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EXISTENCE AND RELAXATION PROBLEMS IN OPTIMAL SHAPE DESIGN

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Abstract. A general abstract theorem on existence of solutions to optimal shape design problems for systems governed by partial differential equations, or variational inequalities or hemivariational inequalities is formulated and two main properties (conditions) responsible for the existence are discussed. When one of them fails one have to make "relaxation" in order to get some generalized optimal shapes. In particular, some relaxation "in state", based on Γ convergence, is presented in details for elliptic, parabolic and hyperbolic PDEs (and then for optimal shape design problems), while the relaxation "in cost functional" is discussed for some special classes of functionals.

1. Introduction

Optimization of shape is one of the most important task in engineering. It is enough to mention looking for optimal shape of airplane wings, or of the submarine under the constraint that its volume is prescribed, the shaping of anode in electromachining problems, or optimal shape of the contact surface in elasticity, and so on.

The mathematical theory of such kind of problems, called also optimal shape design (OSD for short) deals with the existence problems of optimal shapes, their characterization and numerical approximation, or relaxation (i.e. looking

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for some kind of generalized solutions in the case there are no classical ones). It started in 70-ties and developed then quickly with papers and monographs by Céa, Chenais, Miele, Murat and Simon, Pironneau, Buttazzo and Dal Maso, Dal Maso and Garroni, Sokoowski and Zolesio, Neittaanmäkki, Liu and Rubio, Šverák and many others (see References).

The aim of this paper is to present some ideas and results concerning the existence, non-existence and relaxation problems in optimal shape design. For the detailed proofs we refer to original papers.

Thus, in Section 2 after the statement of general OSD problem for systems governed by a relation, which can be a partial differential equation (PDE) or a variational inequality (VI) or hemivariational inequality (HVI), we quote an abstract existence results ([15]) based on direct method of Calculus of Variations and then we discuss two conditions responsible for existence. Lack one of them may produce nonexistence as it is illustrated by an example in Section 5. Starting with preliminaries of Section 3 we concentrate on OSD problems governed by PDE's. In Section 4 we quote relaxation results for elliptic, parabolic and hyperbolic equations. These kinds of problems dealing with determining the minimal classes of equations closed under some passages to the limit (see Remark 2.2) are interesting in themselves and were largely investigated in connection with the homogenization theory. Section 5 is devoted to relaxation of cost functionals and some existence results for OSD of systems governed by evolution PDE's (the results in hyperbolic case extend those obtained in [25] for parabolic case).

We present a unified approach for stationary and evolution case based on γ^A-convergence which was introduced by Dal Maso and Mosco ([31], [42], see also [1], [2]) for symmetric operators A (the case of general A bases on capacitary methods developed by Dal Maso and Garroni [11], [12]). This convergence is related to the Γ-convergence (defined by De Giorgi and Franzoni – see also [2] and [9]) of energy functionals.

Relaxation in state leads to generalized shapes being Borel measures (classical shapes being geometrical domains). Some analogy one can find in constructing reals starting from minimization problems in the set of rational numbers (e.g. $\min_{\mathbf{Q}}(x-\sqrt{2})^2$).

2. Statement of OSD-problems and an existence theorem

2.1. A mathematical model of $(\mathbf{OSD})_{\mathcal{R}}$. A very general OSD problem for systems described by a relation $\mathcal{R}$ can be formulated as follows:

$$(\mathrm{OSD})_{\mathcal{R}} \qquad \begin{cases} \text{find } (\Omega^*, u^*) \in \bigcup\limits_{\Omega\in\mathcal{B}} (\Omega \times S_{\mathcal{R}}(\Omega)) \text{ such that} \\ \mathcal{J}(\Omega^*, u^*) = \min\limits_{\Omega\in\mathcal{B}} \min\limits_{v\in S_{\mathcal{R}}(\Omega)} \mathcal{J}(\Omega, v). \end{cases}$$

The meaning of the symbols above is following:

(1) $\mathcal{B}$ is a fixed subclass of the class $\mathcal{O} = \mathcal{O}_{ad}$ of all admissible shapes (geometrical domains in $\mathbb{R}^N$),

(2) $\mathcal{R}$ stands for the state relation (it can be a PDE or VI or HVI), e.g.

$$\text{(PDE)} \qquad Au = f, \quad u \in V(\Omega),$$

$$\text{(VI)} \qquad a(u, v-u) + \Phi(v) - \Phi(u) \geq \langle f, v-u\rangle \quad \text{for all } v \in K(\Omega),$$

$$\text{(HVI)} \quad a(u, v-u) + \int_\Omega j^0(u, v-u)\,dx \geq \langle f, v-u\rangle \quad \text{for all } v \in K(\Omega),$$

with some differential operator A, bilinear form a, convex function Φ, Clark generalized directional derivative j^0 of superpotential j, and closed convex subset K of a suitable Sobolev space and so on (see [15], [16], [17], [18]),

(3) $S_{\mathcal{R}}(\Omega) \subset V(\Omega)$ denotes the set of solutions to the state relation $\mathcal{R}$,

(4) the function

$$J : \bigcup_{\Omega\in\mathcal{B}} (\Omega \times S_{\mathcal{R}}(\Omega)) \ni (\Omega, u) \to J(\Omega, u) \in \overline{\mathbb{R}}$$

denotes a cost functional.

Definition 2.1. The pair (Ω^*, u^*) above is called the optimal solution for $(OSD)_{\mathcal{R}}$.

Remark 2.1. The solution set $S_{\mathcal{R}}(\Omega)$ reduces to the one element for “well posed” problems as it will be the case below for elliptic, parabolic and hyperbolic PDE’s, (but in general it contains more than one element, e.g. for (HVI) – see [17]). In such a case the double minimization in $(\mathrm{OSD})_{\mathcal{R}}$ also reduces to the single one.

2.2. An abstract existence theorem. Let $\mathcal{V} = \mathcal{V}(G)$ denote a “universal” space of functions (defined on a big set e.g. $G = B(0, R)$ or $G = \mathbb{R}^N$) in which all the sets (of states) $S_{\mathcal{R}}(\Omega)$, $\Omega \in \mathcal{B}$ can be embedded (using for instance a prolongation operator).

We admit the hypothesis:

(H) Assume some topologies $\tau_{\mathcal{O}}$ (on the set of admissible shapes) and $\tau_{\mathcal{V}}$ (on the “universal” set of states) are introduced in such a way that:

(i) The family $\mathcal{B}$ is $\tau_{\mathcal{O}}$-closed subset of $\mathcal{O} = \mathcal{O}_{ad}$.

(ii) The minimizing sequence $(\Omega_n, u_n) \in \bigcup_{\Omega\in\mathcal{B}} (\Omega \times S_{\mathcal{R}}(\Omega))$, $n = 1, 2, \ldots$ is compact in the sense that it possesses a convergent subsequence in the product topology $\tau_{\mathcal{O}} \times \tau_{\mathcal{V}}$.

(iii) The multifunction $\mathcal{O} \ni \Omega \mapsto S_{\mathcal{R}}(\Omega) \subset \mathcal{V}$ (with nonempty values !) is $\tau_{\mathcal{O}}$-usc (upper semicontinuous in Kuratowski sense), i.e. we have implication

$$\{\Omega_n \subset \mathcal{B},\ \Omega_n \xrightarrow{\tau_{\mathcal{O}}} \Omega,\quad u_n \in S_{\mathcal{R}}(\Omega_n),\ u_n \xrightarrow{\tau_{\mathcal{V}}} u\} \Rightarrow u \in S_{\mathcal{R}}(\Omega). \tag{2.1}$$

(iv) The functional

$$J : \bigcup_{\Omega\in\mathcal{B}} (\Omega \times S_{\mathcal{R}}(\Omega)) \ni (\Omega, u) \mapsto J(\Omega, u) \in \overline{\mathbb{R}}$$

is sequentially lsc (lower semicontinuous) in the product topology $\tau_{\mathcal{O}} \times \tau_{\mathcal{V}}$, i.e.

$$J(\Omega, u) \leq \liminf_{n\to\infty} J(\Omega_n, u_n) \quad \text{for all } (\Omega_n, u_n) \to (\Omega, u). \tag{2.2}$$

Now the direct method for OSD problems can be summarized as follows:

Theorem 2.1. *If the hypothesis* (H) *is satisfied, then the problem* $(\mathrm{OSD})_{\mathcal{R}}$ *admits at least one optimal solution.*

The difficulties in applications of this theorem lay in defining the space $\mathcal{V}$ and good topologies satisfying hypothesis (H) (as a priori there is no linear neither convex structure in the set of admissible shapes).

Remark 2.2. The condition (2.1) means some "closedness" property of the class of relations $\{\mathcal{R}\}$ (e.g. it was extensively investigated in papers concerning the asymptotic behavior of some classes of equations in the homogenization theory – an example due to Cioranescu and Murat ([8]) shows the lack of such property (see also [10]).

Relaxation in state means looking for the smallest (in some sense, e.g. see Propositions 4.2–4.5) closed class of relations containing $\mathcal{R}$.

Remark 2.3. The condition (2.2), where integral depends also on the domain of integration, shows that *relaxation in the cost* (i.e. looking for lsc envelope of the cost functional) is more difficult than in classical Calculus of Variations, where the domain is fixed.

Remark 2.4. There is a "conflict" of topologies in the existence result above. Choosing stronger topologies $\tau_{\mathcal{O}} \times \tau_V$ we easier get the lsc property for functional J in (iv), while the compactness properties for minimizing sequences in (ii) we easier obtain for the weaker topologies. Thus, in order to obtain an existence result we have to find a compromise between these two tendencies.

Finally, we would like to mention that such kind of conditions are satisfied in some restrictive classes of shapes considered in the literature where the existence of optimal shapes was proved, e.g. Murat and Simon [21] developed the so called mapping method where the shapes were images of a fixed set by regular transformations, Pironneau [22] considered the shapes contained "between" two fixed sets with the Hausdorff "complementary metric", i.e. $d_{H^c}(\Omega_1, \Omega_2) = d_H(G \setminus \Omega_1, G \setminus \Omega_2)$, Chenais [6] used the "cone property" in order to preserve the shapes from "wild oscillations" of their boundary (which occur in homogenization theory for periodic structures), while Šverák [27] assumed that the complements of admissible shapes had uniformly bounded number of connected components.

3. Preliminaries

Let G be an open and bounded subset of $\mathbb{R}^N$. Admissible shapes are defined as elements of the set $\mathcal{A}(G) := \{\Omega \subset G : \Omega \text{ is open}\}$. So we do not impose other restrictions on shapes, i.e. $\mathcal{B} = \mathcal{A}(G)$.

Given a matrix of functions $a_{i,j} \in L^\infty(G)$, $i, j = 1, \dots, N$, satisfying the usual coercivity assumption

$$\sum_{i,j=1}^{N} a_{i,j}\xi_i\xi_j \geq c|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^N$$

for some $c > 0$, we set

$$Au = -\sum_{i,j=1}^{N} D_i(a_{i,j}(x)D_j u).$$

So we obtain the whole family $A : H_0^1(\Omega) \to H^{-1}(\Omega)$, $(\Omega \in \mathcal{A}(G))$ of linear elliptic operators defined on usual Sobolev spaces and with values in their duals.

After [13] we admit

Definition 3.1. By the harmonic capacity of E with respect to G we mean

$$\operatorname{cap}(E, G) = \inf\{\|u\|_{H_0^1(G)} : u \in H_0^1(G),\ u(x) \geq 1 \text{ on a neighbourhood of } E\}.$$

The set $U \subset G$ is called quasi open if for every $\varepsilon > 0$ there exists $E \subset G$ such that $\operatorname{cap}(E, G) < \varepsilon$ and $U \cup E$ is open. In such a case the set $G \setminus U$ is called quasi closed. We say that some property $P(x)$ holds quasi everywhere in E (q.e. for short) if it is satisfied for all $x \in E$ except a set N with $cap(N, G) = 0$.

Definition 3.2. By $\mathcal{M}_0(G)$ we denote the set of all nonnegative Borel measures on G, possibly infinite and satisfying properties:

(i) $\mu(B) = 0$ for all B with $\operatorname{cap}(B, G) = 0$,
(ii) $\mu(B) = \inf\{\mu(E) : E \text{ is quasi open},\ B \subset E\}$.

As examples of measures in the class $\mathcal{M}_0(G)$ we quote:

(1) $\varphi\mathcal{L}^N \in \mathcal{M}_0(G)$, for all $\varphi \in L^\infty(G)$, $\varphi \geq 0$ ($\mathcal{L}^N$ being the N-dimensional Lebesgue measure),

(2) $\mathcal{H}^\alpha \in \mathcal{M}_0(G)$, for all $N-2 < \alpha \leq N$, $\mathcal{H}^\alpha$ being the α-dimensional Hausdorff measure.
(This is a consequence of the two following implications: $\mathcal{H}^{N-2}(B) < \infty \Rightarrow \mathrm{cap}(B,G) = 0$, $\mathrm{cap}(B,G) = 0 \Rightarrow \mathcal{H}^{N-2+\varepsilon}(B) = 0$, for all $\varepsilon > 0$).

(3) The measure

$$\infty_S(B) = \begin{cases} 0 & \text{if } \mathrm{cap}(B \cap S, G) = 0, \\ \infty & \text{otherwise,} \end{cases}$$

belongs to $\mathcal{M}_0(G)$ for every quasi closed set S, and so does the measure which plays important role in the sequel: $\mu_\Omega = \infty_{G\setminus\Omega}$ for every open $\Omega \subset G$, i.e.

$$\mu_\Omega(B) = \begin{cases} 0 & \text{if } \mathrm{cap}(B \setminus \Omega, G) = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{3.1}$$

4. Relaxation in state

As we have already mentioned in Remark 2.2, the classes of PDE's which appear in natural way in formulating OSD problems are in general not closed under the passage to the limit of their solutions (extended in some way in order they were defined on the same domain). Thus it appears the problem of finding the smallest (in the sense of dense embedding) classes of equations which are closed in suitable topologies. It appears that suitable topology for the shapes will be in all cases (elliptic, parabolic and hyperbolic) the so called γ_A-convergence ($\tau_{\mathcal{O}} = \gamma_A$), while the topology for the solutions will depend on the case.

4.1. Elliptic case. In 1974 it appeared (in connection with the homogenization theory) the paper [8] by Cioranescu-Murat in which the authors proved that in the periodically perforated domains with suitable critical size of holes the solutions of homogeneous Dirichlet problems converge to a function which is the solution of some modified equation with an additional term "venu d'ailleurs".

This and similar examples have led Dal Maso and Mosco (see [13], [14] and also [3], [4], [5]) to considering, together with the class of let say "classical" Dirichlet problems:

$$\begin{cases} Au = f, \\ u \in H_0^1(\Omega), \end{cases} \tag{DP$_\Omega$}$$

($\Omega \in \mathcal{A}(G), f \in H^{-1}(\Omega)$), the new class introduced by them and called the relaxed Dirichlet problems:

$$\begin{cases} Au + \mu u = f, \\ u \in V_\mu(G), \end{cases} \tag{RDP$_\mu$}$$

with $\mu \in \mathcal{M}_0(G)$, $f \in V'_\mu(G)$.

The meaning of the solutions to these problems is precised in the definitions below for which besides the standard Gelfand–Lions triplet of Sobolev spaces $V(\Omega) = H_0^1(\Omega)$, $H(\Omega) = L^2(\Omega)$, $V'(\Omega) = H^{-1}(\Omega)$ (for any $\Omega \in \mathcal{A}(G)$) with dense and continuous embeddings ($V(\Omega) \subset H(\Omega) \subset V'(\Omega)$) we have to introduce the new spaces. Namely, we set

$$V_\mu = V_\mu(G) := H_0^1(G) \cap L_\mu^2(G)), \tag{4.1}$$

where $L_\mu^2(G)$ denotes the space of all square summable functions with respect to the measure μ.

The space V_μ becomes the Hilbert space with the scalar product

$$(u, v)_\mu = \int_G DuDvdx + \int_G uvd\mu. \tag{4.2}$$

Let V'_μ denote the dual space of V_μ and $\langle\,\cdot\,,\,\cdot\,\rangle_\mu$ be the duality pairing. Since in this case V_μ is (in general) not dense in $L^2(G)$ and the latter is not the pivot space we do not identify the isomorphic spaces V_μ and V'_μ.

REMARK 4.1 (see [3]). Even if the transposed mappings to the embeddings of V_μ into $H_0^1(G)$ and into $L^2(G)$ are not injective the spaces $H^{-1}(G)$ and $L^2(G)$ can be considered as the linear subspaces of V'_μ and we have:

$$\langle f, v\rangle_\mu = \langle f, v\rangle_G \quad \text{for } f \in H^{-1}(G), v \in V_\mu$$

and, in particular,

$$\langle f, v\rangle_\mu = \int_G fv\,dx \quad \text{for } f \in L^2(G),\ v \in V_\mu$$

where for any $\Omega \in \mathcal{A}(G)$ we denote by $\langle\,\cdot\,,\,\cdot\,\rangle_\Omega$ the duality pairing between $V(\Omega)$ and $V'(\Omega)$.

DEFINITION 4.1. The function u_Ω is called the solution of $(\mathrm{DP})_\Omega$ if and only if

$$\begin{cases} \langle Au_\Omega, v\rangle_\Omega = \langle f, v\rangle_\Omega & \text{for all } v \in H_0^1(\Omega), \\ u_\Omega \in H_0^1(\Omega), \end{cases}$$

We then set

$$\widetilde{u}_\Omega = \begin{cases} u_\Omega & \text{on } \Omega, \\ 0 & \text{on } G \setminus \Omega. \end{cases}$$

DEFINITION 4.2. The function u_μ is called the solution of $(\mathrm{RDP})_\mu$ with fixed $f \in V'_\mu(G)$ if and only if

$$\begin{cases} \langle Au_\mu, v\rangle_G + \int_G u_\mu v\,d\mu = \langle f, v\rangle_\mu & \text{for all } v \in V_\mu(G), \\ u_\mu \in V_\mu(G). \end{cases}$$

For the justification (with the use of the Riesz–Fréchet representation theorem for the scalar product in $V_\mu(G)$) see [3].

Due to the well known Lax–Milgram lemma one can get

Proposition 4.1. *The* $(\mathrm{RDP})_\mu$ *problem has the unique solution but it cannot be understood in the distributional sense* (*unless* μ *is a Radon measure*) *as in general the set* $C_0^\infty(G)$ *is not contained in* $V_\mu(G)$.

Roughly speaking the idea of elliptic relaxation for symmetric operators, based on Γ convergence (the general case is based on capacitary methods [11], [12]), can be described in three steps.

(1) First: consider the injective mapping $\mathcal{A}(G) \ni \Omega \to F_\Omega \in \mathcal{S}_\Psi(H_0^1(G))$ where

$$F_\Omega(u) = \begin{cases} \langle Au, u\rangle_\Omega - 2\langle f, u\rangle_\Omega & \text{on } H_0^1(\Omega), \\ \infty & \text{on } H_0^1(G) \setminus H_0^1(\Omega), \end{cases}$$

is the energy functional for $(\mathrm{DP})_\Omega$, and the space $\mathcal{S}_\Psi(H_0^1(G))$ (of all w-lsc functionals bounded from below by a w-lsc and coercive function Ψ) is (see [9]) *compactly metrizable by the metric* d_Γ introduced by the De Giorgi Γ convergence.

(2) Next: transport this metric to the set of shapes

$$d(\Omega_1, \Omega_2) = d_\Gamma(F_{\Omega_1}, F_{\Omega_2}).$$

(3) Finally: make standard completion operation of the so obtained metric space $(\mathcal{A}(G), d)$ getting the new space of "generalized shapes" *isomorphic* to the space of measures $\mathcal{M}_0(G)$ with corresponding metric called "γ^A *convergence*".

Hence and from the well known (see [9]) properties of Γ convergence one can justify the formal definition and characterization of γ^A convergence given precisely below and obtain the following statements.

Definition 4.3. Given a sequence $\{\mu_n\}$ and μ in $\mathcal{M}_0(G)$ we admit:

$$\mu_n \xrightarrow{\gamma^A} \mu \overset{\mathrm{df}}{\Longleftrightarrow} u_{\mu_n}(f) \xrightarrow{w-H_0^1(G)} u_\mu(f) \quad \text{for all } f \in L^2(G),$$

where by $u_\mu(f)$ we denote the solution to $(\mathrm{RDP})_\mu$ with the right-hand side f.

Let put $w_\mu = u_\mu(1)$. These solutions ($\mu \in \mathcal{M}_0(G)$) play important role in relaxation as it is seen from remarks below (see [11], [3]).

Remark 4.2. The γ^A convergence defined above is equivalent to the condition

$$w_{\mu_n} := u_{\mu_n}(1) \xrightarrow{w-H_0^1(G)} u_\mu(1) =: w_\mu,$$

as well as (in the case of symmetric A) to the Γ convergence of the energy functionals $F_{\mu_n} \to F_\mu$.

The energy functionals above are defined by

$$F_\mu(u) = \begin{cases} \langle Au, u\rangle_G + \displaystyle\int_G u^2\,d\mu & \text{on } H_0^1(\Omega), \\ \infty & \text{on } L^2(G)\setminus H_0^1(\Omega). \end{cases}$$

REMARK 4.3. Let us fix $\mu \in \mathcal{M}_0(G)$.

(1) The sets

$$R(\mu) := \{x \in G : w_\mu(x) > 0\}, \quad S(\mu) := \{x \in G : w_\mu(x) = 0\}$$

are called the regular and singular sets for μ and, respectively, they are quasi open and quasi closed subsets of G.

(2) The family of functions $\{\varphi w_\mu\}_{\varphi\in C_0^\infty(G)}$ is dense in $V_\mu(G)$.

(3) The closure operations (in the strong topologies) lead to results:

$$\mathrm{cl}_{H_0^1(G)}(V_\mu(G)) = H_0^1(R(\mu)),$$
$$\mathrm{cl}_{L^2_\mu(G)} V_\mu(G) = \{u \in L^2(G) : u(x) = 0 \quad \mu \text{ a.e. on } S(\mu)\}.$$

PROPOSITION 4.2. *The classical Dirichlet problems* $\{(\mathrm{DP})_\Omega\}_{\Omega\in\mathcal{A}(G)}$ *can be embedded in the new class of relaxed Dirichlet problems* $\{(\mathrm{RDP})_\mu\}_{\mu\in\mathcal{M}_0(G)}$ *by means of the measure* μ_Ω *given by* (3.1), *i.e. we have:*

$$u \in H_0^1(\Omega) \textit{is the solution of } (\mathrm{DP})_\Omega \Leftrightarrow u_{\mu_\Omega} = \widetilde{u}_\Omega \textit{ is the solution of } (\mathrm{RDP})_\mu,$$

or in other words

$$\begin{cases} Au = f, \\ u \in H_0^1(\Omega), \\ u(x) = 0 & \textit{q.e. in } G\setminus\Omega, \end{cases} \quad\Leftrightarrow\quad \begin{cases} A\widetilde{u}_\Omega + \mu_\Omega\widetilde{u}_\Omega = f, \\ \widetilde{u}_\Omega \in V_{\mu_\Omega}(G). \end{cases}$$

PROPOSITION 4.3. *The embedding* (*of classical shapes into generalized ones*)

$$\mathcal{A}(G) \ni \Omega \to \mu_\Omega \in \mathcal{M}_0(G)$$

is dense with respect to γ^A *convergence.*

PROPOSITION 4.4 (Compactness). *Every sequence of measures in* $\mathcal{M}_0(G)$ *contains a* γ^A *convergent subsequence.*

PROPOSITION 4.5. *The class* $\{(RDP)_\mu\}_{\mu\in\mathcal{M}_0(G)}$ *is closed under the* γ^A *convergence. We have even stronger implication:*

$$f_n \xrightarrow{s-H^{-1}(G)} f, \quad \mu_n \xrightarrow{\gamma^A} \mu \;\Rightarrow\; u_{\mu_n}(f_n) \xrightarrow{w-H_0^1(G)} u_\mu(f)$$

(*as before* $u_\mu(f)$ *being the solution to* $(\mathrm{RDP})_\mu$ *with the right-hand side* f).

(Thus, the condition (2.1) is satisfied!)

4.2. Parabolic case. In this subsection we base on results by M. Smołka (see Ph. D. Thesis 1999 [24]). J. P. Raymond (U. P. S. Toulouse) has obtained similar results (private communication).

We consider the class of parabolic problems ($\Omega \in \mathcal{A}(G)$, $Q_\Omega = (0,T) \times \Omega$):

$$(\mathrm{PP})_\Omega \qquad \begin{cases} u' + Au = f & \text{in } Q_\Omega, \\ u(0) = u^0 & \text{on } \Omega, \\ u \in W(0,T;\Omega), \end{cases}$$

where the weak solution is understood in the sense of Lions-Magenes ([19]) and

$$W(0,T;\Omega) := \{u \in L^2(0,T;H_0^1(\Omega)); u' \in L^2(0,T;H^{-1}(\Omega))\}$$

(the time derivative u' above is taken in the distributional sense).

The analogous as in elliptic case "closedness" property for the above problems leads to the new class of relaxed parabolic problems with "measure coefficients" ($\mu \in \mathcal{M}_0(G)$):

$$(\mathrm{RPP})_\mu \qquad \begin{cases} u' + Au + \mu u = f, \\ u(0) = u^0, \\ u \in W_\mu(0,T;G). \end{cases}$$

For the existence of solutions to $(\mathrm{RPP})_\mu$ the problem is to find a good evolution triple (and then apply the general theory of PDEs as, for instance in Lions-Magenes [19]). Setting

$$\begin{cases} V_\mu(G) = H_0^1(G) \cap L_\mu^2(G), \\ H_\mu(G) = \text{the } (s - L^2(G))\text{-closure of } V_\mu(G), \\ V_\mu'(G) = \text{the dual of } V_\mu(G) \end{cases}$$

we have $V_\mu(G) \subset H_\mu(G) \subset V_\mu'(G)$ with all the embeddings above being continuous, dense and compact (see [24]).

So (using e.g. the Galerkin method) we can obtain

PROPOSITION 4.6. *The* $(\mathrm{RPP})_\mu$ *posseses the unique solution* (*in the sense of the definition below*)

$$u \in W_\mu = W_\mu(0,T;G) := \{u \in L^2(0,T);V_\mu);\ u' \in L^2(0,T;V'{}_\mu)\}$$

and the last space is continuously imbeded in $C([0,T];H_\mu)$*, so the initial condition has sense.*

DEFINITION 4.4. By the solution to the relaxed problem $(\mathrm{RPP})_\mu$ above we mean a function $u \in W_\mu(0,T;G)$, satisfying the initial condition and the following equation

$$\langle u'(t), v\rangle_\mu + \langle Au(t), v\rangle_G + \int_\Omega u(t) v \, d\mu = \langle f(t), v\rangle_\mu$$

for every $v \in V_\mu$ and a.e. in $(0,T)$ or, equivalently the equation:

$$\int_0^T \langle u'(t), v\psi(t)\rangle_\mu \, dt + \int_0^T \langle Au(t), v\psi(t)\rangle_G \, dt + \int_0^T \int_\Omega u(t,x)v(x)\psi(t) d\mu(x)\, dt = \int_0^T \langle f(t), v\psi(t)\rangle_\mu dt$$

for every $v \in V_\mu$ and $\psi \in C_0^\infty((0,T))$ ($\langle \cdot\, , \cdot \rangle_\mu$ being the duality between V_μ and V'_μ).

Similarly as in elliptic case, we have

Proposition 4.7. *The class of classical parabolic problems can be embedded in the new class of relaxed parabolic problems by means of measures μ_Ω given by* (3.1)*, i.e. for any $\Omega \in \mathcal{A}(G)$ it holds*

$$\begin{cases} u \in W(0,T;\Omega) \\ \textit{is the solution of } (\mathrm{PP})_\Omega \end{cases} \Leftrightarrow \begin{cases} u_{\mu_\Omega} = \widetilde{u} \in W_{\mu_\Omega}(0,T;G) \\ \textit{is the solution of } (\mathrm{RPP})_{\mu_\Omega}, \end{cases}$$

where we admited

$$\widetilde{u} = \begin{cases} u & \textit{on } (0,T)\times\Omega, \\ 0 & \textit{on } (0,T)\times(G\setminus\Omega). \end{cases}$$

We have also the closedness theorem for the new class of $(\mathrm{RPP})_\mu$ problems. Let us consider the whole sequence of problems:

$$(\mathrm{RPP})_{\mu_n} \qquad \begin{cases} u'_n + Au_n + \mu_n u_n = f_n, \\ u_n(0) = u_n^0, \\ u_n \in W_{\mu_n}(0,T;G). \end{cases}$$

We have (see [24])

Theorem 4.1. *Let u and u_n be solutions of* $(\mathrm{RPP})_\mu$ *and* $(\mathrm{RPP})_{\mu_n}$*, respectively and assume $f, f_n \in L^2(Q_G)$, $u^0 \in V_\mu(G)$, $u_n^0 \in V_{\mu_n}(G)$. Suppose*

(i) $\mu_n \xrightarrow{\gamma^A} \mu$,
(ii) $f_n \to f$ *weakly in* $L^2(Q_G)$,
(iii) $u_n^0 \to u^0$ *weakly in* $H_0^1(G)$,
(iv) $||u_n^0||_{L^2_{\mu_n}(G)} \le M$, *for some* $M > 0$ *and for all* $n \in \mathbb{N}$.

Then $u_n \to u$ weakly in $W(0,T;G)$. Moreover,

(1) $u_n \to u$ *strongly in* $L^2(Q_G)$,
(2) $u_n \to u$ *strongly in* $C(0,T;L^2(G))$,
(3) $u_n \to u$ *weakly* $*$ *in* $L^\infty(0,T;H_0^1(G))$,
(4) $u'_n \to u'$ *weakly in* $L^2(Q_G)$,
(5) $u_n(t) \to u(t)$ *weakly in* $H_0^1(G)$ *for all* $t \in [0,T]$.

4.3. Hyperbolic case. For the result of this section one can also consult R. Toader [28] who extended some results of paper [7].

Let us consider the class of hyperbolic problems ($\Omega \in \mathcal{A}(G), Q_\Omega = (0,T)\times\Omega$):

$$(\mathrm{HP})_\Omega \qquad \begin{cases} u'' + Au = f & \text{in } Q_\Omega,\\ u(0) = u^0,\ u'(0) = u^1 & \text{on } \Omega,\\ u = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases}$$

where we extend the solution u to the function $\widetilde{u}$ setting

$$\widetilde{u} = \begin{cases} u & \text{on } (0,T)\times\Omega,\\ 0 & \text{on } (0,T)\times(G\setminus\Omega). \end{cases} \tag{4.3}$$

The same (as in parabolic case) "closedness" property for the above class of problems leads to the new class of relaxed hyperbolic problems with "measure coefficients" ($\mu \in \mathcal{M}_0(G)$):

$$(\mathrm{RHP})_\mu \qquad \begin{cases} u\text{"} + Au + \mu u = f & \text{in } Q_G,\\ u(0) = u^0,\ u'(0) = u^1, \\ u \in W_\mu(0,T;V_\mu,H_\mu). \end{cases}$$

In analogy to the standard notation for the space of solutions to the "classical" $(\mathrm{HP})_\Omega$ problem:

$$\begin{aligned} &W(0,T;H_0^1(\Omega),L^2(\Omega))\\ &\quad = \{u \in L^2(0,T;H_0^1(\Omega)) : u' \in L^2(0,T;L^2(\Omega)),\ u'' \in L^2(0,T;H^{-1}(\Omega))\}, \end{aligned}$$

for the relaxed problem $(\mathrm{RHP})_\mu$ we admit as the space of solutions:

$$\begin{aligned} &W_\mu(0,T;V_\mu(G),H_\mu(G))\\ &\quad = \{u \in L^2(0,T;V_\mu(G)) : u' \in L^2(0,T;H_\mu(G)),\ u\text{"} \in L^2(0,T;V_\mu'(G))\}. \end{aligned}$$

The meaning of such relaxed problem is clarified by the definition:

DEFINITION 4.6. A function $u \in W_\mu(0,T;V_\mu,H_\mu)$ is called the solution of $(\mathrm{RHP})_\mu$ with $f \in L^2(Q_G)$, $u^0 \in V_\mu$, $u^1 \in H_\mu$ if and only if it satisfies the initial conditions above and for every $v \in V_\mu$ the equality

$$\frac{d^2}{dt^2}\int_G u(t)v\,dx + \int_G \sum_{i,j=1}^N a_{i,j}(x)D_iu(t)D_jv\,dx + \int_G u(t)v\,d\mu = \int_G f(t)v\,dx$$

holds in the distributional sense on $[0,T]$, i.e. for every $\psi \in C_0^\infty((0,T))$ we have:

$$\begin{aligned} &\int_0^T \frac{d^2}{dt^2}\langle u(t), v\psi(t)\rangle_G\,dt + \int_0^T \langle Au(t), v\psi(t)\rangle_G\,dt\\ &\qquad + \int_0^T\int_G u(t)v\psi(t)\,d\mu\,dt = \int_0^T \langle f(t), v\psi(t)\rangle_G\,dt \quad \text{for all } v \in V_\mu. \end{aligned}$$

For this class similar results as in parabolic case can be obtained. Namely, we have:

Proposition 4.8 (Existence and regularity). *For given* $\mu \in \mathcal{M}_0(G)$, $f \in L^2(Q_G)$, $u^0 \in V_\mu$, $u^1 \in H_\mu$ *there exists the unique solution* u_μ *of the problem* $(\mathrm{RHP})_\mu$ *and moreover, we have* $u_\mu \in C^0(0,T;V_\mu) \cap C^1(0,T;H_\mu)$ (*so the both initial conditions have sense*).

Proposition 4.9 (Dense embedding). *The class of the classical hyperbolic problems* $\{(\mathrm{HP})_\Omega\}_{\Omega\in\mathcal{A}(G)}$ *can be* (*densely*) *embedded* (*by means of measures* μ_Ω *given by* (3.1)) *into the new class of* $\{(\mathrm{RHP})_\mu\}_{\mu\in\mathcal{M}_0(G)}$, *i.e. for given* $\Omega \in \mathcal{A}(G)$ *and* $\widetilde{u}$ *given by* (4.3) *it holds*

$$\left\{\begin{array}{l} u \in W(0,T;H_0^1(\Omega),L^2(\Omega)) \\ \textit{is the solution of } (\mathrm{HP})_\Omega \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} u_{\mu_\Omega} = \widetilde{u} \in W_{\mu_\Omega}(0,T;V_{\mu_\Omega},H_{\mu_\Omega}) \\ \textit{is the solution of } (\mathrm{RHP})_{\mu_\Omega}. \end{array}\right.$$

We also have the closedness theorem for the new class of $\{(\mathrm{RHP})_\mu\}_{\mu\in\mathcal{M}_0(G)})$ problems (see [28]). Namely, let us consider the whole sequence of problems

$$(\mathrm{RHP})_{\mu_n} \qquad \left\{\begin{array}{l} u_n'' + Au_n + \mu_n u_n = f_n, \\ u_n(0) = u_n^0, u_n'(0) = u_n^1, \\ u_n \in W_{\mu_n}(0,T;V_\mu,H_\mu). \end{array}\right.$$

Theorem 4.2. *Let* $\mu_n, \mu \in \mathcal{M}_0(G)$ *and* u_n *be solutions of* $(\mathrm{RHP})_{\mu_n}$ *problems and assume* $f, f_n \in L^2(Q_G)$, $u_n^0 \in V_{\mu_n}$, $u_n^1 \in H_{\mu_n}$. *Suppose*

(i) $\mu_n \xrightarrow{\gamma^A} \mu$,
(ii) $f_n \to f$ *weakly in* $L^2(Q_G)$,
(iii) $u_n^0 \to u^0$ *weakly in* $H_0^1(G)$,
(iv) $||u_n^0||_{L^2_{\mu_n}(G)} \leq M$, *for all* $n \in \mathbb{N}$, *for some* $M > 0$,
(v) $u_n^1 \to u^1$ *weakly in* $L^2(G)$ *and* $u^1 \in H_\mu$.

Then $u_0 \in V_\mu$ *and*

$$\begin{aligned} u_n \to u \quad & \textit{in } w - * - L^\infty(0,T;H_0^1(G)), \\ u_n' \to u' \quad & \textit{in } w - * - L^\infty(0,T;L^2(G)), \\ ||u_n||_{L^\infty(0,T;V_\mu)} & \leq C, \end{aligned}$$

where u *is the solution of* $(\mathrm{RHP})_\mu$ *and* C *is a constant independent of* n. *Moreover, for every* $\Theta \in H^{-1}(G)$ *it holds*

$$\langle \Theta, u_n(\,\cdot\,)\rangle_G \to \langle \Theta, u(\,\cdot\,)\rangle_G \quad \textit{in } s - C^0([0,T])$$

(*i.e.* $u_n(t) \to u(t)$ *in* $w - H_0^1(G)$, *uniformly in* $t \in [0,T]$).

5. Relaxation in OSD-problems

5.1. An abstract relaxation scheme. Let (Y, σ), (Z, τ) be two topological spaces (τ satisfying the I-axiom of countability for simplicity sake) with continuous and dense embedding $Y \subset Z$. Given a functional $J : Y \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ we admit the definition.

DEFINITION 5.1. The functional $\overline{J} : Z \longrightarrow \overline{\mathbb{R}}$ is called relaxation of J iff $\overline{J}$ is the greatest functional τ_{seq}-lsc and majorized by J on Y, i.e. $\overline{J}|_Y \le J$.

We have for every $z \in Z$

$$\overline{J}(z) = \inf\{\liminf_{n\to\infty} J(y_n) : y_n \to z\}$$

and the following theorem hold:

THEOREM 5.1. *If $\overline{J}$ is coercive, i.e.*

$$\forall t \in \mathbb{R}\ \exists K_t(\mathit{compact}) \subset Z;\ \{z : \overline{J}(z) \le t\} \subset K_t,$$

then it admits its minimum: (exists $z_0 \in Z$ such that)

$$\overline{J}(z_0) = \min_Z \overline{J} = \inf_Y J.$$

Moreover, for any minimizing sequence $\{y_k\}$ of J we have implication

$$y_{k_\nu} \xrightarrow{\tau} z \Rightarrow \overline{J}(z) = \min_Z \overline{J} = \inf_Y J$$

and also some "inverse" implication holds:

$$\overline{J}(z) = \min_Z \overline{J} \Rightarrow \exists(\mathit{m.s.}\ \{y_k\} \subset Y),\ y_k \xrightarrow{\tau} z.$$

(Above m.s. stands for "minimizing sequence").

In $(\mathrm{OSD})_{\mathrm{PDE}}$-problems considered previously we set:

$$Y = \mathcal{A}(G), \quad Z = \mathcal{M}_0(G), \quad (\Omega \sim \mu_\Omega) \quad \sigma = \tau = \gamma^A\text{-convergence}.$$

5.2. Statement of OSD problems for evolution equations. OSD problems for systems governed by elliptic Dirichlet problems (DP) were considered in papers [3], [4], [5], so we omit here this case.

For statement of OSD problems for evolution case it is imporant to define how to generate, respectively, from fixed $f \in L^2(Q_G)$ (the right-hand side of the equation) and $u^0 \in H_0^1(G)$ (the initial condition in (PP) or in (HP) problem) and also from $u^1 \in L^2(G)$ (the initial condition for velocity in (HP) problem) the appropriate values in $L^2(Q_\Omega)$ and the initial conditions on moving domains ($\Omega \in \mathcal{A}(G)$) in the case of classical (OSD) problems or in moving spaces V_μ, ($\mu \in \mathcal{M}_0(G)$), in the case of relaxed (ROSD) problems below. This can be done, respectively, by taking the restrictions $f|_{Q_\Omega}$ and solving an additional relaxed elliptic problem. Namely, let us define operator $P_\mu : H_0^1(G) \to V_\mu$ by setting $P_\mu v = u_\mu$, where u_μ is the solution to the problem:

$$(\mathrm{RDP})_\mu \qquad \begin{cases} Au + \mu u = Av, \\ u \in V_\mu. \end{cases}$$

Since for $\Omega \in \mathcal{A}(G)$ we have $V_{\mu_\Omega} = H_0^1(\Omega)$ (where μ_Ω is given by (3.1)) thus setting $u_\Omega^0 = P_{\mu_\Omega} u^0$ we get the initial condition in the required space also in the classical $(\mathrm{OSD})_{\mathrm{PP}}$ problem below.

This family of operators can be considered also as applications from $H_0^1(G)$ into itself and it has good properties for our purposes. Namely, we have (see [25] and also Proposision 4.5 above):

Proposition 5.1. *The operators P_μ are linear and uniformly bounded, i. e. there exists constant $C > 0$ such that*

$$||P_\mu||_{\mathcal{L}(H_0^1(G), V_\mu)} \le C, \quad \textit{for all } \mu \in \mathcal{M}_0(G).$$

Moreover, if $\mu_n \xrightarrow{\gamma^A} \mu$, then $P_{\mu_n} v \xrightarrow{w-H_0^1(G)} P_\mu v$, for all $v \in H_0^1(G)$.

In the sequel for more compact writing we denote, respectively by $S_{\mathrm{PP}}(\Omega)$, $S_{\mathrm{RPP}}(\mu)$, $S_{\mathrm{HP}}(\Omega)$, $S_{\mathrm{RHP}}(\mu)$ the one element sets of solutions, respectively to $(\mathrm{PP})_\Omega$, $(\mathrm{RPP})_\mu$, and so on (see Remark 2.1).

Now, given $f \in L^2(Q_G)$ and $u^0 \in H_0^1(G)$ we consider the classical and relaxed (OSD) problems:

$$(\mathrm{OSD})_{\mathrm{PP}} \qquad \min_{\Omega \in \mathcal{A}(G)} \{J(\Omega, u) : u \in S_{\mathrm{PP}}(\Omega),\ u(0) = P_{\mu_\Omega} u^0\},$$

$$(\mathrm{ROSD})_{\mathrm{PP}} \qquad \min_{\mu \in \mathcal{M}_0(G)} \{\overline{J}(u_\mu) : u_\mu \in S_{\mathrm{RPP}}(\mu),\ u_\mu(0) = P_\mu u^0\}.$$

Analogous OSD problems can be considered for hyperbolic state equations with a bit more regular initial condition ($u^1 \in H_0^1(G)$)

$$(\mathrm{OSD})_{\mathrm{HP}} \qquad \min_{\Omega \in \mathcal{A}(G)} \{J(\Omega, u) : u \in S_{HP}(\Omega),\ u(0) = P_{\mu_\Omega} u^0, u'(0) = P_{\mu_\Omega} u^1\},$$

$$(\mathrm{ROSD})_{\mathrm{HP}} \qquad \min_{\mu \in \mathcal{M}_0(G)} \{\overline{J}(u_\mu); u_\mu \in S_{RHP}(\mu),\ u_\mu(0) = P_\mu u^0, u_\mu'(0) = P_\mu u^1\}.$$

5.3. Results in OSD for evolution systems. Natural cost functionals for OSD in evolution problems have the form

$$J(\Omega, u) = \int_0^T \int_\Omega j(t, x, u(t,x), Du(t,x))\, dt\, dx + \int_\Omega \widehat{j}(x, u(T,x), Du(T,x))\, dx.$$

Concerning integrands we assume hypothesis H(J):

(J1) $j : Q_G \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $\widehat{j} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Borel functions,

(J2) $j(t, x, \cdot\,, \cdot)$ and $\widehat{j}(x, \cdot\,, \cdot)$ are lsc for almost all $t \in (0, T)$ and $x \in \Omega$, (J3) $j(t,x,s,\cdot\,)$ and $\widehat{j}(x,s,\cdot\,)$ are convex for almost all $t\in(0,T)$, $x\in\Omega$ and all $s\in\mathbb{R}$,

(J4) there exist functions $a_1\in L^1(Q_G)$, $a_2\in L^1(\Omega)$ and constants $b_1,b_2,c_1,c_2\in\mathbb{R}$, such that

$$\begin{aligned} j(t,x,s,\xi) &\geq a_1(t,x)-b_1|s|^2-c_1|\xi|^2,\\ \widehat{j}(x,s,\xi) &\geq a_2(x)-b_2|s|^2-c_2|\xi|^2, \end{aligned}$$

for almost all $t\in(0,T)$, $x\in\Omega$ and all $s\in\mathbb{R}$ i $\xi\in\mathbb{R}^N$.

We formulate first the case the functional does not depend explicitly on the domain of integration, we assume it is fixed and equal to G.

THEOREM 5.2. *Under the above assumptions the relaxed functional for*

$$J:\mathcal{A}(G)\ni\Omega\to J(G,u_\Omega)\in\overline{\mathbb{R}}$$

is given by

$$\overline{J}:M_0(G)\ni\mu\to J(G,u_\mu)\in\overline{\mathbb{R}},$$

where u_Ω is the solution to $(\mathrm{PP})_\Omega$, (*respectively,* $(\mathrm{HP})_\Omega$), *u_μ is the solution to the corresponding relaxed evolution equation* $(\mathrm{RPP})_\mu$, (*respectively,* $(\mathrm{RHP})_\mu$).

PROOF. The proof follows from Theorem 4.1 (respectively, Theorem 4.2) and from already classical results (e.g. see [2] [9]) for lower semicontinuity of the integral functionals appearing in the definition of J. □

Next we can apply general Theorem 5.2 of relaxation getting the following result in the parabolic case (similar result holds for the hyperbolic case so its formulation will be omitted):

THEOREM 5.3. *Under hypothesis* H(J) *the problem* $(\mathrm{ROSD})_{\mathrm{PP}}$ *admits the solution and we have*

$$\min_{\mu\in\mathcal{M}_0(G)} J(G,u_\mu)=\inf_{\Omega\in\mathcal{A}(G)} J(G,u_\Omega).$$

Moreover, for a function $u\in W(0,T;G)$ the following two conditions are equivalent:

(i) *there exists a minimizing sequence $\{\Omega_n\}$ of the problem* $(\mathrm{OSD})_{\mathrm{PP}}$ *such that $u_{\Omega_n}\to u$ weakly in $W(0,T;G)$,*

(ii) *there exists the minimal point of the relaxed problem* $(\mathrm{ROSD})_{\mathrm{PP}}$ *such that $u=u_\mu$.*

PROOF. It follows directly from Theorem 5.1 and Theorem 5.2. □

REMARK 5.1. The calculation of the relaxed functional $\overline{J}$ for general cost functional $J(\Omega,u)$ is much more complicated. We quote here only a simplified version of the example from [5] concerning the elliptic OSD problem. Namely, for

$$J(\Omega, u) = \int_{\Omega} j(u, u_\Omega(x)\, dx,$$

where $u = u_\Omega$ is the solution of $(\mathrm{DP})_\Omega$, under anologous hypothesis as H(J) the γ^A-lsc regularization is given by the formula

$$\overline{J}(u_\mu) = \int_{R(\mu)} j(x, u_\mu(x))\, dx + \inf\left\{ \int_{B\setminus R(\mu)} j(x,0)\, dx;\; B \in \mathcal{B} \right\},$$

where $R(\mu)$ is the regular set of μ (see Remark 4.3),

$$\mathcal{B} := \{B : B \text{ is Borel subset of } G,\; B \supset R(\mu)\}$$

and as usual we admit the convention that inf taken over the empty set is equal to ∞.

5.4. Nonexistence of optimal solutions to OSD problems. The nonexistence in OSD problems could be a consequence of lack either of the "closedness property" (2.1) for state relation or of the "lsc-property" (2.2) for functionals.

For the convenience of the reader we quote an example due to Buttazzo–Dal Maso [3] concerning the elliptic case which was generalized by Smołka [25] to the parabolic case (it can be also modified for the hyperbolic case).

Let w_0 be the solution of the Dirichlet problem on $G = B(0, R) \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta w_0 = 1, \\ w_0 \in H_0^1(G). \end{cases}$$

So $w_0(x) = (R^2 - |x|^2)/4$ for $|x| < R$. We consider:

$$(\mathrm{OSD})_{\mathrm{DP}} \qquad \min_{\Omega \in \mathcal{A}(G)} \int_G |2u_\Omega - w_0|^2\, dx$$

where

$$u_\Omega = \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } G \setminus \Omega, \end{cases}$$

and u is the solution to the following Dirichlet problem

$$(\mathrm{DP})_\Omega \qquad \begin{cases} -\Delta u = 1, \\ u \in H_0^1(\Omega). \end{cases}$$

Thus, above we have $\mathcal{R} = DP$, $S_{DP}(\Omega) = \{u\}$, $\mathcal{V} = H_0^1(G)$ and the cost functional ("deviation" from the fixed function) does depend on the shape Ω only by the solution to the state equation, i.e.

$$J(\Omega, u) = J(u_\Omega) = \int_G |2u_\Omega - w_0|^2\, dx.$$

There is no optimal solution to the above $(\mathrm{OSD})_{\mathrm{DP}}$. Indeed, one can construct a minimizing sequence $\{(\Omega_n, u_n)\}$ for J and prove $J(u_{\Omega_n}) \to 0$, $u_{\Omega_n} \to u$ in the weak topology of $H^1_0(G)$, but **there is no** $\Omega \in \mathcal{A}(G)$ such that $u \in S_{\mathrm{DP}}(\Omega)$. So it is not satisfied the "closedness" condition (2.1).

On the contrary, there exist optimal solution to the relaxed problem:

$$(\mathrm{ROSD})_{\mathrm{DP}} \qquad \min_{\mu \in \mathcal{M}_0(G)} \overline{J}(u_\mu),$$

where $\overline{J}$ is the γ-lsc regularization of J (here $\overline{J} = J$), and u_μ is the solution of

$$(\mathrm{RDP})_\mu \qquad \begin{cases} -\Delta u + \mu u = 1, \\ u \in V_\mu(G) \qquad (V_\mu(G) := H^1_0(G) \cap L^2_\mu(G)). \end{cases}$$

Namely, the optimal solution is given by the measure ("generalized" shape) $\mu^* = (1/w_0(x))\mathcal{L}^2$ belonging to $\mathcal{M}_0(G)$ and the corresponding $u_{\mu^*} = w_0/2$ being the solution to $(\mathrm{RDP})_{\mu^*}$, i.e. we have $J(u_{\mu^*}) = 0$.

Finally, we would like to underline that there are still many open questions in this field, especially for problems with Neumann type boundary conditions and the relaxation for OSD problems with states described by (HVI)s. For the characterization of optimal solutions to OSD problems we refer to [3], [21], [25], [26], where some necessary conditions for optimality are given.

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