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DOMAIN IDENTIFICATION PROBLEM FOR ELLIPTIC HEMIVARIATIONAL INEQUALITIES

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Abstract. The domain identification problems for the elliptic hemivariational inequalities are studied. These problems are formulated as the optimal control problems with admissible domains as controls. The existence of optimal shapes is obtained by the direct method of calculus of variations for a l.s.c. cost functional.

1. Introduction

In this paper we study the domain identification problems for the hemivariational inequalities. These problems are usually formulated as the optimal control ones where the role of the set of controls is played by a class of admissible shapes. This class consists of all subsets of $\mathbb{R}^N$ which have a suitable property that will be made more precise later. As a state variable we consider a solution of the hemivariational inequality which is solved in an admissible domain. The problem under consideration consists in minimization of a cost functional on a set of admissible control-state pairs. The purpose of this paper is twofold. First, dealing with the same class of domains as in Chenais [2] we extend the main result of [2] to a class of Neumann problems for elliptic equations with multivalued nonlinearities. Such problems are expressed as the hemivariational inequalities

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since they involve nonconvex superpotentials which are only locally Lipschitz functions. A second goal is to give another approach to the problems studied by Denkowski and Migórski in [5] who applied the mapping method to the domain identification problems with a different class of admissible shapes.

The hemivariational inequalities have been introduced and studied since 1981 by P. D. Panagiotopoulos as the mathematical models of many problems coming from mechanics, engineering and economics. The hemivariational inequalities represent the principle of virtual work or power. They are variational formulations of problems with nonconvex energy functions and they are derived with the help of the generalized subdifferential in the sense of Clarke [3]. In mechanics the laws in the subdifferential form describe the relations between stresses and strains, between boundary displacements and reactions or between forces and fluxes.

The results on existence of solutions for the elliptic hemivariational inequalities have been delivered by Rauch [16] and Panagiotopoulos [14] who used a regularization technique, by Chang [1] who applied the deformation lemma and by Naniewicz and Panagiotopoulos [12] where the theory of pseudomonotone operators of Browder–Hess was adopted. The evolution hemivariational inequalities have been considered only recently, see [9] and [10] and the references therein. The literature on the mathematical theory of shape optimization problems is very large. We mention only that such problems for partial differential equations were studied by Pironneau [15], Murat and Simon [11], Sokolowski, Zolesio, while the variational inequalities were treated by Liu and Rubio [8], Tiihonen [17] and Neittaanmäki [13]. For systems governed by the hemivariational inequalities, the existence results for domain optimization problems were obtained by Denkowski and Migórski [4] and [5], and Gasiński [7].

In the present paper we consider the optimal shape design problem for the elliptic hemivariational inequality. In this problem the class of admissible geometric domains consists of all open subsets of a given bounded open set D in $\mathbb{R}^N$ satisfying the cone property. This class of controls is equipped with the $L^2(D)$-topology of the characteristic functions of its elements. It is known that the considered class of admissible shapes is compact in this topology and that the open sets satisfying the cone property are the uniform Lipschitz sets (for details see [2]). The domain identification problem is formulated as the double minimization one, since the elliptic hemivariational inequality usually does not possess a unique solution. The main theorem being the generalization of Theorem IV.1 in [2] deals with the existence of solutions to optimal shape design problem. We apply the direct method of the calculus of variations for a lower semicontinuous cost functional which is in a general integral form. The crucial point in obtaining the existence of optimal shapes is a result on the dependence of solutions of the hemivariational inequality with respect to the admissible domains.

The plan of the paper is as follows. In Section 2 we present some relevant definitions and results which will be needed in the sequel. In Section 3 we consider a class of hemivariational inequalities. For this model we obtain the existence of solutions and we show some a priori estimates. We provide also a result on the closedness of the graph of the mapping which to every admissible shape assigns the solution set of hemivariational inequality (see Theorem 3.1). In Section 4 we prove the main result of this paper on the existence of optimal domains for systems governed by hemivariational inequalities.

2. Preliminaries

In this section we fix the notation and recall some relevant definitions and results which will be needed in next sections.

Let D be a given bounded open set in $\mathbb{R}^N$. Let θ, h and r be three given constants such that $\theta \in (0, \pi/2)$, $h > 0$, $r > 0$ and $2r \leq h$. By $\Pi = \Pi(\theta, h, r)$ we denote the family of all open subsets of D satisfying the cone property with constants θ, h and r. Recall (see [2]) that a subset Ω of $\mathbb{R}^N$ is said to satisfy the cone property with constants θ, h and r if and only if

$$\forall z \in \partial\Omega \ \exists C_z = C(\xi_z, \theta, h) \text{ such that } \forall y \in B(z, r) \cap \Omega \quad y + C_z \subset \Omega,$$

where $C(\xi_z, \theta, h) = \{x \in \mathbb{R}^N : \langle x, \xi_z \rangle > |x| \cos\theta,\ |x| < h\}$ is the cone of angle θ, height h and axis ξ_z ($|\xi_z| = 1$) (here the symbols $\langle\,\cdot\,,\,\cdot\,\rangle$, $|\cdot|$ denote the usual inner product and the norm in $\mathbb{R}^N$, respectively) and $B(z, r)$ denotes the open ball of radius r and center z in $\mathbb{R}^N$.

For $\Omega \in \Pi$ we denote by $\mathbb{I}_\Omega$ the characteristic function of Ω in D. We introduce on the set Π the following topology: we say that a sequence $\{\Omega_n\} \subset \Pi$ converges to Ω in Π if and only if $\mathbb{I}_{\Omega_n} \to \mathbb{I}_\Omega$ in $L^2(D)$. It is known (see Theorems III.1 and III.2 of [2]) that the set Π is closed and relatively compact in such topology. We know also that the family Π satisfies the "uniform extension property", i.e. we have

Theorem 2.1 (Theorem II.1 of [2]). *There exists a positive constant $K = K(\theta, h, r)$ depending on $\Omega \in \Pi(\theta, h, r)$ through θ, h and r only, and such that for any $\Omega \in \Pi$ there exists a linear continuous extension operator $p_\Omega : H^1(\Omega) \to H^1(\mathbb{R}^N)$ such that $\|p_\Omega\| \leq K$.*

In the sequel, for Ω in $\mathbb{R}^N$ we consider the space $V(\Omega) = H^1(\Omega)$ with norm $\|\cdot\|_{V(\Omega)}$ and we denote by $(\,\cdot\,,\,\cdot\,)_{L^2(\Omega)}$ the inner product on $L^2(\Omega)$.

We recall the definition of the generalized gradient of Clarke for a locally Lipschitz function. Let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $j(\,\cdot\,, \xi)$ is measurable on Ω for all $\xi \in \mathbb{R}$, $j(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Omega$. Then for a.e. $x \in \Omega$ the symbol $\partial j : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ denotes Clarke's generalized subdifferential of j with respect to ξ (see [3]) defined as follows

$$\partial j(x,\xi) = \{\zeta \in \mathbb{R} : j^0(x,\xi;\eta) \geq \zeta\eta, \text{ for all } \eta \in \mathbb{R}\}$$

for all $\xi \in \mathbb{R}$, where $j^0(\cdot\,,\cdot\,;\cdot)$ is the generalized directional derivative of j at ξ in the direction η given by

$$j^0(x,\xi;\eta) = \limsup_{h\to 0\ \lambda\searrow 0} \frac{j(x,\xi+h+\lambda\eta) - j(x,\xi+h)}{\lambda},$$

for all $\xi, \eta \in \mathbb{R}$, a.e. $x \in \Omega$.

Finally, we shall also use another extension operator $L^2(\Omega) \ni w \mapsto \overline{w} \in L^2(D)$ which extends function w by zero outside Ω.

3. Formulation of hemivariational inequality

The aim of this section is to consider a class of hemivariational inequalities. We present a result on existence of solutions, we show the a priori estimate for solutions and we give the theorem on the dependence of the solution set on the domain.

Let Ω be an open bounded subset of $\mathbb{R}^N$ and let $V(\Omega) = H^1(\Omega)$. We consider the following problem: find $u \in V(\Omega)$ such that there exists $\chi \in L^2(\Omega)$ and

$$\text{(P)} \qquad \begin{cases} a_\Omega(u,v) + (\chi, v)_{L^2(\Omega)} = (f,v)_{L^2(\Omega)} & \text{for all } v \in V(\Omega), \\ \chi(x) \in \partial j(x, u(x)) & \text{a.e. } x \in \Omega, \end{cases}$$

where a_Ω, f and j are prescribed data.

Now, we present the assumptions guaranteeing the existence of at least one solution of the problem (P). We make the following hypotheses on the data:

$\underline{\mathrm{H}(a)}$: $a_\Omega : V(\Omega) \times V(\Omega) \to \mathbb{R}$ is a form such that

$$a_\Omega(u,v) = \int_\Omega \bigg(\sum_{i,j=1}^N a_{ij}(x) D_i u D_j v + a_0(x) uv \bigg)\, dx,$$

where $a_{ij}, a_0 \in L^\infty(\Omega)$, $a_0 \geq c_1 > 0$ and there exists a constant $\alpha_0 > 0$ such that $\sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2$ for all $\xi \in \mathbb{R}^N$, a.e. $x \in \Omega$.

$\underline{\mathrm{H}(j)}$: $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying the following conditions:

(i) $\Omega \ni x \mapsto j(x,\xi)$ is measurable on Ω for all $\xi \in \mathbb{R}$,

(ii) $\mathbb{R} \ni \xi \mapsto j(x,\xi)$ is locally Lipschitz on $\mathbb{R}$ for a.e. $x \in \Omega$,

(iii) $j(\cdot\,, 0) \in L^1(\Omega)$,

(iv) (the growth condition) there exists a constant $c_0 > 0$ such that $|\zeta| \leq c_0(1 + |\xi|)$ for all $\zeta \in \partial j(x,\xi)$, $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$,

(v) (the generalized sign condition) there exists a nonnegative function

$\alpha \in L^2(\Omega)$ such that $j^0(x,\xi;-\xi) \le \alpha(x)|\xi|$ for all $\xi \in \mathbb{R}$ and a.e. $x \in \Omega$.

The following existence result can be obtained as a corollary of Theorem 4.25 of Naniewicz and Panagiotopoulos [12].

LEMMA 3.1. *If the hypotheses* H(a), H(j) *hold and* $f \in L^2(\Omega)$, *then the problem* (P) *admits at least one solution.*

In what follows, the solution set of problem (P) will be denoted by $S(\Omega)$. We also say that a pair (u,χ) solves the problem (P), if $u \in S(\Omega)$ and $\chi \in L^2(\Omega)$ is the corresponding selection of $\partial j(\,\cdot\,,u(\,\cdot\,))$ which appears in (P).

LEMMA 3.2. *Under the hypotheses of Lemma* 3.1, *if* $(u,\chi) \in S(\Omega) \times L^2(\Omega)$ *is a solution to* (P), *then there exists a positive constant* c *such that*

$$\|u\|_{V(\Omega)} \le c(1 + \|f\|_{L^2(\Omega)}), \tag{1}$$

$$\|\chi\|_{L^2(\Omega)} \le c(1 + \operatorname{meas}(\Omega) + \|f\|_{L^2(\Omega)}). \tag{2}$$

PROOF. It is enough to observe that from coercivity of $a_\Omega(\,\cdot\,,\,\cdot\,)$ and the generalized sign condition of j, we easily get

$$\begin{aligned}
\min(\alpha_0,c_1)\|u\|^2_{V(\Omega)} &\le a_\Omega(u,u) \le \int_\Omega j^0(x,u(x);-u(x))\,dx + (f,u)_{L^2(\Omega)} \\
&\le \int_\Omega \alpha(x)|u(x)|\,dx + (f,u)_{L^2(\Omega)} \\
&\le \|\alpha\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)} \\
&\le c(1 + \|f\|_{L^2(\Omega)})\|u\|_{V(\Omega)},
\end{aligned}$$

with some constant $c > 0$. Hence (1) follows. Moreover, we notice that $|\chi(x)| \le c_0(1 + |u(x)|)$ and this implies

$$\|\chi\|^2_{L^2(\Omega)} \le 2c_0^2 \int_\Omega (1 + |u(x)|^2)\,dx \le c(1 + \operatorname{meas}(\Omega) + \|u\|^2_{V(\Omega)}).$$

So, $\|\chi\|_{L^2(\Omega)} \le c\,(1 + \operatorname{meas}(\Omega) + \|u\|_{V(\Omega)})$. From this inequality and (1) we get the desired estimate (2). $\square$

Let D be an open bounded set in $\mathbb{R}^N$ and let $\Omega_0 \subset D$. Given three numbers $\theta \in (0,\pi/2)$, $h > 0$ and $r > 0$ such that $2r \le h$, we consider a set $\Pi_0 = \Pi_0(\theta,h,r)$ of all elements of $\Pi(\theta,h,r)$ which contain Ω_0. In the sequel, we require θ, h and r to be uniform for all the domains of Π. We consider the class Π with the $L^2(D)$-topology of the characteristic functions of its elements, i.e. $\Omega_n, \Omega \in \Pi$, $\Omega_n \to \Omega$ in Π if and only if $\mathbb{I}_{\Omega_n} \to \mathbb{I}_\Omega$ in $L^2(D)$.

Now, let us consider a sequence of problems:

$$(\mathrm{P}_n) \qquad \begin{cases} a_{\Omega_n}(u_n,v) + (\chi_n,v)_{L^2(\Omega_n)} = (f,v)_{L^2(\Omega_n)} & \text{for all } v \in V(\Omega_n), \\ \chi_n(x) \in \partial j(x,u_n(x)) & \text{a.e. } x \in \Omega_n, \end{cases}$$

where $\Omega_n \in \Pi_0$, $V(\Omega_n) = H^1(\Omega_n)$ for all $n \in \mathbb{N}$ and $f \in L^2(D)$.

We assume that the prescribed data a_{Ω_n} and j are as follows:

$\underline{\mathrm{H}(a)_1}$: the forms $a_{\Omega_n} : V(\Omega_n) \times V(\Omega_n) \to \mathbb{R}$ satisfy H(a), where $a_{ij}, a_0 \in L^\infty(D)$, Ω is replaced by Ω_n, and the constants α_0, c_1 are independent of n.

$\underline{\mathrm{H}(j)_1}$: the function $j : D \times \mathbb{R} \to \mathbb{R}$ satisfies H(j) with Ω being replaced by D.

The following result on the dependence of solutions of hemivariational inequality (P) with respect to the domain is a crucial point in obtaining the existence of optimal shapes.

THEOREM 3.1. *Let us assume that* $\mathrm{H}(a)_1$, $\mathrm{H}(j)_1$ *hold and* $f \in L^2(D)$. *Then the map* $\Pi_0 \ni \Omega \mapsto S(\Omega) \subset V(\Omega)$ *has a closed graph in the following sense: if* $\Omega_n, \Omega \in \Pi_0$ *and* $\Omega_n \to \Omega$ *in* Π *as* $n \to \infty$, *then for every solution* $(u_n, \chi_n) \in S(\Omega_n) \times L^2(\Omega_n)$ *of* (P_n), *there exists a subsequence* $\{n_k\} \subset \{n\}$ *such that*

$$p_{n_k} u_{n_k} \to \xi \quad \text{weakly in } H^1(D),$$

(*here* $p_n = p_{\Omega_n}$ *denotes the extension operator from Theorem* 2.1),

$$\overline{\chi}_{n_k} \to \zeta \quad \text{weakly in } L^2(D),$$

and the pair $(u, \chi) := (\xi_{|\Omega}, \zeta_{|\Omega})$ *is a solution to the problem* (P).

PROOF. First applying Lemma 3.1 and Lemma 3.2 we have that $S(\Omega_n) \neq \emptyset$ for all $n \in \mathbb{N}$ and the following estimate holds

$$\|u_n\|_{V(\Omega_n)} + \|\chi_n\|_{L^2(\Omega_n)} \le c(1 + \mathrm{meas}(\Omega_n) + \|f\|_{L^2(\Omega_n)}) \tag{3}$$

with a constant $c > 0$ independent of n. From Theorem 2.1 and (3) it follows that

$$\|p_n u_n\|_{H^1(D)} \le \|p_n u_n\|_{H^1(\mathbb{R}^N)} \le K\|u_n\|_{V(\Omega_n)} \le Kc(1 + \mathrm{meas}(D) + \|f\|_{L^2(D)}).$$

Hence $\{p_n u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1(D)$ and there exists a subsequence $\{n_k\} \subset \{n\}$ such that

$$p_{n_k} u_{n_k} \to \xi \quad \text{weakly in } H^1(D) \text{ with } \xi \in H^1(D). \tag{4}$$

Consequently, $p_{n_k} u_{n_k} \to \xi$ in $L^2(D)$, and also for the next subsequence (still denoted as before), we have

$$(p_{n_k} u_{n_k})(x) \to \xi(x) \quad \text{as } n \to \infty \text{ for a.e. } x \in D, \tag{5}$$

$$|(p_{n_k} u_{n_k})(x)| \le h(x) \quad \text{for all } n \in \mathbb{N},\ \text{a.e. } x \in D \text{ with } h \in L^2(D). \tag{6}$$

On the other hand the estimate (3) implies that $\{\overline{\chi}_n\}_{n\in\mathbb{N}}$ is bounded in $L^2(D)$. So, we may assume, by passing to a subsequence if necessary, that

$$\overline{\chi}_{n_k} \to \zeta \quad \text{weakly in } L^2(D) \text{ with some } \zeta \in L^2(D). \tag{7}$$

We put $u := \xi_{|\Omega}$ and $\chi := \zeta_{|\Omega}$. So, we have $u \in V(\Omega)$ and $\chi \in L^2(\Omega)$. We show that (u, χ) is a solution of the corresponding limit hemivariational inequality. For simplicity, we continue to write $\{u_n\}$ for a subsequence $\{u_{n_k}\}$.

Let $v \in H^1(D)$. Since $u_n \in S(\Omega_n)$, we have

$$\int_{\Omega_n} \Big(\sum_{i,j=1}^N a_{ij}(x) D_i u_n D_j v + a_0(x) u_n v \Big)\, dx + \int_{\Omega_n} \chi_n v\, dx = \int_{\Omega_n} f v\, dx. \tag{8}$$

Let us notice that

$$(f, v)_{L^2(\Omega_n)} = (f, v \cdot \mathbb{I}_{\Omega_n})_{L^2(D)} \to (f, v \cdot \mathbb{I}_\Omega)_{L^2(D)} = (f, v)_{L^2(\Omega)},$$

and similarily $(\chi_n, v)_{L^2(\Omega_n)} \to (\chi, v)_{L^2(\Omega)}$, as $n \to \infty$. Thanks to (4), we obtain

$$\begin{aligned} \lim_{n\to\infty} \int_{\Omega_n} \Big(\sum_{i,j=1}^N a_{ij}(x) D_i u_n D_j v + a_0(x) u_n v \Big)\, dx \\ = \int_{\Omega} \Big(\sum_{i,j=1}^N a_{ij}(x) D_i u D_j v + a_0(x) u v \Big)\, dx. \end{aligned}$$

Hence, using the last three convergences and passing to the limit in (8), we get

$$\int_{\Omega} \Big(\sum_{i,j=1}^N a_{ij}(x) D_i u D_j v + a_0(x) u v \Big)\, dx + (\chi, v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}$$

for all $v \in H^1(D)$, and consequently, $u \in S(\Omega)$.

In order to complete the proof we need to show that $\chi(x) \in \partial j(x, u(x))$ for a.e. $x \in \Omega$. To this end, let $w \in L^2(\Omega)$. Since $\chi_n(x) \in \partial j(x, u_n(x))$ a.e. $x \in \Omega_n$, we conclude that

$$\int_{\Omega_n} \chi_n(x) \overline{w}(x)\, dx \le \int_{\Omega_n} j^0(x, u_n(x); \overline{w}(x))\, dx.$$

Since

$$\int_{\Omega_n} j^0(x, u_n(x); \overline{w}(x))\, dx = \int_D j^0(x, (p_n u_n)(x); \overline{w}(x)) \mathbb{I}_{\Omega_n}(x)\, dx,$$

from (7) we immediately have

$$\begin{aligned} (\chi, w)_{L^2(\Omega)} &= \lim_{n\to\infty} \int_{\Omega_n} \chi(x) \overline{w}(x)\, dx \\ &\le \limsup_{n\to\infty} \int_D j^0(x, (p_n u_n)(x); \overline{w}(x)) \mathbb{I}_{\Omega_n}(x)\, dx. \end{aligned} \tag{9}$$

From H(j)(iv), it follows that

$$\begin{aligned} j^0(x, (p_n u_n)(x); \overline{w}(x)) &= \max\{\zeta \cdot \overline{w}(x) : \zeta \in \partial j(x, (p_n u_n)(x))\} \\ &\le c_0\,(1 + |(p_n u_n)(x)|) |\overline{w}(x)| \quad \text{a.e. } x \in D. \end{aligned}$$

Thus, applying the Fatou lemma to a sequence of measurable functions

$$g_n(x) = j^0(x, (p_n u_n)(x); \overline{w}(x)) - c_0\,(1 + |(p_n u_n)(x)|)|\overline{w}(x)|,$$

we have

$$\limsup_{n\to\infty} \int_D g_n(x)\mathbb{I}_{\Omega_n}(x)\,dx \le \int_D \limsup_{n\to\infty} g_n(x)\mathbb{I}_{\Omega_n}(x)\,dx. \tag{10}$$

On the other hand, since $\mathbb{I}_{\Omega_n} \to \mathbb{I}_\Omega$ in $L^2(D)$, we may also suppose that

$$\mathbb{I}_{\Omega_n}(x) \to \mathbb{I}_\Omega(x) \quad \text{a.e. } x \in D \ \text{ for a subsequence.} \tag{11}$$

So, combining this with (10), we get

$$\begin{aligned}
\limsup_{n\to\infty} \int_D & j^0(x, (p_n u_n)(x); \overline{w}(x))\mathbb{I}_{\Omega_n}(x)\,dx \\
&- c_0 \lim_{n\to\infty} \int_D (1 + |(p_n u_n)(x)|)|\overline{w}(x)|\mathbb{I}_{\Omega_n}(x)\,dx \\
&\le \int_D \limsup_{n\to\infty} j^0(x, (p_n u_n)(x); \overline{w}(x))\mathbb{I}_{\Omega_n}(x)\,dx \\
&- c_0 \int_D \lim_{n\to\infty} (1 + |(p_n u_n)(x)|)|\overline{w}(x)|\mathbb{I}_{\Omega_n}(x)\,dx.
\end{aligned} \tag{12}$$

Now, from the convergences (11), (5) and the estimate (6), we have

$$\lim_{n\to\infty} \int_D (1 + |(p_n u_n)(x)|)|\overline{w}(x)|\mathbb{I}_{\Omega_n}(x)\,dx = \int_D (1 + |\xi(x)|)|\overline{w}(x)|\mathbb{I}_\Omega(x)\,dx. \tag{13}$$

By the upper semicontinuity of generalized directional derivative with respect to the second variable (see Proposition 2.1.1 in [3]), we have

$$\limsup_{n\to\infty} j^0(x, (p_n u_n)(x); \overline{w}(x)) \le j^0(x, \xi(x); \overline{w}(x)) \quad \text{a.e. } x \in D.$$

Multiplying this inequality by the characteristic function $\mathbb{I}_\Omega$ and integrating over D, we obtain

$$\int_D \limsup_{n\to\infty} j^0(x, (p_n u_n)(x); \overline{w}(x))\mathbb{I}_\Omega(x)\,dx \le \int_D j^0(x, \xi(x); \overline{w}(x))\mathbb{I}_\Omega(x)\,dx.$$

Hence, substituting (13) into (12), we deduce

$$\begin{aligned}
\limsup_{n\to\infty} \int_D & j^0(x, (p_n u_n)(x); \overline{w}(x))\mathbb{I}_{\Omega_n}(x)\,dx \\
&\le \int_D j^0(x, \xi(x); \overline{w}(x))\mathbb{I}_\Omega(x)\,dx = \int_\Omega j^0(x, u(x); w(x))\,dx.
\end{aligned}$$

So, from (9) we get $(\chi, w)_{L^2(\Omega)} \le \int_\Omega j^0(x, u(x); w(x))\,dx$ and this by the arbitrariety of w implies that $\chi(x) \in \partial j(x, u(x))$ a.e. $x \in \Omega$. This completes the proof. $\square$

4. Shape optimization problem

In this section we state the main result of this paper on the existence of optimal shapes for systems governed by hemivariational inequalities. The optimal shape design problem consists in solving the following control one:

$$
\text{(OSD)} \qquad \begin{cases} \text{find } (\Omega^*, u^*) \in \bigcup\limits_{\Omega\in\Pi_0} (\Omega \times S(\Omega)) \text{ such that} \\ J(\Omega^*, u^*) = \min\limits_{\Omega\in\Pi_0} \min\limits_{u\in S(\Omega)} J(\Omega, u), \end{cases}
$$

in which the control is the set Ω changing in the family $\Pi_0 \subset \Pi$ of admissible shapes. The cost functional J has the following form:

$\underline{\text{H}(J)}$:
$$
J(\Omega, u) = \int_\Omega g(x, u(x))\, dx,
$$

where $g: \mathbb{R}^N \times \mathbb{R} \to \overline{\mathbb{R}}$ is a nonnegative normal integrand (cf. [6]) which satisfies the following growth condition:

$$
g(x, v) \le k(x)(1 + |v|^r) \quad \text{for all } v \in \mathbb{R} \text{ and a.e. } x \in \mathbb{R}^N
$$

with $k \in L^1(\mathbb{R}^N)$ and $r \ge 1$.

Lemma 4.1. *The cost functional J which satisfies the hypothesis $H(J)$ is lower semicontinuous in the* $\Pi_0 \times (\text{weak-}H^1(D))$*-topology.*

Proof. Let $\{\Omega_n\}_{n\in\mathbb{N}} \subset \Pi_0$ be a sequence of admissible domains such that $\Omega_n \to \Omega$ in Π, i.e. $\mathbb{I}_{\Omega_n} \to \mathbb{I}_\Omega$ in $L^2(D)$. From Theorem 3.1 it follows that the corresponding sequence of solutions $u_n \in S(\Omega_n)$ converges in terms of their extensions to $H^1(D)$, that is

$$
p_n u_n \to \xi \text{ weakly in } H^1(D) \text{ and } \xi_{|\Omega} =: u \in S(\Omega).
$$

Next, we have

$$
g(x, \xi(x))\mathbb{I}_\Omega(x) \le \liminf_{n\to\infty} g(x, (p_n u_n)(x))\mathbb{I}_{\Omega_n}(x) \quad \text{a.e. } x \in D. \tag{14}
$$

Indeed, by the lower semicontinuity of $g(x, \cdot\,)$ and the convergence $(p_n u_n)(x) \to \xi(x)$ for a.e. $x \in D$ (which holds at least for a subsequence still denoted as before), it follows that

$$
g(x, \xi(x)) \le \liminf_{n\to\infty} g(x, (p_n u_n)(x)) \quad \text{a.e. } x \in D.
$$

From this we deduce that

$$
\begin{aligned}
\mathbb{I}_\Omega(x) g(x, \xi(x)) &\le \mathbb{I}_\Omega(x) \liminf_{n\to\infty} g(x, (p_n u_n)(x)) \\
&= \mathbb{I}_\Omega(x) \liminf_{n\to\infty} g(x, (p_n u_n)(x)) \\
&\quad + \liminf_{n\to\infty} [g(x, (p_n u_n)(x))(\mathbb{I}_{\Omega_n}(x) - \mathbb{I}_\Omega(x))] \\
&= \liminf_{n\to\infty} g(x, (p_n u_n)(x))\mathbb{I}_{\Omega_n}(x),
\end{aligned}
$$

because of the growth condition in the hypothesis H(J). Finally, integrating (14) over D and applying Fatou's formula, we obtain that

$$J(\Omega, u) = \int_D g(x, \xi(x))\mathbb{I}_\Omega(x)\,dx \le \int_D \liminf_{n\to\infty} g(x, (p_n u_n)(x))\mathbb{I}_{\Omega_n}(x)\,dx$$
$$\le \liminf_{n\to\infty} \int_D g(x, (p_n u_n)(x))\mathbb{I}_{\Omega_n}(x)\,dx = \liminf_{n\to\infty} J(\Omega_n, u_n)$$

which finishes the proof. □

We can now formulate the main result of this paper.

Theorem 4.1. *Under the hypotheses of Theorem* 3.1 *and* H(J) *the problem* (OSD) *admits at least one solution.*

Proof. We apply the direct method of the calculus of variations. Let $\{(\Omega_n, u_n)\}_{n\in\mathbb{N}}$ such that $\Omega_n \in \Pi_0$ and $u_n \in S(\Omega_n)$ for all $n \in \mathbb{N}$ be a minimizing sequence for (OSD). The compactness of the sequence $\{(\Omega_n, u_n)\}$ in the $\Pi_0 \times$ (weak-$H^1(D)$)-topology follows from the fact that Π_0 is a compact set, as a closed subset of compact set Π (see Theorem III.1 and Theorem III.2 of [2]). This implies that there exists a subsequence of $\{\Omega_n\}$ (still denoted as before) and a set $\Omega^* \in \Pi_0$ such that $\Omega_n \to \Omega^*$ in Π.

Now, applying Theorem 3.1 we obtain that a limit u^* of a subsequence of $u_n \in S(\Omega_n)$ is a solution of the corresponding hemivariational inequality considered in Ω^*, i.e. $u^* \in S(\Omega^*)$. Hence the pair (Ω^*, u^*) is admissible for (OSD). We conclude from the lower semicontinuity of J (compare Lemma 4.1) that (Ω^*, u^*) is also an optimal pair for (OSD). This completes the proof. □

Remark 4.1. The above theorem still holds if the class Π_0 is replaced by another family of admissible shapes:

$$\Pi_C = \left\{\Omega \in \Pi : \int_\Omega h(x)\,dx = C\right\},$$

where $h \in L^1(D)$ is a given function and C is a prescribed constant. In this case using the dominated convergence theorem it is clear that the set Π_C is also compact as a closed subset of Π. We proceed analogously as in the proof of Theorem 4.1.

Example 4.1. Let D, Ω_0 be two given bounded open sets in $\mathbb{R}^N$ such that $\Omega_0 \subset D$, and let $f \in L^2(D)$, $u_d \in L^2(\Omega_0)$ be given functions. We consider the following problem:

$$(\mathrm{P}^*) \qquad \text{find } \Omega^* \in \Pi_0 \text{ such that } u^*|_{\Omega_0} = u_d \text{ and } u^* \in S(\Omega^*),$$

i.e. we are looking for a domain Ω^* in a convenient class of open sets for which the restriction to Ω_0 of a solution $u^* \in S(\Omega^*)$ is equal to the desired function u_d.

In order to obtain an existence result for this problem, we consider the cost functional J of the form

$$J(\Omega, u) = \|u_{|\Omega_0} - u_d\|^2_{L^2(\Omega_0)} = \int_D |(p_\Omega u)(x) - (p_{\Omega_0} u_d)(x)|^2 \mathbb{I}_{\Omega_0}(x)\, dx.$$

This functional correponds to minimization of deviation from the desired state u_d. It is easy to check that the function $g(x, v) := |v - (p_{\Omega_0} u_d)(x)|^2$ satisfies the hypothesis $H(J)$ and so the functional J is lower semicontinuous in the $\Pi_0 \times$ (weak-$H^1(D)$)-topology. From Theorem 4.1 we get the existence of a minimum for J. Moreover, if the computed minimum is zero, then the corresponding domain is a solution. Otherwise, the original problem (P^*) has no solution in Π_0.

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