [2], [4], see also [17]. In this paper, we propose a general unifying theory which combines these approaches to the coincidence inclusion

$$F(x) \in \Phi(x) \tag{1}$$

where, roughly speaking, Φ may be approximated by single-valued maps (e.g. Φ attains convex values), and F is such that $F^{-1}(\{y\})$ is acyclic for each y. For $F = \mathrm{id}$, one arrives at the first approach described above, and for single-valued functions Φ, one is led to the second approach. For $F = \mathrm{id}$ and single-valued Φ, one arrives at the classical fixed point theory. For single-valued φ, a coincidence index for the equation

$$F(x) = \varphi(x) \tag{2}$$

for compact φ was introduced in [24] and further developed in [16]; this index is used in [21] to obtain an index for multivalued acyclic maps. Meanwhile, a general index theory for the coincidence problem (2) is known, at least in the compact case [14] (see also [23]). This index was generalized for the inclusion (1) in [5], but under the rather artificial assumption that x be taken from a finite-dimensional space.

By our approach, x can be taken from an arbitrary metric space. Moreover, we do not restrict our attention to maps F for which $F^{-1}(\{x\})$ is acyclic. Instead, we just assume that F is given such that a coincidence index for the equation (2) exists which satisfies certain axioms and which is applicable for compact (or even just finite-dimensional) functions φ. In Section 2, we then extend this index to the more general inclusion (1) for compact multi-valued maps Φ, and in Section 3, we extend that index in turn to (single- or) multi-valued maps Φ which only satisfy certain "a priori" compactness assumptions.

In other words: We provide a general scheme which allows to extend any coincidence index (in finite dimensions and for single-valued maps) to the multivalued and, more important, to the noncompact case. The basic ideas of such an extension scheme are of course well-known:

In the case $F = \mathrm{id}$, this scheme corresponds to the extension of the classical Brouwer degree to the Schauder (resp. the Nussbaum-Sadovskiĭ) degree [29], [32] (in case of single-valued maps) and to e.g. the degree from [26] (resp. [11], [31]) (in case of multi-valued maps).

However, in the case $F \neq \mathrm{id}$, some technical difficulties arise, and it appears that no attempts have been made so far to overcome them *systematically* (certain special cases have been studied already [14], [16]).

Moreover, we add some refinements to the "known" scheme so that even in the case $F = \mathrm{id}$, we gain new results: The fixed point index obtained by our scheme in this case requires less restrictive compactness assumptions than the fixed point indices considered in literature so far (even in the single-valued case). Note that for this special case, we need besides some elementary arguments only the "standard" fixed point index on convex finite-dimensional sets which can be obtained immediately from the Brouwer degree by a simple retraction argument.

The main idea for the noncompact case is to consider only those maps Φ for which it is possible to find a certain set (a so-called *fundamental set*) which contains all information of Φ which is important for coincidence points, and such that $\Phi \circ F^{-1}$ is compact on this set. For $F = \mathrm{id}$, the concept of fundamental sets was apparently first introduced in [38] (see also [22]) and later developed by V. V. Obukhovskiĭ and others, even in the context of multivalued maps [2], [4]; for acyclic multivalued maps, see also [3], [21], [30], [39]. The class of maps Φ which are admissible for this theory (the so-called *fundamentally restrictible maps*) contains in particular the so-called condensing maps. Actually, the so-called *ultimate range* introduced by Sadovskiĭ [32] (see also [1]) as a tool for the degree theory of condensing maps (for the multivalued case, see e.g. [11], [31]) is nothing else but a special fundamental set. A new point in our concept is that we do not require the existence of a fundamental set on the whole domain of definition but only on a certain subset. This generalization is technical and appears artificial, but it has an important advantage:

For applications to equations containing integrals or derivatives of vector functions, one can usually estimate measures of noncompactness only for *countable* sets (see e.g. [28], [37]). For this reason, it is of interest to have a theory also for countably condensing maps. The first results in this direction were given in [6], [7], [19], [27], [33]. In [36], a fixed point index for countably condensing maps was introduced which was based on fundamental sets. However, the assumptions needed in [36] to define this index appear not very natural (although they are sufficiently general for most purposes). In Section 4, we will see that our index (for $F = \mathrm{id}$) actually generalizes the index from [36] and requires in contrast to the latter only "natural" assumptions on countable sets.

The most basic coincidence/fixed point theorems obtained by our index are presented in Section 5.

2. Compact multivalued maps

Let X be a metric space, Y be a closed convex subset of some locally convex Hausdorff vector space Z, and $F : X \to Y$ be continuous and proper (i.e. preimages of compact sets are compact).

We call a triple (φ, Ω, K) *finitely F-admissible*, if $K \subseteq Y$ is compact and convex and contained in a finite-dimensional subspace of Z, $\Omega \subseteq K$ is open in K, and $\varphi : X \to K$ is continuous and such that $F(x) = \varphi(x) \in \overline{\Omega}$ implies $F(x) \in \Omega$.

DEFINITION 2.1. We say that F provides the *coincidence index* ind_F (on Y), if there is a map ind_F from the system of finitely F-admissible triples into a ring with 1 (typically $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{Z}_2$) such that for any finitely F-admissible triple (φ, Ω, K) the following holds:

- (i) (Coincidence point property) If $\operatorname{ind}_F(\varphi, \Omega, K) \neq 0$, then the coincidence equation $F(x) = \varphi(x) \in \Omega$ has a solution.
- (ii) (Normalization) If $\varphi(x) \equiv c \in \Omega$, then $\operatorname{ind}_F(\varphi, \Omega, K) = 1$.
- (iii) (Homotopy invariance) If $H : [0,1] \times X \to K$ is continuous and the triple $(H(\lambda, \cdot\,), \Omega, K)$ is finitely F-admissible for each $\lambda \in [0,1]$, then

$$\operatorname{ind}_F(H(0, \cdot\,), \Omega, K) = \operatorname{ind}_F(H(1, \cdot\,), \Omega, K). \tag{3}$$

- (iv) (Permanence) If $K_0 \subseteq K$ is closed and convex with $\varphi(X) \subseteq K_0$, then

$$\operatorname{ind}_F(\varphi, \Omega, K) = \operatorname{ind}_F(\varphi, \Omega \cap K_0, K_0). \tag{4}$$

We say that ind_F satisfies the *excision property*, if additionally for each finitely F-admissible triple (φ, Ω, K) the following holds:

- (v) (Excision) If $\Omega_0 \subseteq \Omega$ is open in K and such that $F(x) = \varphi(x) \in \overline{\Omega}$ implies $F(x) \in \Omega_0$, then

$$\operatorname{ind}_F(\varphi, \Omega, K) = \operatorname{ind}_F(\varphi, \Omega_0, K).$$

We call ind_F *additive*, if additionally the following property is satisfied for each finitely F-admissible triple (φ, Ω, K):

- (vi) (Additivity) If $\Omega_1, \Omega_2 \subseteq \Omega$ are disjoint and open in K and such that $F(x) = \varphi(x) \in \overline{\Omega}$ implies $F(x) \in \Omega_1 \cup \Omega_2$, then

$$\operatorname{ind}_F(\varphi, \Omega, K) = \operatorname{ind}_F(\varphi, \Omega_1, K) + \operatorname{ind}_F(\varphi, \Omega_2, K).$$

Of course, the excision property is the special case of the additivity with $\Omega_2 = \emptyset$. Observe that the normalization implies in view of the coincidence point property that $F : X \to Y$ is surjective.

We separated the excision property/additivity from the other axioms, since we conjecture that there are examples of coincidence point indices which fail to have this property (because they are e.g. obtained by homotopic instead of homologic methods); a hint in this direction is given by the class of 0-epi maps [12] (see also [18]) which might be considered as a "homotopic" analogue to degree theory but for which the excision property does not hold in general (however, if one tries to define an index in $\mathbb{Z}_2$ analogous to the definition of 0-epi maps, one also runs into problems in connection with the permanence property, so we can not provide a particular example up to now).

Example 2.1. If $X = Y = Z$ is a locally convex metric space, and $F = \mathrm{id}$, it is well-known that an unique additive index with values in $\mathbb{Z}$ with the above properties exists: This is the classical fixed point index of φ (sometimes also called degree of $\mathrm{id} - \varphi$) on Ω relative to K.

Example 2.1 is generalized by the following result which is essentially a reformulation of a special case of [14, Theorem (47.8)].

Recall that a continuous proper surjection $F : X \to Y$ is called *Vietoris*, if for each $y \in Y$ the set $F^{-1}(\{y\})$ is acyclic with respect to the Čech cohomology with coefficients in $\mathbb{Q}$.

Theorem 2.1 (Górniewicz). *Let X be a metric space, Z be a locally convex metric vector space, and $Y \subseteq Z$ be closed and convex. Then any Vietoris map $F : X \to Y$ provides an additive coincidence index with values in $\mathbb{Z}$.*

Proof. Let a finitely F-admissible triple (φ, Ω, K) be given. Using the notation of [14, Theorem (47.8)], the triple (K, Ω, Φ) belongs to the class $\mathcal{B}$, where Φ is the morphism determined by the pair (F, φ) (roughly speaking: $\Phi = \varphi \circ F^{-1}$; note that F^{-1} is upper semicontinuous, since Y is metrizable [13]). We may thus define $\mathrm{ind}_F(\varphi, \Omega, K)$ as the index from [14, Theorem (47.8)] for the triple (K, Ω, Φ). The desired properties of our index follow from the properties of the latter index (the coincidence point property corresponds to the "existence" property from [14], and the permanence property corresponds to the "contraction" property from [14]). □

Remark 2.1. In the proof of Theorem 2.1, we employed that the solutions of (2) correspond to the fixed points of the multivalued map $\Phi = \varphi \circ F^{-1}$. We recall that, conversely, given an upper semicontinuous compact (finite-dimensional) multivalued map Φ with nonempty acyclic values, one may define X as the graph of Φ, and let F and φ be the projection on the first and second component, respectively; then F is a Vietoris map, φ is continuous and compact (resp. finite-dimensional), and $\Phi = \varphi \circ F^{-1}$. See [25] for a further discussion on the connection of fixed points of multivalued maps and coincidence points.

We emphasize once more that we do not restrict our attention to Vietoris maps: By our approach, any extension of Theorem 2.1 to *some* (non-Vietoris) function F will immediately lead to a corresponding index for the more general equation (1) (for certain noncompact maps Φ) for that function F. More general classes of functions F which provide a coincidence index can be found e.g. by considering the fixed point index of so-called decompositions, see [13, Theorem (51.10)]. However, the case of Vietoris maps is the most important example for our theory (and the obtained results are new in this case, even for $F = \mathrm{id}$).

For a set A in a metric space and $r > 0$, we use the notation $B_r(A) = \{x : \mathrm{dist}(x, A) < r\}$ and $B_r(x) = B_r(\{x\})$.

For a multivalued map $\Phi : D \to 2^Z$ and $A \subseteq D$, we use the convenient notation $\Phi(A) := \bigcup\{\Phi(x) : x \in A\}$.

Let $\Phi : D \to 2^Z$ be some multivalued map where D is a metric space. Let $O \subseteq Z$ be some neighbourhood of 0, and $\varepsilon > 0$. We call a single-valued continuous map $\varphi : D \to Z$ an *(ε, O)-approximation* for Φ, if the inclusion $\varphi(x) \in \Phi(B_\varepsilon(x)) + O$ holds for all $x \in D$. If additionally the range of φ is contained in a finite-dimensional subspace of Z, we call φ a *finite (ε, O)-approximation*.

The following lemma is one of the main reasons why we will restrict ourselves to the case of convex-valued maps: The corresponding result in [15] for e.g. maps with R_δ-values requires that D be an ANR and, moreover, it is not clear whether a "simultaneous approximation" as in the following lemma is possible. Parts of the proof of this lemma are inspired by the proof of [8, Theorem 24.2].

For $M \subseteq Z$, let $\mathcal{C}(M)$ denote the system of all nonempty closed and convex subsets of M.

Lemma 2.1. *Let D be a compact metric space, and $\Phi : D \to \mathcal{C}(Z)$ be upper semicontinuous. Let $D_1, \dots, D_n \subseteq D$ be closed subsets such that for each pair $j \neq k$ one of the relations $D_j \cap D_k = \emptyset$, $D_j \subseteq D_k$, or $D_k \subseteq D_j$ holds. Then for each $\varepsilon > 0$, and each neighbourhood $O \subseteq Z$ of 0, there is some finite (ε, O)-approximation $\varphi : D \to \operatorname{conv}(\Phi(D))$ for Φ such that simultaneously $\varphi|_{D_j} : D_j \to \operatorname{conv}(\Phi(D_j))$ is a finite (ε, O)-approximation for $\Phi|_{D_j}$ $(j = 1, \dots, n)$.*

Proof. Since Z is locally convex, we may assume that O is convex. Choose open sets $U_j \supseteq D_j$ such that $U_j \cap D_k = \emptyset$ whenever $D_j \cap D_k = \emptyset$.

Let $r(x) > 0$ be the supremum of all numbers $\rho \in (0, \varepsilon]$ such that $\Phi(B_{2\rho}(x)) \subseteq \Phi(x) + O$ (here we used that Φ is upper semicontinuous). Without loss of generality, we may assume $D_n = D$ and that the sets D_j are ordered such that $j < k$ implies $D_j \subseteq D_k$ or $D_j \cap D_k = \emptyset$. Now we define by induction on $k = 0, \dots, n$ finite sets $P_k \subseteq D$ and open sets $B_k \subseteq D$ as follows: Put $P_0 = B_0 = \emptyset$. If P_j and B_j are already defined for $j < k$, then $A_k = D_k \setminus \bigcup\{B_j : j < k\}$ is compact. Hence, we find a finite set $P_k \subseteq A_k$ such that $B_k = \bigcup\{B_{r(z)}(z) \cap U_k : z \in P_k\}$ contains A_k.

We have $D_k \subseteq \bigcup\{B_j : j \leq k\}$, because any element from D_k either belongs to $A_k \subseteq B_k$ or to $\bigcup\{B_j : j < k\}$. Put $O_k = B_k \setminus \bigcup\{D_j : j < k\}$. By what we just proved it follows on the one hand that $O_k \supseteq B_k \setminus \bigcup\{B_j : j < k\}$, and then on the other hand that $\bigcup O_k \supseteq \bigcup B_k \supseteq D_n = D$. We may conclude that the sets $V_{k,z} = B_{r(z)}(z) \cap U_k \setminus \bigcup\{D_j : j < k\}$ $(z \in P_k,\ k = 1, \dots, n)$ constitute a (finite) open cover of D. Let $\varphi_{k,z}$ be a partition of unity with $\operatorname{supp} \varphi_{k,z} \subseteq V_{k,z}$. For each k and each $z \in P_k$ choose some $y_{k,z} \in \Phi(z)$. We claim that

$$\varphi(x) = \sum_{k=1}^{n} \sum_{z \in P_k} \varphi_{k,z}(x) y_{k,z}$$

has the desired properties. Given $x \in D_j$, the set $I = \{(k,z) : \varphi_{k,z}(x) \neq 0\}$ is contained in $\{1,\dots,n\} \times D_j$. Indeed, since $x \in \operatorname{supp} \varphi_{k,z} \subseteq V_{k,z}$ and $V_{k,z} \cap D_j = \emptyset$ for $j < k$, we must have $j \geq k$. Moreover, since $x \in U_k \cap D_j$, our choice of U_k implies that $D_k \cap D_j \neq \emptyset$, and so in view of our order that $D_j \supseteq D_k$. Hence, $z \in P_k \subseteq A_k \subseteq D_k \subseteq D_j$, as claimed.

It readily follows that $\varphi(x) \in \operatorname{conv}(\{y_{k,z} : (k,z) \in I\}) \subseteq \operatorname{conv}(\Phi(D_j))$.

Fix some $(k_0, z_0) \in I$ with $r(z_0) = \max\{r(z) : (k,z) \in I\}$. Then we have for any $(k,z) \in I$ in view of $x \in V_{k,z} \cap V_{k_0,z_0}$ the estimate $d(z,z_0) \leq d(z,x) + d(x,z_0) \leq 2r(z_0)$, and so $y_{k,z} \in \Phi(B_{2r(z_0)}(z_0)) \subseteq \Phi(z_0) + O$. Since $\Phi(z_0) + O$ is convex (recall that O is convex), we thus find $\varphi(x) \in \operatorname{conv}(\{y_{k,z} : (k,z) \in I\}) \subseteq \Phi(z_0) + O$. Since $z_0 \in D_j$ (as we have proved above) and $z_0 \in B_\varepsilon(x)$ (because $r(z_0) \leq \varepsilon$), it follows that $\varphi|_{D_j}$ is an (ε, O)-approximation for $\Phi|_{D_j}$. □

The following lemma implies that convex-valued maps are appropriate in the sense of [15] (even if D is not necessarily an ANR).

Lemma 2.2. *Let D be a compact metric space, $\Phi : D \to \mathcal{C}(Z)$ be upper semicontinuous, $O_0 \subseteq Z$ be some neighbourhood of 0, and $\delta > 0$. Then there is some neighbourhood $O \subseteq Z$ of 0 and some $\varepsilon > 0$ such that whenever φ, ψ are two finite (ε, O)-approximations for Φ, then the homotopy $h(\lambda, x) = \lambda\varphi(x) + (1-\lambda)\psi(x)$ has the property that $h(\lambda, \cdot\,)$ is a δ-approximation for Φ for each $\lambda \in [0,1]$.*

Proof. We may assume that O_0 is convex. Let $O \subseteq Z$ be some neighbourhood of 0 with $O + O \subseteq O_0$. Let $r(x) > 0$ be the supremum of all numbers $\rho \in (0, \delta]$ such that $\Phi(B_\rho(x)) \subseteq \Phi(x) + O$. Let $\varepsilon > 0$ be the Lebesgue number of the covering $(B_{r(x)}(x))_x$ of the compact set D, i.e. we find for any $x \in D$ some $x_0 \in D$ with $B_\varepsilon(x) \subseteq B_{r(x_0)}(x_0)$, and so $\Phi(B_\varepsilon(x)) \subseteq \Phi(x_0) + O$. If φ, ψ are two finite (ε, O)-approximations for Φ, and $\lambda \in [0,1]$, then $\varphi(x), \psi(x) \in \Phi(B_\varepsilon(x)) + O \subseteq \Phi(x_0) + O + O \subseteq \Phi(x_0) + O_0$. Since $\Phi(x_0) + O_0$ is convex, we have $h(\lambda, x) \in \operatorname{conv}(\{\varphi(x), \psi(x)\}) \subseteq \Phi(x_0) + O_0$. Note now that $d(x, x_0) < r(x_0) \leq \delta$. □

If Z is metrizable, the following result is easily proved by contradiction and a sequential argument. One could try to prove the general case similarly by using nets. However, the following proof avoids the axiom of choice:

Lemma 2.3. *Let I be a compact metric space, $\Omega \subseteq Y$, $D = F^{-1}(\overline{\Omega})$, and $H : I \times D \to 2^Z$ be upper semicontinuous with closed values. Let $M \subseteq Z$ be compact such that the inclusion $F(x) \in H(I \times \{x\}) \cap M$ has no solution. Then there is some neighbourhood $O \subseteq Z$ of 0 and some $\varepsilon > 0$ such that for any (ε, O)-approximation h for H the inclusion $F(x) \in h(I \times \{x\}) \cap M$ has no solution.*

Proof. We consider the compact metric space $A = I \times F^{-1}(M)$. Given $(\lambda, x) \in A$, the set $H(\lambda, x) \cap M$ is closed and does not contain $F(x)$. Hence, we find some neighbohrood $O_0 \subseteq Z$ of 0 with $F(x) \notin H(\lambda, x) + O_0$. Let $O \subseteq Z$ be some neighbourhood of 0 with $O+O-O \subseteq O_0$. Since H is upper semicontinuous, we find some $\rho \in (0,1)$ such that $H(B_\rho(\lambda, x)) \subseteq H(\lambda, x) + O$. Since F is continuous, we may assume that the relation $(\lambda_0, x_0) \in B_\rho(\lambda, x)$ implies $F(x_0) \in F(x) + O$.

This proves that for each $(\lambda, x) \in A$ we find some $\rho \in (0,1)$ with the property that there is some neighbourhood $O \subseteq Z$ of 0 with $F(x_0) \notin H(B_\rho(\lambda, x)) + O$ whenever $(\lambda_0, x_0) \in B_\rho(\lambda, x)$. Let $r(\lambda, x)$ denote the half of the supremum of all those numbers ρ. Then A is covered by the sets $B_{r(\lambda,x)}(\lambda, x)$. Choose a finite subcovering $B_1 = B_{r(\lambda_1,x_1)}(\lambda_1, x_1), \ldots, B_n = B_{r(\lambda_n,x_n)}(\lambda_n, x_n)$. For $i = 1, \ldots, n$, we find an open neighbourhood $O_i \subseteq Z$ of 0 such that the relation $(\lambda, x) \in B_i$ implies $F(x) \notin H(B_i) + O_i$. Let $\varepsilon > 0$ be the Lebesgue number of the cover $(B_i)_i$, and put $O = O_1 \cap \cdots \cap O_n$.

Now if h is some (ε, O)-approximation h for H, we can not have $F(x) = h(\lambda, x) \in M$: Otherwise, we find some i with $B_\varepsilon(\lambda, x) \subseteq B_i$. Then $h(\lambda, x) = F(x) \notin H(B_i) + O \supseteq H(B_\varepsilon(\lambda, x)) + O$ which contradicts the fact that h is an (ε, O)-approximation for H. □

If Z is metrizable (and $I = [0,1]$), the following lemma is essentially [15, (4.3)]. We assume in this lemma that the space $I \times D$ is equipped with the metric $d((\lambda, x), (\lambda_0, x_0)) = \max\{d(\lambda, \lambda_0), d(x, x_0)\}$.

Lemma 2.4. *Let I be some metric space, D some compact metric space, and $H : I \times D \to 2^Z$ be upper semicontinuous. Let $O \subseteq Z$ be a neighbourhood of 0, $\varepsilon > 0$, and $\lambda_0 \in I$. Then there is some neighbourhood $O_1 \subseteq Z$ of 0 and some $\varepsilon_1 > 0$ such that if $h : I \times D \to Z$ is an (ε_1, O_1)-approximation for H, then $h(\lambda_0, \cdot\,)$ is an (ε, O)-approximation for $H(\lambda_0, \cdot\,)$.*

Proof. Let $O_1 \subseteq Z$ be a neighbourhood of 0 with $O_1 + O_1 \subseteq O$. Since H is upper semicontinuous, we may define for each $\lambda \in D$ the value $r(x)$ as the supremum of all $\rho \in (0, \varepsilon]$ with $H(B_\rho(\lambda_0, x)) \subseteq H(\lambda_0, x) + O_1$. Then the open sets $B_{r(x)}(x)$ cover the compact set D. Let $B_{r(x_1)}(x_1), \ldots, B_{r(x_n)}(x_n)$ be a finite subcover, and $\varepsilon_1 \le \min\{r(x_1), \ldots, r(x_n)\}$ be smaller than the corresponding Lebesgue number. Then we find for each $x \in D$ some k with $B_{\varepsilon_1}(x) \subseteq B_{r(x_k)}(x_k)$. If h is an (ε_1, O_1)-approximation of H, we find points $(\lambda, z) \in B_{\varepsilon_1}(\lambda_0, x)$ with $h(\lambda_0, x) \in H(\lambda, z) + O_1$. By definition of the metric in $I \times D$, we have $z \in B_{\varepsilon_1}(x) \subseteq B_{r(x_k)}(x_k)$ and $d(\lambda, \lambda_0) < \varepsilon_1 \le r(x_k)$, and so $(\lambda, z) \in B_{r(x_k)}(\lambda_0, x_k)$. Thus, $H(\lambda, z) \in H(B_{r(x_k)}(\lambda_0, x_k)) \subseteq H(\lambda_0, x_k) + O_1$, and so $h(\lambda_0, x) \in H(\lambda_0, x_k) + O_1 + O_1 \subseteq H(\lambda_0, x_k) + O$. Since $d(x, x_k) < r(x_k) \le \varepsilon$, the claim follows. □

Definition 2.2. Let I be some metric space, $K \subseteq Y$ closed and convex, $\Omega \subseteq K$ open in K, and $D_0 = F^{-1}(\overline{\Omega}) \subseteq D \subseteq X$. Let $H : I \times D \to 2^Z$ such that $H : I \times D_0 \to \mathcal{C}(K)$ is upper semicontinuous and $\overline{\operatorname{conv}}\,(H(I \times D_0))$ is compact. If the relation $F(x) \in H(I \times \{x\}) \cap \overline{\Omega}$ implies $F(x) \in \Omega$, we call the triple (H, Ω, K) *compactly F-admissible.*

We emphasize that we do not require that Ω be bounded (however, $H(I \times D_0)$ must be bounded, of course).

If no confusion arises, we identify functions $\Phi : D \to 2^Z$ with $H : I \times D \to 2^Z$ where e.g. $I = \{0\}$ and $H(0, x) = \Phi(x)$. Similarly, we identify as usual single-valued functions φ with the corresponding multivalued function $\Phi(x) = \{\varphi(x)\}$. In this sense, each finitely F-admissible triple is compactly F-admissible.

Theorem 2.2. *Let $F : X \to Y$ be continuous and proper and provide a coincidence index* ind_F *(for single-valued maps). Then* ind_F *has an extension to a coincidence index* ind_F *for multivalued maps defined on all compactly F-admissible triples. This index has the following properties. If (Φ, Ω, K) is a compactly F-admissible triple, then we have:*

(i) (Localization) *If $\Psi : D_0 \to 2^Z$ with $F^{-1}(\overline{\Omega}) \subseteq D_0 \subseteq X$ and $\Psi|_{F^{-1}(\overline{\Omega})} = \Phi|_{F^{-1}(\overline{\Omega})}$, then (Ψ, Ω, K) is compactly F-admissible, and*

$$\operatorname{ind}_F(\Phi, \Omega, K) = \operatorname{ind}_F(\Psi, \Omega, K).$$

(ii) (Coincidence point property) *If $\operatorname{ind}_F(\Phi, \Omega, K) \neq 0$, then the inclusion $F(x) \in \Phi(x) \cap \Omega$ has a solution.*

(iii) (Normalization) *If $\Phi(x) \equiv \{c\}$ with $c \in \Omega$, then $\operatorname{ind}_F(\Phi, \Omega, K) = 1$.*

(iv) (Homotopy invariance) *If $H : [0, 1] \times D \to 2^Z$ is such that (H, Ω, K) is compactly F-admissible, then $(H(\lambda, \cdot\,), \Omega, K)$ is compactly F-admissible for $0 \le \lambda \le 1$, and*

$$\operatorname{ind}_F(H(0, \cdot\,), \Omega, K) = \operatorname{ind}_F(H(1, \cdot\,), \Omega, K).$$

(v) (Permanence) *If $K_0 \subseteq K$ is closed and convex with $\Phi(F^{-1}(\overline{\Omega})) \subseteq K_0$, then $(\Phi, \Omega \cap K_0, K_0)$ is compactly F-admissible, and*

$$\operatorname{ind}_F(\Phi, \Omega, K) = \operatorname{ind}_F(\Phi, \Omega \cap K_0, K_0).$$

If the given index satisfies the excision property, then we have also:

(vi) (Excision) *If $\Omega_0 \subseteq \Omega$ is open in K and such that $F(x) \in \Phi(x) \cap \overline{\Omega}$ implies $F(x) \in \Omega_0$, then (Φ, Ω_0, K) is compactly F-admissible, and*

$$\operatorname{ind}_F(\Phi, \Omega, K) = \operatorname{ind}_F(\Phi, \Omega_0, K).$$

If the given index is additive, then we have also:

(vii) (Additivity) *If* $\Omega_1, \Omega_2 \subseteq \Omega$ *are disjoint and open in* K *and such that* $F(x) \in \Phi(x) \cap \overline{\Omega}$ *implies* $F(x) \in \Omega_1 \cup \Omega_2$*, then* (Φ, Ω_1, K) *and* (Φ, Ω_2, K) *are compactly* F*-admissible, and*

$$\operatorname{ind}_F(\Phi, \Omega, K) = \operatorname{ind}_F(\Phi, \Omega_1, K) + \operatorname{ind}_F(\Phi, \Omega_2, K).$$

Proof. The definition of the index for a compactly F-admissible triple (Φ, Ω, K) is as follows: Let $K_0 \subseteq K$ be some compact and convex set which contains $\overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega})))$. Then $D = F^{-1}(K_0 \cap \overline{\Omega})$ is compact, and $\Phi|_D : D \to \mathcal{C}(K)$ is upper semicontinuous. Apply Lemma 2.3 with $M = (\overline{\Omega} \cap K_0) \setminus \Omega$ to find some $\delta > 0$ and some neighbourhood $O_0 \subseteq Z$ of 0 such that for any (δ, O_0)-approximation φ for $\Phi|_D$ the equation $F(x) = \varphi(x) \in M$ has no solution. Applying Lemma 2.2, we find some neighbourhood $O \subseteq Z$ of 0 and some $\varepsilon > 0$ such that whenever φ, ψ are two finite (ε, O)-approximation for $\Phi|_D$ with values in K_0, then the homotopy $h(\lambda, x) = (1-\lambda)\varphi(x) + \lambda\psi(x)$ is such that for each $\lambda \in [0,1]$ the map $h(\lambda, \cdot\,)$ is a (δ, O_0)-approximation for $\Phi|_D$. By Lemma 2.1, there exists some finite (ε, O)-approximation $\varphi : D \to K_0$ for $\Phi|_D$. Then we extend φ to some continuous function $\varphi : X \to K_0$ with values in some finite-dimensional subspace S: Since φ attains its values in a finite-dimensional space, such an extension exists by the Tietze–Urysohn theorem (which in contrast to Dugundji's extension theorem [9] does not require the (uncountable) axiom of coice). Now we put

$$\operatorname{ind}_F(\Phi, \Omega, K) := \operatorname{ind}_F(\varphi, \Omega \cap K_0 \cap S, K_0 \cap S). \tag{5}$$

This is well-defined: Note first that $\Omega \cap K_0 \cap S$ is in fact open in the closed and convex set $K_0 \cap S$. Moreover, since φ is in particular a (δ, O_0)-approximation, the relation $F(x) = \varphi(x) \in \overline{\Omega \cap K_0 \cap S}$ implies (since $F(x) \notin M$ by our choice of δ and O_0) that $F(x) \in \Omega \cap K_0 \cap S$. Hence, $(\varphi, \Omega \cap K_0 \cap S, K_0 \cap S)$ is in fact finitely F-admissible.

The above definition is independent of the particular choice of S: If we let S_0 denote the linear hull of the range of φ, we have $K_0 \cap S_0 \subseteq K \cap S$, and the permanence property implies

$$\operatorname{ind}_F(\varphi, \Omega \cap K_0 \cap S, K_0 \cap S) = \operatorname{ind}_F(\varphi, \Omega \cap K_0 \cap S_0, K_0 \cap S_0).$$

But the right-hand side is independent of S. Moreover, the above definition is independent of the particular choice of φ: If ψ is another finite (ε, O)-approximation for Φ on D with values in K_0, extend ψ to a continuous map $\psi : X \to K_0$ with values in a finite-dimensional space, and put $h(\lambda, x) = \lambda\varphi(x) + (1-\lambda)\psi(x)$. Then $h : X \to K_0 \cap S$ for some finite-dimensional space $S \subseteq Z$. Since by our choice of ε and O the map $h(\lambda, \cdot\,)|_D$ is a (δ, O_0)-approximation for Φ, the same argument as above shows that the triple $(h(\lambda,\,\cdot\,),\Omega\cap K_0\cap S,K_0\cap S)$ is finitely F-admissible. The homotopy invariance thus implies

$$\operatorname{ind}_F(\varphi,\Omega\cap K_0\cap S,K_0\cap S)=\operatorname{ind}_F(\psi,\Omega\cap K_0\cap S,K_0\cap S),$$

and the independence of our definition from the choice of φ is established. Note now that each (ε,O)-approximation is also an (ε',O')-approximation for $0<\varepsilon'<\varepsilon$ and neighbourhoods $O'\subseteq O$ of 0 and that we also *find* (ε',O')-approximations φ' by Lemma 2.1.

It remains to prove that our definition is independent from the particular choice of K_0. To see this, we show that we get the same value for $\operatorname{ind}_F(\Phi,\Omega,K)$ if we make for K_0 the particular choice $K_0'=\overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega})))$: Denote the corresponding sizes constructed above by D', δ', ε', O', and O_0'. It follows from the definition that $K_0'\subseteq K_0$, and so $D'\subseteq D$. Applying Lemma 2.1 with $\tilde{\varepsilon}=\min\{\varepsilon,\varepsilon'\}$ and $\tilde{O}=O\cap O'$, we find some finite (ε,O)-approximation $\varphi:D\to K_0'$ for $\Phi|_D$ such that $\varphi|_{D'}$ is simultaneously a finite (ε',O')-approximation for $\Phi|_{D'}$. Extend φ to a continuous function $\varphi:X\to K_0'$ with values in some finite-dimensional subspace $S\subseteq Z$. The permanence property implies

$$\operatorname{ind}_F(\varphi,\Omega,K)=\operatorname{ind}_F(\varphi,\Omega\cap K_0\cap S,K_0\cap S)=\operatorname{ind}_F(\varphi,\Omega\cap K_0'\cap S,K_0'\cap S),$$

and the independence of our definition from the set K_0 is proved.

The localization property follows immediately from the fact that our definition of the index depends only on the restriction of Φ to the set $F^{-1}(\overline{\Omega})$. Moreover, our newly defined index is an extension of the given index, and so in particular, the normalization property is satisfied. Indeed, if $\Phi(x)=\{\varphi(x)\}$ is single-valued and (φ,Ω,K) is finitely F-admissible, we may choose $K_0=K$ and $S=\operatorname{span}(K_0)$ in our above definition, and then find

$$\operatorname{ind}_F(\Phi,\Omega,K)=\operatorname{ind}_F(\varphi,\Omega\cap K_0\cap S,K_0\cap S)=\operatorname{ind}_F(\varphi,\Omega,K),$$

where the last equality holds by the permanence property.

The fixed point property follows from Lemma 2.3: Let (Φ,Ω,K) be compactly admissible and such that the inclusion $F(x)\in\Phi(x)\cap\overline{\Omega}$ has no solution. Choose K_0, D, ε and O as in the above definition of the index. Applying Lemma 2.3 with $M_0=K_0\cap\overline{\Omega}$, we may assume that for any (ε,O)-approximation φ for $\Phi|_D$ the equation $F(x)=\varphi(x)\in M_0$ has no solution. Hence, if φ and S is as in the above definition of the index, the fixed point property of the given index implies that the right-hand side of (5) vanishes.

To prove the homotopy invariance, let $H:[0,1]\times D\to 2^Z$ with $D=F^{-1}(\overline{\Omega})$ be such that (H,Ω,K) is compactly F-admissible. Put $K_0=\overline{\operatorname{conv}}\,(H([0,1]\times D))$, and $D_0=F^{-1}(K_0\cap\overline{\Omega})$. By our definition of the index, there is some $\varepsilon>0$ and some neighbourhood $O \subseteq Z$ of 0 such that

$$\text{(6)} \qquad \operatorname{ind}_F(H(\lambda,\,\cdot\,),\Omega,K) = \operatorname{ind}_F(\varphi_\lambda,\Omega\cap K_0\cap S_\lambda,K_0\cap S_\lambda) \qquad (\lambda=0,1)$$

holds whenever $\varphi_\lambda : D \to K_0$ is continuous with values in a finite dimensional space $S_\lambda \subseteq Z$ such that the restriction $\varphi_\lambda|_{D_0}$ is a finite (ε,O)-approximation for $H(\lambda,\,\cdot\,)|_{D_0}$. By Lemma 2.4, we find some $\varepsilon_1 > 0$ and some neighbourhood $O_1 \subseteq Z$ of 0 such that for any (ε_1,O_1)-approximation h for $H|_{[0,1]\times D_0}$ the function $h(\lambda,\,\cdot\,)$ is a finite (ε,O)-approximation for $H(\lambda,\,\cdot\,)|_{D_0}$ $(\lambda=0,1)$. Applying Lemma 2.3 with $M = (\overline{\Omega}\setminus\Omega)\cap K_0$, we find some $\varepsilon_2 > 0$ and some neighbourhood $O_2 \subseteq Z$ of 0 such that for any (ε_2,O_2)-approximation h for $H|_{[0,1]\times D_0}$ the inclusion $F(x) \in h([0,1]\times\{x\})\cap M$ has no solution. By Lemma 2.1, we find some finite $(\min\{\varepsilon_1,\varepsilon_2\},O_1\cap O_2)$-approximation h for $H|_{[0,1]\times D_0}$. Extend h to a continuous function $h : [0,1]\times X \to K_0$ with values in some finite-dimensional subspace $S \subseteq Z$. Then (6) implies

$$\operatorname{ind}_F(H(\lambda,\,\cdot\,),\Omega,K) = \operatorname{ind}_F(h(\lambda,\,\cdot\,),\Omega\cap K_0\cap S,K_0\cap S) \quad (\lambda=0,1).$$

Moreover, the relation $F(x) \in h([0,1]\times\{x\})\cap(\overline{\Omega\cap K_0\cap S})$ implies first $F(x) \notin M$ and then by definition of M also $F(x) \in \Omega\cap K_0\cap S$. Hence, the triple $(h(\lambda,\,\cdot\,),\Omega\cap K_0\cap S,K_0\cap S)$ is finitely F-admissible for $0\le\lambda\le 1$, and so the homotopy invariance of the given index implies

$$\operatorname{ind}_F(h(0,\,\cdot\,),\Omega\cap K_0\cap S,K_0\cap S) = \operatorname{ind}_F(h(1,\,\cdot\,),\Omega\cap K_0\cap S,K_0\cap S).$$

Now (4) follows.

Concerning the permanence property, let (Φ,Ω,K) be compactly F-admissible, and $K_1 \subseteq K$ be closed and convex with $K_0 = \overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega}))) \subseteq K_1$. Choose D, δ, ε, O_0, O, φ, and S as in the above definition of the index. Then (5) holds. Moreover, since $K_0 \subseteq K_1$, we may choose the same values δ, ε, O_0, O, φ, and S in the above definition if we replace Ω by $\Omega\cap K_1$. Hence,

$$\operatorname{ind}_F(\Phi,\Omega\cap K_1,K_1) = \operatorname{ind}_F(\varphi,\Omega\cap K_0\cap S,K_0\cap S),$$

which in view of (5) implies the permanence property.

Now we prove the additivity (the excision property follows analogously with the choice $\Omega_2 = \emptyset$): Let (Φ,Ω_0,K) be compactly F-admissible, and $\Omega_1,\Omega_2 \subseteq \Omega_0$ be disjoint and open in K and such that $F(x) \in \Phi(x)\cap\overline{\Omega}_0$ implies $F(x) \in \Omega_1\cup\Omega_2$. Put $K_0 = \overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega}_0)))$. Choose D_i, δ_i, ε_i, $O_{0,i}$ and O_i as in the above definition of the index for the triple (Φ,Ω_i,K) $(i=0,1,2)$. Applying Lemma 2.3 with $M = K_0\cap\overline{\Omega}_0\setminus(\Omega_1\cup\Omega_2)$, we find some neighbourhood $O \subseteq Z$ of 0 and some $\varepsilon > 0$ such that the relation $F(x) = \varphi(x) \in M$ has no solution for each (ε,O)-approximation φ for $\Phi|_{D_0}$. We may assume that $O \subseteq O_i$ and $\varepsilon \le \varepsilon_i$ $(i=0,1,2)$. In view of Lemma 2.1, we find some function φ such that $\varphi|_{D_i}$ is a finite (ε,O)-approximation for $\Phi|_{D_i}$ $(i=0,1,2)$. Extend φ to a continuous function $\varphi : X \to K_0$ with values in some finite-dimensional subspace $S \subseteq Z$. Then our definition of the index implies

$$\operatorname{ind}_F(\Phi, \Omega_i, K) = \operatorname{ind}_F(\varphi, \Omega_i \cap K_0 \cap S, K_0 \cap S) \quad (i = 0, 1, 2).$$

Since the relation $F(x) = \varphi(x) \in \overline{\Omega_0 \cap K_0 \cap S}$ implies $F(x) = (\Omega_1 \cup \Omega_2) \cap K_0 \cap S$, the additivity of the given index implies

$$\operatorname{ind}_F(\Phi, \Omega_0, K) = \operatorname{ind}_F(\varphi, \Omega_1 \cap K_0 \cap S, K_0 \cap S) + \operatorname{ind}_F(\varphi, \Omega_2 \cap K_0 \cap S, K_0 \cap S),$$

and the additivity of our new index follows. □

3. Weakly admissible maps

In the following, we shall need a stronger form of the permanence property:

Lemma 3.1. *The index* ind_F *from Theorem* 2.2 *has the following stronger permanence property: Let* (Φ, Ω, K) *be compactly* F*-admissible, and* $K_0 \subseteq K$ *be compact and convex with* $\Phi(F^{-1}(\overline{\Omega} \cap K_0)) \subseteq K_0$ *and such that* $F(x) \in \overline{\Omega} \cap \operatorname{conv}(\Phi(x) \cup K_0)$ *implies* $F(x) \in K_0$*. Then* $(\Phi, \Omega \cap K_0, K_0)$ *is compactly* F*-admissible, and* (3) *holds.*

Proof. We have $\Phi : D \to Y$ with $F^{-1}(\overline{\Omega}) \subseteq D \subseteq Y$. By definition, K_0 contains the set $S = \{F(x) : F(x) \in \Phi(x) \cap \overline{\Omega}\}$. Hence, if $K_0 = \emptyset$, we have $S = \emptyset$, and so both sides of (3) vanish by the coincidence point property. Thus, we have only to consider the case $K_0 \neq \emptyset$. Put $D_0 = F^{-1}(\overline{\Omega} \cap K_0)$, and let Φ_0 denote the restriction of Φ to D_0. Then $\Phi_0 : D_0 \to \mathcal{C}(K_0)$ is semicontinuous, and so we may extend Φ_0 to an upper semiconitinuous map $\Phi_0 : D \to \mathcal{C}(K_0)$: Since K_0 is nonempty, compact, and convex, and $D_0 \subseteq D$ is closed, this is possible by Ma's extension theorem [26, Theorem 2.1].

Consider now the upper semicontinuous map $H(\lambda, x) = \lambda\Phi_0(x) + (1-\lambda)\Phi(x)$. Note that $\Phi(x), \Phi_0(x) \in \mathcal{C}(K_0)$ implies $H(\lambda, x) \in \mathcal{C}(K_0)$. We show now that $(H(\lambda, \cdot\,), \Omega, K)$ is compactly F-admissible for $0 \leq \lambda \leq 1$. Indeed, assume that $F(x) \in H([0,1] \times \{x\}) \cap \overline{\Omega}$. Then $F(x) \in \overline{\Omega} \cap \operatorname{conv}(\Phi(\{x\}) \cup K_0)$, and so the assumption implies $F(x) \in K_0$. Hence, $x \in D_0$ which in turn implies $\Phi_0(x) = \Phi(x)$, and so $F(x) \in H([0,1] \times \{x\}) \cap \overline{\Omega} = \Phi(x) \cap \overline{\Omega}$ which implies $F(x) \in \Omega$, because (Φ, Ω, K) is F-admissible. Moreover, $\overline{\operatorname{conv}}\,(H([0,1] \times D)) = \overline{\operatorname{conv}}\,(\Phi(D) \cup K_0) = \overline{\operatorname{conv}}\,(\Phi(D))$ is compact. The homotopy invariance thus implies

$$\operatorname{ind}_F(\Phi, \Omega, K) = \operatorname{ind}_F(H(0, \cdot\,), \Omega, K) = \operatorname{ind}_F(H(1, \cdot\,), \Omega, K) = \operatorname{ind}_F(\Phi_0, \Omega, K),$$

where all sizes are defined. Since $\Phi_0(D) \subseteq K_0$, we find in view of the permanence property and the localization property that

$$\operatorname{ind}_F(\Phi_0, \Omega, K) = \operatorname{ind}_F(\Phi_0, \Omega \cap K_0, K_0) = \operatorname{ind}_F(\Phi, \Omega \cap K_0, K_0).$$

Combining the above equations, we find (3). □

In the situation of Example 2.1 ($F = \mathrm{id}$), the proof of Lemma 3.1 is well-known in principle and is implicitly used in all definitions of a fixed point index for noncompact maps in one form or another. However, we never found an explicit formulation of Lemma 3.1 even for this special case. In the case $F = \mathrm{id}$, the sets K_0 satisfying the assumption of Lemma 3.1 are usually called fundamental for Φ. For reasons that will become clear later, we are interested in a generalization of this definition when we replace $\overline{\Omega}$ by some other set.

Definition 3.1. Let $\Omega \subseteq Z$, and $H : I \times D \to 2^Z$ where I is some nonempty set, and $F^{-1}(\overline{\Omega}) \subseteq D \subseteq X$. Let $K \subseteq Y$ be closed and convex with $H(I \times F^{-1}(\overline{\Omega})) \subseteq K$. Given $V \subseteq K$, we say that a set $U \subseteq K$ is *V-fundamental* for H on $O \subseteq \overline{\Omega}$ (with respect to F and K), if

(i) $\overline{\mathrm{conv}}\, U = U \supseteq V$,
(ii) $H(I \times (F^{-1}(O \cap U))) \subseteq U$, and
(iii) Whenever $(\lambda, x) \in I \times D$ satisfies $F(x) \in O \cap \mathrm{conv}(H(\lambda, x) \cup U)$, then $F(x) \in U$.

We call H *fundamentally V-restrictible on O* (to U), if there is some V-fundamental set U such that $\overline{\mathrm{conv}}\,(H(I \times (F^{-1}(O \cap U))) \cup V)$ is compact.

In case $V = \emptyset$, we call U *fundamental* for H, resp. we call H *fundamentally restrictible.*

As before, we identify functions $\Phi : D \to 2^Z$ with functions $H : I \times D \to 2^Z$.

The most important case in the previous definition is $V = \emptyset$. However, to *verify* that a given function is fundamentally restrictible, it is sometimes convenient to consider also other sets V in view of the following observations:

Lemma 3.2. *The intersection U_0 of any nonempty family $\mathfrak{U}$ of V-fundamental sets (on O) is V-fundamental. Moreover, if U_1 is V-fundamental on O, then*

$$U_2 = \overline{\mathrm{conv}}\,(H(I \times (F^{-1}(O \cap U_1)))) \cup V)$$

is V-fundamental on O and satisfies $U_2 \subseteq U_1$.

Proof. Clearly, $U_0 = \overline{\mathrm{conv}}\, U_0 \supseteq V$. We have for any $U \in \mathfrak{U}$ in view of $U_0 \subseteq U$ that $H(I \times (F^{-1}(O \cap U_0))) \subseteq H(I \times (F^{-1}(O \cap U))) \subseteq U$. Hence, $H(I \times (F^{-1}(O \cap U_0))) \subseteq U_0$. Moreover, if $F(x) \in O \cap \mathrm{conv}(H(\lambda, x) \cup U_0)$, then we have for any $U \in \mathfrak{U}$ that $F(x) \in O \cap \mathrm{conv}(H(\lambda, x) \cup U)$, which implies $F(x) \in U$. Hence, $F(x) \in U_0$.

Since U_1 is V-fundamental on O, we have $H(I \times (F^{-1}(O \cap U_1))) \subseteq U_1$ and $U_1 = \overline{\mathrm{conv}}\, U_1 \supseteq V$. This implies $U_2 \subseteq U_1$. Hence, $H(I \times (F^{-1}(O \cap U_2))) \subseteq H(I \times (F^{-1}(O \cap U_1))) = U_2$. Moreover, if $F(x) \in O \cap \mathrm{conv}(H(\lambda, x) \cup U_2)$, then $F(x) \in O \cap \mathrm{conv}(H(\lambda, x) \cup U_1)$, and so $F(x) \in U_1$. Consequently, $F(x) \in O \cap U_1$, i.e. $x \in F^{-1}(O \cap U_1)$ which in turn implies $F(x) \in O \cap \mathrm{conv}(H(I \times F^{-1}(O \cap U_1)) \cup U_2) \subseteq O \cap \mathrm{conv}(U_2 \cup U_2) \subseteq U_2$, as desired. □

Proposition 3.1. *For each $V \subseteq K$ and each $O \subseteq \overline{\Omega}$ there is a smallest V-fundamental set U_V on O (for H). This set satisfies*

$$U_V = \overline{\operatorname{conv}}\,(H(I \times (F^{-1}(O \cap U_V))) \cup V). \tag{7}$$

The function H is V-fundamentally restrictible if and only if U_V is compact, i.e. if and only if there is some compact V-fundamental set U.

Proof. Let $\mathfrak{U}$ denote the family of all V-fundamental sets. Since $K \in \mathfrak{U}$, we have $\mathfrak{U} \neq \emptyset$, and so Lemma 3.2 implies that $U_V = \bigcap \mathfrak{U}$ is the smallest V-fundamental set. Lemma 3.2 implies also that the set $U_2 = \overline{\operatorname{conv}}\,(H(I \times (F^{-1}(\overline{\Omega \cap U_V}))) \cup V)$ is V-fundamental and satisfies $U_2 \subseteq U_V$. Since U_V is the *smallest* V-fundamental set, we also have the converse inclusion $U_V \subseteq U_2$. The second statement is an immediate consequence of (7). □

Definition 3.2. Let I be some metric space, $K \subseteq Y$ closed and convex, $\Omega \subseteq K$ open in K, and $H : I \times D \to 2^Z$ with $D_0 = F^{-1}(\overline{\Omega}) \subseteq D \subseteq X$ such that the restriction $H : I \times D_0 \to \mathcal{C}(Y)$ is upper semicontinuous, and $H(I \times D_0) \subseteq K$. Then we call the triple (H, Ω, K) *weakly F-admissible*, if there is some set $\Omega_0 \subseteq \Omega$ which is open in K such that the relation $F(x) \in H(I \times \{x\}) \cap \overline{\Omega}$ implies $F(x) \in \Omega_0$ and such that H is fundamentally restrictible on $\overline{\Omega}_0$. If even the choice $\Omega_0 = \Omega$ is possible, then (H, Ω, K) is called *F-admissible*.

We point out that Ω and Ω_0 may also be unbounded.

Clearly, each compactly F-admissible triple is F-admissible, and each F-admissible triple is weakly F-admissible.

In the classical situation $F = \mathrm{id}$ of Example 2.1, the novelty of *weakly* id-admissible triples lies in the fact that we do not require that H is fundamentally restrictible on $\overline{\Omega}$ but only on the possibly smaller set $\overline{\Omega}_0$ (which is of course a weaker condition). We will see in the proof of Theorems 4.1 and 4.3, how one can take advantage of this fact.

Now we are in a position to formulate our main result. We note that the strong permanence property proved below contains Lemma 3.1 as a special case. Moreover, this property implies that the set K_0 in Lemma 3.1 need actually not be compact.

Theorem 3.1. *Assume that $F : X \to Y$ is continuous and proper and provides a coincidence index ind_F (resp. a coincidence index which satisfies the excision property). Then ind_F has an extension to a coincidence index ind_F defined on all (weakly) F-admissible triples such that for each (weakly) F-admissible triple (Φ, Ω, K) the following properties are satisfied:*

(i) (Localization) *If $\Psi : D_0 \to 2^Z$ with $F^{-1}(\overline{\Omega}) \subseteq D_0 \subseteq X$ and $\Psi|_{F^{-1}(\overline{\Omega})} = \Phi|_{F^{-1}(\overline{\Omega})}$, then (Ψ, Ω, K) is F-admissible, and*

$$\mathrm{ind}_F(\Phi, \Omega, K) = \mathrm{ind}_F(\Psi, \Omega, K).$$

(ii) (Coincidence point property) *If* $\operatorname{ind}_F(\Phi,\Omega,K)\neq 0$, *then the inclusion* $F(x)\in\Phi(x)\cap\Omega$ *has a solution.*

(iii) (Normalization) *If* $\Phi(x)\equiv\{c\}\in\Omega$, *then* $\operatorname{ind}_F(\Phi,\Omega,K)=1$.

(iv) (Homotopy invariance) *If* (H,Ω,K) *is* (*weakly*) F*-admissible, then, for* $0\leq\lambda\leq 1$, $(H(\lambda,\,\cdot\,),\Omega,K)$ *is* (*weakly*) F*-admissible and*

$$\operatorname{ind}_F(H(0,\,\cdot\,),\Omega,K)=\operatorname{ind}_F(H(1,\,\cdot\,),\Omega,K). \tag{8}$$

(v) (Strong permanence) *If* $K_0\subseteq K$ *is fundamental for* Φ *on* $\overline{\Omega}$, *then* $(\Phi,\Omega\cap K_0,K_0)$ *is* (*weakly*) F*-admissible, and*

$$\operatorname{ind}_F(\Phi,\Omega,K)=\operatorname{ind}_F(\Phi,\Omega\cap K_0,K_0). \tag{9}$$

If ind_F *satisfies the excision property, we may consider throughout weakly* F*-admissible triples, and* ind_F *also has the following properties in this case:*

(vi) (Extended permanence) *If there are sets* $K_0\subseteq K$ *and* $\Omega_0\subseteq\Omega$ *such that* Ω_0 *is open in* K, *the relation* $F(x)\in\Phi(x)\cap\overline{\Omega}$ *implies* $F(x)\in\Omega_0$, *and if* K_0 *is fundamental for* Φ *on* $\overline{\Omega}_0$ *and* $\Phi(F^{-1}(\overline{\Omega\cap K_0}))\subseteq K_0$, *then* $(\Phi,\Omega\cap K_0,K_0)$ *is weakly* F*-admissible, and* (9) *holds.*

(vii) (Excision) *If* (Φ,Ω,K) *is weakly* F*-admissible and* $\Omega_0\subseteq K$ *is open in* K *and such that* $F(x)\in\Phi(x)\cap\overline{\Omega}$ *implies* $F(x)\in\Omega_0$, *then* (Φ,Ω_0,K) *is weakly* F*-admissible, and*

$$\operatorname{ind}_F(\Phi,\Omega,K)=\operatorname{ind}_F(\Phi,\Omega_0,K).$$

If the given index ind_F *is even additive, we also have:*

(viii) (Additivity) *If* $\Omega_1,\Omega_2\subseteq K$ *are disjoint and open in* K *and such that* $F(x)\in\Phi(x)\cap\overline{\Omega}$ *implies* $F(x)\in\Omega_1\cup\Omega_2$, *then* (Φ,Ω_1,K) *and* (Φ,Ω_2,K) *are weakly* F*-admissible, and*

$$\operatorname{ind}_F(\Phi,\Omega,K)=\operatorname{ind}_F(\Phi,\Omega_1,K)+\operatorname{ind}_F(\Phi,\Omega_2,K). \tag{10}$$

Proof. The index for a (weakly) F-admissible triple (Φ,Ω,K) is defined as follows: Put $S=\{F(x):F(x)\in\Phi(x)\cap\overline{\Omega}\}$. By assumption, we find some $\Omega_0\subseteq\Omega$ which is open in K such that $\Omega_0\supseteq S$ and such that Φ is fundmantally restrictible on $\overline{\Omega}_0$ to some set U. If ind_F does not satisfy the excision property (and thus we consider only F-admissible triples), we require $\Omega_0=\Omega$.

There exists some convex and compact set $K_0\subseteq K$ with $K_0\supseteq\Phi(F^{-1}(\overline{\Omega}_0\cap U))$ such that $\Phi(F^{-1}(\overline{\Omega}_0\cap K_0))\subseteq K_0$. Indeed, one possible choice is $K_0=\overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega}_0\cap U)))$, because Lemma 3.2 implies for this choice $K_0\subseteq U$. For such a set K_0, we define

$$\begin{aligned}\operatorname{ind}_F(\Phi,\Omega,K)&:=\operatorname{ind}_F(\Phi,\Omega_0\cap K_0,K_0)\\&=\operatorname{ind}_F(\Phi|_{F^{-1}(\overline{\Omega_0\cap K_0})},\Omega_0\cap K_0,K_0),\end{aligned} \tag{11}$$

where the index on the right-hand side is the index for compactly F-admissible triples from Theorem 2.2. Note that the triple $(\Phi|_{F^{-1}(\overline{\Omega_0 \cap K_0})}, \Omega_0 \cap K_0, K_0)$ is in fact compactly F-admissible (and so the second equality in (11) holds by the localization property). Indeed, $\Omega_1 := \Omega_0 \cap K_0$ is open in K_0, and the range of the function $\Psi = \Phi|_{F^{-1}(\overline{\Omega_0 \cap K_0})}$ is contained in $\Phi(F^{-1}(\overline{\Omega}_0 \cap K_0)) \subseteq K_0$ (note that K_0 is compact and convex). Finally, the relation $F(x) \in \Psi(x) \cap \overline{\Omega}_1$ implies $F(x) \in S \cap \overline{\Omega}_1 \subseteq \Omega_0 \cap K_0 = \Omega_1$.

The above definition is independent of the particular choice of Ω_0, U, and K_0. Let Ω_1, U_1, and K_1 be some (possibly different) sets with open $\Omega_1 \subseteq K$, $S \subseteq \Omega_1 \subseteq \Omega$ such that U_1 is fundamental for Φ on $\overline{\Omega}_1$, $K_1 \subseteq Y$ is convex and compact with $K_1 \supseteq \Phi(F^{-1}(\overline{\Omega}_1 \cap U_1))$ and $\Phi(F^{-1}(\overline{\Omega}_1 \cap K_1)) \subseteq K_1$ (assume $\Omega_1 = \Omega$ if ind_F does not satisfy the excision property).

The set $\Omega_2 = \Omega_0 \cap \Omega_1$ is open in K with $S \subseteq \Omega_2$. The excision property of ind_F thus implies

$$\operatorname{ind}_F(\Phi, \Omega_i \cap K_i, K_i) = \operatorname{ind}_F(\Phi, \Omega_2 \cap K_i, K_i) \quad (i = 0, 1). \tag{12}$$

If ind_F does not satisfy the excision property, we have $\Omega_i = \Omega_2$, and so (12) holds also. Put $U_2 = U \cap U_1$. Since $\overline{\Omega}_2 \subseteq \overline{\Omega}_0 \cap \overline{\Omega}_1$, the sets U and U_1 are both fundamental on $\overline{\Omega}_2$. Lemma 3.2 thus implies that U_2 and then also $K_2 = \overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega}_2 \cap U_2)))$ is fundamental on $\overline{\Omega}_2$. The latter implies in particular $\Phi(F^{-1}((\overline{\Omega_2 \cap K_i}) \cap K_2)) \subseteq \Phi(F^{-1}(\overline{\Omega}_2 \cap K_2)) \subseteq K_2$ $(i = 0, 1)$, and the relation $F(x) \in (\overline{\Omega_2 \cap K_i}) \cap \operatorname{conv}(\Phi(\{x\}) \cup K_2)$ implies $F(x) \in \overline{\Omega}_2 \cap \operatorname{conv}(\Phi(\{x\}) \cup K_2)$ which in turn implies $F(x) \in K_2$. Since $(\Phi, \Omega_2 \cap K_i, K_i)$ is compactly F-admissible (recall (12)), we find by Lemma 3.1 that

$$\operatorname{ind}_F(\Phi, \Omega_2 \cap K_i, K_i) = \operatorname{ind}_F(\Phi, (\Omega_2 \cap K_i) \cap K_2, K_i \cap K_2) \qquad (i = 0, 1). \tag{13}$$

Note that $K_2 \subseteq \overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega}_0 \cap U \cap \overline{\Omega}_1 \cap U_1))) \subseteq K_0 \cap K_1$. Hence, (12) and (13) together imply

$$\operatorname{ind}_F(\Phi, \Omega_0 \cap K_0, K_0) = \operatorname{ind}_F(\Phi, \Omega_2 \cap K_2, K_2) = \operatorname{ind}_F(\Phi, \Omega_1 \cap K_1, K_1).$$

This shows that the definition of the index is in fact independent of the particular choice of the sets Ω_0, U, and K_0.

The localization property of the newly defined index follows immediately from the definition. Moreover, if (Φ, Ω, K) is compactly F-admissible, we may choose $\Omega_0 = \Omega$, $U = K$, and $K_0 = \overline{\operatorname{conv}}\,(\Phi(\overline{\Omega}))$; by the above definition, we have

$$\operatorname{ind}_F(\Phi, \Omega, K) = \operatorname{ind}_F(\Phi, \Omega \cap K_0, K_0),$$

and the permanence property thus shows that our newly defined index ind_F is in fact an extension of the given index (i.e. with the same values for compactly F-admissible triples). In particular, the use of the same symbol ind_F for the new index is justified, and ind_F satisfies the normalization property. The coincidence point property is almost trivial: If $\operatorname{ind}_F(\Phi,\Omega,K)\neq 0$, choose $\Omega_0\subseteq\Omega$ and K_0 as in the definition of the index above. Then (11) implies $\operatorname{ind}_F(\Phi,\Omega_0\cap K_0,K_0)\neq 0$, and by the coincidence point property for compactly F-admissible triples, the equation $F(x)\in\Phi(x)\cap(\overline{\Omega_0\cap K_0})$ has a solution, and thus also $F(x)\in\Phi(x)\cap\overline{\Omega}$.

To see the homotopy invariance, let H be a homotopy such that (H,Ω,K) is weakly F-admissible (resp. F-admissible if ind_F does not satisfy the excision property). By assumption, we find some $\Omega_0\subseteq\Omega$ which is open in K and which contains $S=\{F(x):F(x)\in H([0,1]\times\{x\})\cap\overline{\Omega}\}$ such that H is fundamentally restrictible on $\overline{\Omega}_0$ to some set U (if ind_F does not satisfy the excision property, put $\Omega_0=\Omega$). Lemma 3.2 implies that $K_0=\overline{\operatorname{conv}}\,(H([0,1]\times(F^{-1}(\overline{\Omega}_0\cap U))))$ satisfies $K_0\subseteq U$. Thus, we have for fixed $\lambda\in[0,1]$ that $\overline{\operatorname{conv}}\,(H(\{\lambda\}\times(F^{-1}(\overline{\Omega}_0\cap K_0))))\subseteq K_0$. Since additionally $\Omega_0\subseteq\Omega$ contains $\{F(x):F(x)\in H(\lambda,x)\cap\overline{\Omega}\}$ and U is fundamental for each $H(\lambda,\,\cdot\,)$ on $\overline{\Omega}_0$, we thus find by our definition of ind_F that

$$\operatorname{ind}_F(H(\lambda,\,\cdot\,),\Omega,K)=\operatorname{ind}_F(H(\lambda,\,\cdot\,),\Omega_0\cap K_0,K_0),$$

where the triple $(H(\lambda,\,\cdot\,),\Omega_0\cap K_0,K_0)$ is compactly F-admissible. In particular, $H([0,1]\times F^{-1}(\overline{\Omega_0\cap K_0}))\subseteq K_0$, and since K_0 is compact and convex, the homotopy invariance for compactly F-admissible triples implies

$$\operatorname{ind}_F(H(0,\,\cdot\,),\Omega_0\cap K_0,K_0)=\operatorname{ind}_F(H(1,\,\cdot\,),\Omega_0\cap K_0,K_0).$$

Hence, (8) follows.

Now we prove the strong permanence property (and the extended permanence property if ind_F satisfies the excision property): Let (Φ,Ω,K) be (weakly) F-admissible, and $\Omega_0\subseteq\Omega$ be open in K and contain $S=\{F(x):F(x)\in\Phi(x)\cap\overline{\Omega}\}$ (if ind_F does not satisfy the excision property, assume $\Omega_0=\Omega$). Moreover, assume that $K_0\subseteq K$ is fundamental for Φ on $\overline{\Omega}_0$ and satisfies $\Phi(F^{-1}(\overline{\Omega\cap K_0}))\subseteq K_0$.

Since (Φ,Ω,K) is (weakly) F-admissible, there is some $\Omega_1\subseteq\Omega$ which is open in K and contains S such that Φ is fundamentally restrictible to some set U_1 (assume $\Omega_1=\Omega$ if ind_F does not satisfy the excision property). Putting $K_1=\overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega}_1\cap U_1)))$, we find by the definition of the index that

$$\operatorname{ind}_F(\Phi,\Omega,K)=\operatorname{ind}_F(\Phi,\Omega_1\cap K_1,K_1). \tag{14}$$

The set $\Omega_2=\Omega_0\cap\Omega_1$ is open in K, and we have $S\subseteq\Omega_2$. In particular, the relation $F(x)\in\Phi(x)\cap(\overline{\Omega_1\cap K_1})$ implies $F(x)\in\Omega_2\cap K_1$. The excision property of the index for compactly F-admissible triples thus implies

$$\operatorname{ind}_F(\Phi,\Omega_1\cap K_1,K_1)=\operatorname{ind}_F(\Phi,\Omega_2\cap K_1,K_1) \tag{15}$$

(if ind_F does not satisfy the excision property, this equality is trivial, since then $\Omega_1=\Omega_2$). Let us now prove that the triple $(\Phi,\Omega\cap K_0,K_0)$ is (weakly) F-admissible. Recall first that by assumption $\Phi(F^{-1}(\overline{\Omega\cap K_0}))\subseteq K_0$. Moreover, the relation $F(x) \in \Phi(x) \cap (\overline{\Omega \cap K_0})$ implies $F(x) \in S \cap K_0 \subseteq \Omega_2 \cap K_0$. The sets U_1 and K_0 are both fundamental on $\overline{\Omega_2 \cap K_0} \subseteq \overline{\Omega}_0 \cap \overline{\Omega}_1$, and so Lemma 3.2 shows that $U_2 = U_1 \cap K_0$ is fundamental on $\overline{\Omega_2 \cap K_0}$. Finally, since $U_2 \subseteq U_0 \cap U_1$, the set $K_2 = \overline{\operatorname{conv}}\,(\Phi(F^{-1}(\overline{\Omega}_2 \cap U_2)))$ is contained in $K_0 \cap K_1$ and thus in particular compact. Hence, the triple $(\Phi, \Omega \cap K_0, K_0)$ is in fact (weakly) F-admissible. Moreover, the definition of the index implies

$$\operatorname{ind}_F(\Phi, \Omega \cap K_0, K_0) = \operatorname{ind}_F(\Phi, \Omega_2 \cap K_2, K_2). \tag{16}$$

The triple $(\Phi, \Omega_2 \cap K_1, K_1)$ is compactly F-admissible (recall (15)). Since U_2 is fundamental on $\overline{\Omega}_2$, Lemma 3.2 implies that also K_2 is fundamental on $\overline{\Omega}_2$. In particular, K_2 is fundamental on $\overline{\Omega_2 \cap K_1} \subseteq \overline{\Omega}_2$. Lemma 3.1 thus implies

$$\operatorname{ind}_F(\Phi, \Omega_2 \cap K_1, K_1) = \operatorname{ind}_F(\Phi, \Omega_2 \cap K_2, K_2). \tag{17}$$

Combining the above equations (14)–(17), we find (9).

Now we prove the additivity of the index (the proof of the excision property is analogous with $\Omega_2 = \emptyset$ in the following arguments): Let (Φ, Ω, K) be weakly F-admissible. Put $S = \{F(x) : F(x) \in \Phi(x) \cap \overline{\Omega}\}$. By assumption, we find some $\Omega_0 \subseteq \Omega$ which is open in K such that $\Omega_0 \supseteq S$ and such that Φ is fundamantally restrictible on $\overline{\Omega}_0$ to some set U. By the definition of the index, we find some convex and compact set $K_0 \subseteq K$ with $K_0 \supseteq \Phi(F^{-1}(\overline{\Omega}_0 \cap U))$ and $\Phi(F^{-1}(\overline{\Omega}_0 \cap K_0)) \subseteq K_0$, and then (11) holds. Let $\Omega_1, \Omega_2 \subseteq K$ be disjoint and open in K with $S \subseteq \Omega_1 \cup \Omega_2$. For $i = 1, 2$, we have then $(\Omega_1 \cup \Omega_2) \cap \overline{\Omega}_i \subseteq \Omega_i$. The set $\Omega_{i,0} = \Omega_i \cap \Omega_0$ is open in K, and $S \cap \overline{\Omega}_i \subseteq (\Omega_0 \cap (\Omega_1 \cup \Omega_2)) \cap \overline{\Omega}_i \subseteq \Omega_0 \cap \Omega_i = \Omega_{0,i}$. Since $\overline{\Omega}_{0,i} \subseteq \overline{\Omega}_0$, the function Φ is fundamentally restrictible on $\overline{\Omega}_{0,i}$ to U, and so the triple (Φ, Ω_i, K) is F-admissible. Moreover, $\overline{\Omega}_{0,i} \subseteq \overline{\Omega}_0$ also implies $K_0 \supseteq \Phi(F^{-1}(\overline{\Omega}_{i,0} \cap U))$ and $\Phi(F^{-1}(\overline{\Omega}_{i,0} \cap K_0)) \subseteq K_0$. The definition of the index thus shows that

$$\operatorname{ind}_F(\Phi, \Omega_i, K) = \operatorname{ind}_F(\Phi, \Omega_{i,0} \cap K_0, K_0) \quad (i = 1, 2).$$

The sets $\Omega_{i,1} = \Omega_{i,0} \cap K_0$ are open in K_0. Moreover, the relation $F(x) \in \Phi(x) \cap (\overline{\Omega_0 \cap K_0})$ implies $F(x) \in S \cap K_0 \subseteq (S \cap (\Omega_1 \cup \Omega_2) \cap \Omega_0) \cap K_0 \subseteq \Omega_{1,1} \cup \Omega_{2,1}$. Hence, the additivity of the index for compactly F-admissible triples implies

$$\operatorname{ind}_F(\Phi, \Omega_0 \cap K_0, K_0) = \operatorname{ind}_F(\Phi, \Omega_{1,1}, K_0) + \operatorname{ind}_F(\Phi, \Omega_{2,1}, K_0).$$

In view of (11), the above equalities imply (10). □

Remark 3.1. Our proof actually shows the following extension of Theorems 2.2 and 3.1: Let $\mathcal{O}$ be a system of subsets of Y with the following properties

(i) If $\Omega_1, \Omega_2 \in \mathcal{O}$, then $\Omega_1 \cap \Omega_2 \in \mathcal{O}$.
(ii) If $K \subseteq Y$ is closed and convex and $\Omega \cap K$ is open in K, then $\Omega \cap K \in \mathcal{O}$.

For example, $\mathcal{O}$ may be the system of all convex subsets of Y or, more general, the system of finite unions of convex subsets of Y. For many natural functions F, the set $F^{-1}(\overline{\Omega})$ is then always an ANR for $\Omega \in \mathcal{O}$ (recall the remarks in front of Lemma 2.1).

Call a finitely F-admissible triple (φ, Ω, K) *finitely* $(F, \mathcal{O})$*-admissible*, if $\Omega \in \mathcal{O}$. Similarly, call a compactly F-admissible triple (Φ, Ω, K) *compactly* $(F, \mathcal{O})$*-admissible*, if $\Omega \in \mathcal{O}$, and a (weakly) F-admissible triple (Φ, Ω, K) *(weakly)* $(F, \mathcal{O})$*-admissible*, if $\Omega \in \mathcal{O}$ and additionally the set $\Omega_0 \subseteq \Omega$ in Definition 3.2 may be chosen such that $\Omega_0 \in \mathcal{O}$.

Assume that the index from Definition 2.1 is only defined for finitely $(F, \mathcal{O})$-admissible triples (of course, the additivity, resp. the excision property, is then required only for sets $\Omega_i \in \mathcal{O}$). Then Theorem 2.2 still holds for compactly $(F, \mathcal{O})$-admissible triples, and Theorem 3.1 holds for (weakly) $(F, \mathcal{O})$-admissible triples (for the extended permanence property and for the additivity, resp. excision property, we require additionally that $\Omega_i \in \mathcal{O}$).

REMARK 3.2. Drop for a moment the general axiom of choice, and consider instead a weaker form, the axiom of dependent choices [20], which allows countably many recursive or nonrecursive choices. Then all results in this paper still hold if we assume in addition that Y has the following "continuous extension property" (which is required in the proof of Lemma 3.1 to drop Ma's extension theorem):

If $D_0 \subseteq X$ is compact, and $\Phi : D_0 \to \mathcal{C}(Y)$ is upper semicontinuous and $K_0 \subseteq Y$ is nonempty, compact and convex with $\Phi(D_0) \subseteq K_0$, then Φ has an extension to an upper semicontinuous function $\Phi : X \to \mathcal{C}(K_0)$.

If Z is metrizable, this property is always satisfied [34] (observe that D_0 is separable and K_0 is complete). This is the reason why we formulated Lemma 3.1 only for *compact* fundamental sets K_0. It is somewhat surprising that the strong permanence property in Theorem 3.1 then implies that actually Lemma 3.1 is valid even if K_0 is not compact.

If one is only interested in a coincidence index for *single-valued* functions Φ, one could replace the above "continuous extension property" by the requirement that for each nonempty convex and compact set $K_0 \subseteq Y$ there exists a retraction $\rho : Y \to K_0$ onto K_0. In fact, for single-valued Φ we can then in the proof Lemma 3.1 just put $\Phi_0 = \rho \circ \Phi$ (because then $\Phi_0|_{D_0} = \Phi|_{D_0}$). Note that this argument fails for multivalued Φ, because it is not clear whether $\rho \circ \Phi$ attains convex values.

4. Countable compactness conditions

We provide now some convenient tests which allow to verify that a given triple (H, Ω, K) is (weakly) F-admissible:

We consider the following situation: Let X be some metric space, Z some locally convex *metrizable* space, $Y \subseteq X$ closed and convex, and assume that $F : X \to Y$ is continuous and proper and provides some coincidence index. Let $K \subseteq Y$ be closed and convex, $\Omega \subseteq K$ open in K, and I be a compact metric space. Let $H : I \times D \to \mathcal{K}(K)$ be upper semicontinuous where $\mathcal{K}(K)$ denotes the system of all nonempty convex compact subsets of K.

The following result is the most important test for weak F-admissibility if $V = \emptyset$ and $U = K$:

Theorem 4.1. *Consider the above situation. Suppose that the relation* $F(x) \in H(I \times \{x\}) \cap \overline{\Omega}$ *implies* $F(x) \in \Omega$.

Let $V \subseteq K$ *be such that* $\overline{\mathrm{conv}}\, V$ *is compact, and assume there is some set* $U \subseteq K$ *which is* V*-fundamental on* Ω *with the following property: For any countable* $C \subseteq F^{-1}(U \cap \Omega)$ *the relation*

$$\overline{F(C)} = \overline{\mathrm{conv}(H(I \times C) \cup V) \cap \Omega} \tag{18}$$

implies that $\overline{\mathrm{conv}}\,(H(I \times C))$ *is compact. Then the triple* (H, Ω, K) *is weakly* F*-admissible. If* Z *is even a Fréchet space* (*i.e. complete*)*, one may alternatively assume that* (18) *implies that* $\overline{C}$ *is compact.*

If we want to conclude that (H, Ω, K) is even F-admissible, we have to replace (18) by a less natural inclusion; moreover, we have to consider subsets C of $F^{-1}(U \cap \overline{\Omega})$ (and not only of $F^{-1}(U \cap \Omega)$). This is the price we have to pay if we do not want to use the more technical condition of *weakly* F-admissible triples in our general theory of Section 3 (we are forced to do so if the index does not satisfy the excision property):

Theorem 4.2. *Consider the situation described at the beginning of this section. Suppose that the relation* $F(x) \in H(I \times \{x\}) \cap \overline{\Omega}$ *implies* $F(x) \in \Omega$. *Let* $V \subseteq K$ *be such that* $\overline{\mathrm{conv}}\, V$ *is compact, and assume there is some set* $U \subseteq K$ *which is* V*-fundamental for* H *on* $\overline{\Omega}$ *with the following property: For any countable* $C \subseteq F^{-1}(U \cap \overline{\Omega})$ *the relation*

$$\mathrm{conv}(H(I \times C) \cup V) \cap \overline{\Omega} \subseteq \overline{F(C)} \subseteq \overline{\mathrm{conv}}\,(H(I \times C) \cup V) \cap \overline{\Omega} \tag{19}$$

implies that $\overline{\mathrm{conv}}\,(H(I \times C))$ *is compact. Then the triple* (H, Ω, K) *is* F*-admissible. If* Z *is even a Fréchet space, one may alternatively assume that* (19) *implies that* $\overline{C}$ *is compact.*

We note that Theorem 4.2 is sharp in the sense that if (H, Ω, K) is F-admissible, then Proposition 3.1 implies that there is a fundamental set U for which $\overline{\mathrm{conv}}\,(H(I \times F^{-1}(U \cap \overline{\Omega}))) = U$ is compact, and so the compactness assumptions of Theorem 4.2 are trivially satisfied.

Roughly speaking, one might interpret the conditions of the previous results as conditions on the map $F^{-1}\circ H$. One may also formulate the conditions in terms of the map $H(I\times F^{-1}(\,\cdot\,))$. This is somewhat more technical. In the following result, one should think of only a small number of different sets G_i, say $G_1=Y$ and possibly $G_2=\Omega$, $G_3=\overline{\Omega}$.

THEOREM 4.3. *Consider the situation described at the beginning of this section. Suppose that the relation* $F(x)\in H(I\times\{x\})\cap\overline{\Omega}$ *implies* $F(x)\in\Omega$. *Let* $V\subseteq K$ *be precompact, and assume there is some set* $U\subseteq K$ *which is* V*-fundamental on* Ω *with the following property: There are sets* $G_1,G_2,\ldots\subseteq Y$ *such that* $G_n\cap H(I\times(\overline{F^{-1}(\{u\})\cap M}))$ *is separable and such that for any countable* $C\subseteq U$ *the relations*

$$C\subseteq\overline{\operatorname{conv}}\,(H(I\times(F^{-1}(C\cap\Omega)))\cup V), \tag{20}$$

$$G_n\cap\operatorname{conv}(H(I\times(F^{-1}(C\cap\Omega)))\cup V)\subseteq\overline{G_n\cap C}\quad(n=1,2,\ldots) \tag{21}$$

imply that $\overline{C}$ *is compact. Then* (H,Ω,K) *is weakly* F*-admissible. If* Z *is even a Fréchet space, one may alternatively assume that* (20) *and* (21) *imply that* $\overline{F^{-1}(C\cap\Omega)}$ *is compact.*

The analogous result for F-admissible triples reads as follows:

THEOREM 4.4. *Consider the situation described at the beginning of this section. Suppose that the relation* $F(x)\in H(I\times\{x\})\cap\overline{\Omega}$ *implies* $F(x)\in\Omega$. *Let* $V\subseteq K$ *be precompact, and assume there is some set* $U\subseteq K$ *which is* V*-fundamental on* $\overline{\Omega}$ *with the following property: There are sets* $G_1,G_2,\ldots\subseteq Y$ *such that* $G_n\cap H(I\times(\overline{F^{-1}(\{u\})\cap M}))$ *is separable for each* $u\in U$ *and each* n *and such that for any countable* $C\subseteq U$ *the relations*

$$C\subseteq\overline{\operatorname{conv}}\,(H(I\times(F^{-1}(C\cap\overline{\Omega})))\cup V), \tag{22}$$

$$G_n\cap\operatorname{conv}(H(I\times(F^{-1}(C\cap\overline{\Omega})))\cup V)\subseteq\overline{G_n\cap C}\quad(n=1,2,\ldots) \tag{23}$$

imply that $\overline{C}$ *is compact. Then* (H,Ω,K) *is* F*-admissible. If* Z *is even a Fréchet space, one may alternatively assume that* (22) *and* (23) *imply that* $\overline{F^{-1}(C\cap\overline{\Omega})}$ *is compact.*

For $F=\mathrm{id}$ and in Fréchet spaces, Theorems 4.2 and 4.4 reduce to the main results from [36].

For applications, the second inclusion from (19) is the most important one. To understand this, let us formulate the condition from Theorems 4.1 and 4.2 in terms of measures of noncompactness: We call a function γ a *monotone measure of noncompactness on* K (in the sense of [1], [32]) if γ associates to each $A\subseteq K$ a value $\gamma(A)$ in some partially ordered set with the properties:

(i) $\gamma(A)=\gamma(\overline{\operatorname{conv}}\,A)$, and

(ii) $\gamma(A_1) \leq \gamma(A_2)$ whenever $A_1 \subseteq A_2$.

We say that H is *countably F-compact* on O, if for each countable $C \subseteq F^{-1}(O)$ which is not precompact there is some monotone measure of noncompactness γ on K such that $\gamma(H(I \times C)) \not\geq \gamma(F(C))$. Roughly speaking, this condition means that H is "more compact than F is proper". For example, in the situation of Example 2.1 (i.e. $F = \mathrm{id}$), this condition is satisfied if O is bounded and H is countably condensing with respect to the family of monotone measures of noncompactness (in the sense of [36]).

Corollary 4.1. *Consider the situation described at the beginning of this section with a Fréchet space Z. Suppose that the relation $F(x) \in H(I \times \{x\}) \cap \overline{\Omega}$ implies $F(x) \in \Omega$. Then we have:*

(i) *If H is countably F-compact on Ω, then (H, Ω, K) is weakly F-admissible.*

(ii) *If H is countably F-compact on $\overline{\Omega}$, then (H, Ω, K) is F-admissible.*

Proof. Apply Theorem 4.1 (resp. Theorem 4.2) with $V = \emptyset$. Let a countable set $C \subseteq F^{-1}(\Omega)$ (resp. $F^{-1}(\overline{\Omega})$) satisfy (18) (resp. (19)). Then we have $F(C) \subseteq \overline{\mathrm{conv}}\,(H(I \times C))$, and so we have for any monotone measure of noncompactness γ that

$$\gamma(F(C)) \leq \gamma(\overline{\mathrm{conv}}\,(H(I \times C))) = \gamma(H(I \times C)).$$

Since H is countably F-compact, this implies that C is precompact, and so $\overline{C}$ is compact, as desired. □

To prove Theorems 4.1–4.4, we prove some "more general" results which are of independent interest (we will use them also in some other papers).

We call a subset M of a (not necessarily complete) metric space *precompact* if its completion is compact, i.e. if any sequence in M contains a Cauchy subsequence. If even $\overline{M}$ is compact, i.e. if any sequence in M contains a convergent subsequence, we call M *relatively compact*.

A modification of the following result has been implicitly proved in [34]. For the reader's convenience, we recall the proof.

We use the notation $F^{-1}(M) = \{x : F(x) \in M\}$, even if the set M is not necessarily contained in the range of F.

Proposition 4.1. *Let X be a metric space, Z a locally convex metric space, and $F : X \to Z$. Let I be some set, $D \subseteq X$, and $H : I \times D \to 2^Z$ such that any $H(I \times \{x\})$ is separable. Let $V \subseteq Z$ be separable, and $M, N \subseteq X$ with $M \cap N \subseteq D$ satisfy*

$$F^{-1}(\overline{\mathrm{conv}}\,(H(I \times (M \cap N)) \cup V)) = M. \tag{24}$$

Moreover, assume that for any countable $C \subseteq M \cap N$ *the relation*

$$\operatorname{conv}(H(I \times C) \cup V) \cap F(N) \subseteq \overline{F(C)} \subseteq \overline{\overline{\operatorname{conv}}\,(H(I \times C) \cup V) \cap F(N)} \tag{25}$$

implies that $\overline{\operatorname{conv}}\,(H(I \times C) \cup V)$ *is compact* (*resp.* C *is precompact, relatively compact*). *Then* $\overline{\operatorname{conv}}\,(H(I \times (M \cap N)))$ *is compact* (*resp.* $M \cap N$ *is precompact, relatively compact*).

Proof. Assume that $\overline{\operatorname{conv}}\,(H(I \times (M \cap N))$ is not compact. Then this set contains a sequence $y_1, y_2, \dots$ without a convergent subsequence. Since each y_n is the limit of a sequence of (finite) convex combinations of elements from $H(I \times (M \cap N))$, we find some countable set $C_0 \subseteq M \cap N$ such that $y_1, y_2, \dots \in H(I \times C_0)$. Similarly, if $M \cap N$ is not precompact (resp. not relatively compact), we find some countable $C_0 \subseteq M \cap N$ which is not precompact (resp. relatively compact). The statement follows in all cases, if we can show that there is a countable set $C \supseteq C_0$ which is contained in $M \cap N$ and additionally satisfies (25). To construct this set C, we define by induction on $n = 0, 1, \dots$ countable sets C_n according to the following conditions:

$$C_n \subseteq C_{n+1} \subseteq M \cap N, \tag{26}$$

$$F(C_n) \subseteq \overline{\operatorname{conv}}\,(H(I \times C_{n+1}) \cup V), \tag{27}$$

$$\overline{\operatorname{conv}}\,(H(I \times C_n) \cup V) \cap F(N) \subseteq \overline{F(C_{n+1})}. \tag{28}$$

This is possible: If C_n is already defined, we have by (24) that $C_n \subseteq M \subseteq F^{-1}(\overline{\operatorname{conv}}\,(H(I \times (M \cap N)) \cup V))$. Hence, $F(C_n) \subseteq \overline{\operatorname{conv}}\,(H(I \times (M \cap N)) \cup V)$. Thus, any of the countable many elements from $F(C_n)$ is the limit of a sequence of (finite) convex combinations of elements from $H(I \times (M \cap N)) \cup V$. We thus find a countable $A_n \subseteq M \cap N$ such that $F(C_n) \subseteq \overline{\operatorname{conv}}\,(H(I \times A_n) \cup V)$. Note that (24) implies in view of (26) that the set $H_n = \overline{\operatorname{conv}}\,(H(I \times C_n) \cup V)$ satisfies $F^{-1}(H_n) \subseteq M$, and so $H_n \cap F(N) \subseteq F(M \cap N)$. Since $H_n \cap F(N)$ is separable, we thus find a countable $B_n \subseteq M \cap N$ such that $F(B_n)$ is dense in $H_n \cap F(N)$. Now the set $C_{n+1} = A_n \cup B_n \cup C_n$ satisfies (26)–(28).

The set $C = \bigcup C_n$ contains C_0 by construction and satisfies (25), as desired. Indeed, (27) implies in view of $C \subseteq N$ that for any n the relation

$$F(C_n) \subseteq \overline{\operatorname{conv}}\,(H(I \times C) \cup V) \cap F(N)$$

holds which implies the second inclusion in (25). For the first inclusion, note that $H_n = \overline{\operatorname{conv}}\,(H(I \times C_n) \cup V)$ is by (26) an increasing sequence of convex sets. Hence $\bigcup H_n$ is convex, and so we have by (28) that

$$\operatorname{conv}\Big(\bigcup H_n\Big) \cap F(N) = \bigcup H_n \cap F(N) = \bigcup (H_n \cap F(N)) \subseteq \bigcup \overline{F(C_{n+1})} \subseteq \overline{F(C)},$$

and the first inclusion of (26) is proved. □

Also the following observation is essentially from [34].

Corollary 4.2. *Consider the situation of Proposition* 4.1. *If* $\overline{\mathrm{conv}}\,(H(I \times (M \cap N)) \cup V) \setminus F(N)$ *is closed, one may equivalently replace* (25) *by the single equality*

$$\overline{F(C)} = \overline{\mathrm{conv}(H(I \times C) \cup V) \cap F(N)}. \tag{29}$$

Proof. For $C \subseteq M \cap N$ the relation (25) and (29) are actually equivalent. Indeed, put $K_0 = \overline{\mathrm{conv}}\,(H(I \times (M \cap N)) \cup V)$ and $A = \mathrm{conv}(H(I \times C) \cup V)$. We have $A \subseteq (A \cap F(N)) \cup (K_0 \setminus F(N))$. Since $K_0 \setminus F(N)$ is closed, this implies $\overline{A} \subseteq (\overline{A \cap F(N)}) \cup (K_0 \setminus F(N))$, and so $\overline{A} \cap F(N) \subseteq \overline{A \cap F(N)}$. It follows that $\overline{A} \cap F(N) = \overline{A \cap F(N)}$, and so (25) implies (29); the converse is trivial. □

Proposition 4.2. *Let X be a metric space, Z a locally convex metric space, and $F : X \to Z$. Let I be some compact metric space, $D \subseteq X$, and $H : I \times D \to 2^Z$ be upper semicontinuous with compact values $H(\lambda, x)$. Let $V \subseteq Z$ be precompact, and $U, O \subseteq Z$ be such that $F^{-1}(O \cap U) \subseteq D$ and*

$$U = \overline{\mathrm{conv}}\,(H(I \times F^{-1}(O \cap U)) \cup V) \tag{30}$$

holds. Assume that for any countable $C \subseteq F^{-1}(O \cap U)$ the relation

$$\mathrm{conv}(H(I \times C) \cup V) \cap O \cap F(X) \subseteq \overline{F(C)} \subseteq \overline{\overline{\mathrm{conv}}\,(H(I \times C) \cup V) \cap O \cap F(X)} \tag{31}$$

implies that $\overline{\mathrm{conv}}\,(H(I \times C) \cup V)$ is compact (*resp. $\overline{C}$ is compact*). *Then U is compact* (*resp. $\overline{F^{-1}(O \cap U)}$ is compact*). *If $\overline{F^{-1}(O \cap U)} \subseteq D$ and U is complete, then U is compact in both cases.*

Proof. Apply Proposition 4.1 with $M = F^{-1}(U)$ and $N = F^{-1}(O)$. Note first that $H(I \times \{x\})$ is separable for each $x \in D$, since an upper semicontinuous map in metric spaces with separable values sends separable sets into separable sets [36]. Since $M \cap N = F^{-1}(O \cap U)$, we have $U = \overline{\mathrm{conv}}\,(H(I \times (M \cap N)) \cup V)$, and so $F^{-1}(\overline{\mathrm{conv}}\,(H(I \times (M \cap N)) \cup V)) = F^{-1}(U) = M$. Hence, (24) holds. Note that (25) is equivalent to (31), since $F(N) = O \cap F(X)$. Proposition 4.2 thus implies that $\overline{\mathrm{conv}}\,(H(I \times (M \cap N))) = \overline{\mathrm{conv}}\,(H(I \times F^{-1}(O \cap U))) = U$ is compact (resp. $\overline{M \cap N} = \overline{F^{-1}(O \cap U)}$ is compact). If $D_0 = \overline{F^{-1}(O \cap U)} \subseteq D$, then $H(I \times D_0)$ is compact since upper semicontinuous maps with compact values send compact sets into compact sets. Consequently, $H(I \times F^{-1}(O \cap U))$ is relatively compact and in particular precompact. It follows that $U = \overline{\mathrm{conv}}\,(H(I \times F^{-1}(O \cap U)) \cup V)$ is precompact; if U is complete, it must be compact. □

In view of Corollary 4.2 we find:

Corollary 4.3. *Consider the situation of Proposition* 4.2. *If* $F(X)\cap O$ *is open in* K, *one may equivalently replace* (31) *by the single equality*

$$\overline{F(C)} = \overline{\operatorname{conv}(H(I\times C)\cup V)\cap O\cap F(X)}. \tag{32}$$

The previous results imply Theorems 4.1 and 4.2, as we will see. Before we give the proofs, let us show the "dual" versions of these results which will imply Theorems 4.3 and 4.4.

For a multivalued map $F : X\to 2^Z$, we use the notation $F^{-1}(M) = \{x : F(x)\subseteq M\}$.

Proposition 4.3. *Let* X *be a metric space,* Z *a locally convex metric space, and* $F : X\to 2^Z$. *Let* I *be some set,* $M\subseteq X$, *and* $H : I\times M\to 2^Z$. *Let* $U, V\subseteq Z$ *be such that* (30) *holds, and let* $G_1, G_2, \dots\subseteq Z$ *be such that* $G_n\cap(H(I\times(\overline{F^{-1}(\{u\})\cap M}))\cup V)$ *is separable for each* $u\in U$ *and each* n. *Then* U *is compact if for any countable* $C\subseteq U$ *satisfying*

$$C\subseteq\overline{\operatorname{conv}}\,(H(I\times(F^{-1}(C)\cap M))\cup V), \tag{33}$$

$$G_n\cap\operatorname{conv}(H(I\times(F^{-1}(C)\cap M))\cup V)\subseteq\overline{G_n\cap C}\quad (n=1,2,\dots), \tag{34}$$

the set $\overline{C}$ *is compact. Similarly,* $F^{-1}(U)\cap M$ *is precompact* (*resp. relatively compact*) *if for any countable* $C\subseteq U$ *satisfying* (33) *and* (34) *the set* $F^{-1}(C)\cap M$ *is precompact* (*resp. relatively compact*).

Proof. Assume that U is not compact (resp. $F^{-1}(U)\cap M$ is not precompact, relatively compact). Choose some countable $C_0\subseteq U$ such that $\overline{C_0}$ is not compact (resp. $F^{-1}(C_0)\cap M$ is not compact, relatively compact). As in the proof of Proposition 4.1, we will define a countable set $C\supseteq C_0$ which satisfies $C\subseteq U$, (33) and (34). To this end, we define recursively countable sets C_n satisfying the inclusions

$$C_n\subseteq C_{n+1}\subseteq U, \tag{35}$$

$$C_n\subseteq\overline{\operatorname{conv}}\,(H(I\times(F^{-1}(C_{n+1})\cap M))\cup V), \tag{36}$$

and, with the shortcut $H_n=\overline{\operatorname{conv}}\,(H(I\times(F^{-1}(C_n)\cap M))\cup V)$,

$$G_k\cap H_n\subseteq\overline{G_k\cap C_{n+1}}\quad (k=1,2,\dots). \tag{37}$$

This is possible: If C_n is already defined, we have $C_n\subseteq U\subseteq\overline{\operatorname{conv}}\,(H(I\times(F^{-1}(U)\cap M))\cup V)$ by (30). Hence, any of the countable many elements from C_n may be approximated by a sequence of (finite) convex combinations of elements from $H(I\times(\overline{F^{-1}(U)\cap M}))\cup V$, i.e. we find some countable $A_n\subseteq U$ with $C_n\subseteq\overline{\operatorname{conv}}\,(H(I\times(F^{-1}(A_n)\cap M))\cup V)$. Since $G_k\cap H_n$ is separable by assumption, we find countable sets $B_{n,k}\subseteq G_k\cap H_n$ which are dense in $G_k\cap H_n$. Since (30) implies $H_n \subseteq \overline{\operatorname{conv}}\,(H(I\times(F^{-1}(C_n)\cap M))\cup V) \subseteq U$, we have $B_{n,k} \subseteq U$. Hence, the countable set $C_{n+1} = A_n \cup \bigcup_k B_{n,k} \cup C_n$ satisfies (35), and the relations $C_{n+1} \supseteq A_n$ and $C_{n+1} \supseteq B_{n,k}$ imply (36) and (37).

The set $C = \bigcup C_n$ is countable. We have by (36) that for each n

$$C_n \subseteq \overline{\operatorname{conv}}\,(H(I \times (F^{-1}(C) \cap M)) \cup V),$$

and so (33) holds. Moreover, (35) implies that H_n is an increasing sequence of convex sets, and so $\bigcup H_n$ is convex. For fixed k, the relation (37) thus implies

$$\begin{aligned} G_k \cap \operatorname{conv}\left(\bigcup_{n=1}^{\infty} H_n\right) &= G_k \cap \bigcup_{n=1}^{\infty} H_n \\ &= \bigcup_{n=1}^{\infty} (G_k \cap H_n) \subseteq \bigcup_{n=1}^{\infty} (\overline{G_k \cap C_{n+1}}) \subseteq \overline{G_k \cap C}, \end{aligned}$$

and so C satisfies also (34). □

For Fréchet spaces Z the following observation follows from Mazur's lemma. However, we need this also if Z is not necessarily complete:

Lemma 4.1. *Let Z be a topological (Hausdorff) vector space, and $A, B \subseteq Z$ be compact and convex. Then $\overline{\operatorname{conv}}\,(A\cup B) = \operatorname{conv}(A\cup B)$ is compact and convex.*

Proof. Consider the continuous map $f : A \times B \times [0,1] \to Y$, defined by $f(a,b,\lambda) = \lambda a + (1-\lambda)b$. Since A and B are convex, the range of the continuous function f contains (and thus is equal to) $\operatorname{conv}(A \cup B)$. We point out that the compactness of a finite product $A \times B \times [0,1]$ of compact spaces can be proved without appealing to the axiom of choice [35] (although the proof is much more cumbersome with this restriction than other proofs of Tychonoff's theorem). □

The tests for F-admissible triples follow now immediately from our previous results:

Proof of Theorem 4.2. We prove that H is fundamentally V-restrictible on $\overline{\Omega}$. In view of Proposition 3.1, we have to prove to this end that the smallest V-fundamental set U_V is compact. Replacing U by U_V if necessary, it is no loss of generality to assume $U = U_V$. Hence (7) holds which implies (30) with $O = \overline{\Omega}$. Now the compactness of U follows from Proposition 4.2. Here we used the fact that $O \cap F(X) = \overline{\Omega} \cap Y = \overline{\Omega}$ and that the compactness of $A = \overline{\operatorname{conv}}\,(H(I \times C))$ implies the compactness of $\overline{\operatorname{conv}}\,(H(I \times C) \cup V) \subseteq \overline{\operatorname{conv}}\,(A \cup \overline{\operatorname{conv}}\, V)$ in view of Lemma 4.1. For the second statement note that the continuity of F implies that $F^{-1}(\overline{\Omega})$ is closed, and so $\overline{F^{-1}(O \cap U)} \subseteq F^{-1}(\overline{\Omega}) \subseteq D$. □

Proof of Theorem 4.4. The proof of the first statement is analogous to the proof of Theorem 4.2. The only difference is that we apply now Proposition 4.3. Under the assumptions of the second statement, Proposition 4.3

implies that $D_0 = \overline{F^{-1}(U \cap \overline{\Omega})}$ is compact. As in the proof of Theorem 4.2, we have $D_0 \subseteq D$, and so (30) implies that U is contained in the compact set $\overline{\mathrm{conv}}\,(H(I \times D_0) \cup V)$. □

For the proof of the corresponding results for *weakly* F-admissible triples, we need an additional observation.

LEMMA 4.2. *Let X, Y be metric spaces, and $F : X \to Y$ be continuous and proper. Let I be a compact metric space, $C \subseteq Y$ be closed, and $H : I \times F^{-1}(C) \to 2^Y$ be upper semicontinuous with closed values. Then the set $S = \{F(x) : F(x) \in H(I \times \{x\}) \cap C\}$ is closed.*

PROOF. Let $y_n \in S$ converge to some y. We find $(\lambda_n, x_n) \in I \times F^{-1}(C)$ such that $y_n = F(x_n) \in H(\lambda_n, x_n)$. Since F is proper and $\{y, y_1, y_2 \ldots\}$ is compact, the sequence x_n contains a convergent subsequence. Passing to this subsequence, we may assume that $x_n \to x$ converges. Similarly, we may assume that $\lambda_n \to \lambda$ converges. The continuity of F implies that $F(x) = y$ and that $F^{-1}(C)$ is closed. In particular, we must have $F(x) \in C$. We claim that $F(x) \in H(\lambda, x)$ which then implies $y = F(x) \in S$. But if $F(x) \notin H(\lambda, x)$, we find, since $H(\lambda, x)$ is closed, disjoint neighbourhoods U, V of $y = F(x)$ and $H(\lambda, x)$. Since H is upper semicontinuous and $y_n \in U$ and $H(\lambda_n, x_n) \subseteq V$ for sufficiently large n, we have $y_n = F(x_n) \notin H(\lambda_n, x_n)$ for sufficiently large n, a contradiction. □

PROOF OF THEOREMS 4.1 AND 4.3. The same arguments as in the proof of Theorem 4.2 (resp. Theorem 4.4) show that H is V-fundamentally restrictible to Ω (for Theorem 4.1 recall Corollary 4.3).

This observation already implies the statements: Indeed, by Lemma 4.2, the set $S = \{F(x) : F(x) \in H(I \times \{x\}) \cap \overline{\Omega}\}$ is closed, and by assumption $S \subseteq \Omega$. Consequently, $\overline{S} \subseteq \Omega$, and since K is a normal space, we find some $\Omega_0 \supseteq \overline{S}$ which is open in K and satisfies the inclusion $\overline{\Omega}_0 \subseteq \Omega$. Since H is V-fundamentally restrictible to Ω, this inclusion implies that H is V-fundamentally restrictible to $\overline{\Omega}_0$, and so (H, Ω, K) is weakly F-admissible. □

5. Coincidence point theorems

THEOREM 5.1. *Let Z be a locally convex metric space, $K \subseteq Y \subseteq Z$ be closed and convex subsets, and $\Omega \subseteq K$ nonempty and open in K. Let X be a metric space, and $F : X \to Y$ continuous and proper and provide a coincidence point index with the excision property (e.g. F is Vietoris). Suppose that $H : [0,1] \times F^{-1}(\overline{\Omega}) \to \mathcal{K}(K)$ is upper semicontinuous with the following properties:*

(i) *The range of $H(0, \cdot)$ is contained in a compact convex subset $V_0 \subseteq \overline{\Omega}$.*
(ii) *Whenever $F(x) \in \partial\Omega := \overline{\Omega} \setminus \Omega$, we have $F(x) \notin H([0,1] \times \{x\})$.*

(iii) *There is some set* $V \subseteq K$ *with compact* $\overline{\text{conv}}\, V$ *such that for any countable* $C \subseteq \Omega$ *the relation*

$$\overline{F(C)} = \overline{\text{conv}(H(I \times C) \cup V) \cap \Omega}$$

implies that $\overline{\text{conv}}\,(H(I \times C))$ *is compact.*

Then, for $\Phi = H(1, \cdot\,)$*, the inclusion* $F(x) \in \Phi(x)$ *has a solution* $x \in \Omega$*. Moreover,* (Φ, Ω, K) *is weakly* F*-admissible, and for the index from Theorem* 3.1 *we have* $\text{ind}_F(\Phi, \Omega, K) = 1$.

Proof. By Theorem 4.1, the triple (H, Ω, K) is weakly F-admissible. In view of the homotopy invariance and the coincidence point property we thus only have to prove that $\text{ind}_F(\Psi, \Omega, K) = 1$ where $\Psi = H(0, \cdot\,)$. In view of the (strong) permanence property, we have

$$\text{ind}_F(\Psi, \Omega, K) = \text{ind}_F(\Psi, \Omega \cap V_0, V_0). \tag{38}$$

Let Ψ_0 denote the restriction of Ψ to $D = F^{-1}(V_0)$. Note that V_0 and thus D are compact sets. In particular, D is separable, and V_0 is convex and complete, and so we may extend Ψ_0 to an upper semicontinuous map $\Psi_0 : X \to \mathcal{K}(V_0)$ (the general axiom of choice is not required for this, recall Remark 3.2). Note that the relation $F(x) \in \Psi_0(x)$ implies $F(x) \in V_0$ and thus $x \in D$ which in turn implies $F(x) \in V_0 \cap \Psi_0(x) = \overline{\Omega} \cap \Psi(x)$, and so $F(x) \in \Omega$ by (ii), and thus even $F(x) \in \Omega \cap V_0$. Putting $\Omega_1 = V_0$, we thus have proved that the triple (Ψ_0, Ω_1, V_0) is compactly F-admissible, and that in view of the excision and localization properties the equalities

$$\text{ind}_F(\Psi_0, \Omega_1, V_0) = \text{ind}_F(\Psi_0, \Omega \cap V_0, V_0) = \text{ind}_F(\Psi, \Omega \cap V_0, V_0)$$

hold. Fix some $y_0 \in V_0$ and consider the convex homotopy $H_0(\lambda, x) = \lambda y_0 + (1 - \lambda)\Psi_0(x)$ with values in $\mathcal{K}(V_0)$. Since $\Omega_1 = V_0$ is closed, the relation $F(x) \in \overline{\Omega}_1 \cap H_0(\lambda, x)$ trivially implies $F(x) \in \Omega_1$, and so (H, Ω_1, V_0) is compactly F-admissible. The homotopy invariance and normalization property thus show

$$\text{ind}_F(\Psi_0, \Omega_1, V_0) = \text{ind}(H(1, \cdot\,), \Omega_1, V_0) = 1.$$

Combining the previous formulas, we find $\text{ind}_F(\Psi, \Omega, K) = 1$, as desired. □

One could simplify the proof of Theorem 5.1 dramatically, if one would require $V_0 \subseteq \Omega$ and not only $V_0 \subseteq \overline{\Omega}$: In the former case, the statement follows immediately from (38) by considering the compact homotopy $H_0(\lambda, x) = \lambda y_0 + (1 - \lambda)\Psi(x)$ with fixed $y_0 \in V_0$. However, if $V_0 \cap \partial\Omega \neq \emptyset$ this homotopy may fail to be compactly F-admissible.

For $F = \text{id}$ (i.e. when we start with the "classical" fixed point index ind_F), Theorem 5.1 generalizes the main fixed point theorem from [36].

Corollary 5.1. *Let Z be a locally convex metric space, $K \subseteq Y \subseteq Z$ be closed and convex subsets, and $\Omega \subseteq K$ be open in K. Let X be a metric space, and $F : X \to Y$ provide a coincidence index with the excision property* (*e.g. F is Vietoris*). *Let $\Phi : F^{-1}(\overline{\Omega}) \to \mathcal{K}(K)$ be upper semicontinuous. Assume there is some $x_0 \in \Omega$ with the following properties:*

(i) *The Leray–Schauder boundary condition holds on $\partial\Omega := \overline{\Omega} \setminus \Omega$:*

$$\Phi(x) - x_0 \not\ni \lambda(F(x) - x_0) \quad (F(x) \in \partial\Omega,\ \lambda \geq 1).$$

(ii) *If $C \subseteq \Omega$ is countable and satisfies*

$$\overline{F(C)} = \overline{\operatorname{conv}(\Phi(C) \cup \{x_0\}) \cap \Omega} \tag{39}$$

then $\overline{\operatorname{conv}}\,(\Phi(C))$ is compact.

Then the inclusion $F(x) \in \Phi(x)$ has a solution in Ω. Moreover, (Φ, Ω, K) is weakly F-admissible, and we have $\operatorname{ind}_F(\Phi, \Omega, K) = 1$ *for the index from Theorem* 3.1.

Proof. Put $V = \{x_0\}$ and $H(\lambda, x) = \lambda\Phi(x) + (1-\lambda)x_0$ in Theorem 5.1, and observe that $\operatorname{conv}(H([0,1] \times C)) = \operatorname{conv}(\Phi(C) \cup \{x_0\})$ for each $C \subseteq \Omega$. Note that the compactness of $A = \overline{\operatorname{conv}}\,(\Phi(C))$ implies the compactness of $\overline{\operatorname{conv}}\,(H([0,1] \times C)) \subseteq \overline{\operatorname{conv}}\,(A \cup \{x_0\})$ in view of Lemma 4.1. □

For the case that F is a Vietoris map (recall Theorem 2.1), Corollary 5.1 generalizes the main coincidence theorems from [16]. For $F = \mathrm{id}$, Corollary 5.1 contains the two (single-valued) fixed point theorems from [27] (see also [8, Theorem 18.1 and Theorem 18.2]) both as special cases. Moreover, Corollary 5.1 generalizes also a result from [36] (for $F = \mathrm{id}$) where equation (39) was replaced by

$$\overline{\Omega} \cap \operatorname{conv}(\Phi(C) \cup \{x_0\}) \subseteq \overline{C} \subseteq \overline{\Omega} \cap \overline{\operatorname{conv}}\,(\Phi(C) \cup \{x_0\}) \tag{40}$$

which is more restrictive and not so "natural". The deeper reason for this discrepancy is that with (40) the assumptions imply even the id-admissibility of the triple (H, Ω, K) in the proof while (39) implies only the *weak* F-admissibility which is apparently a new concept.

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