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PERIODIC SOLUTIONS OF DIFFERENTIAL INCLUSIONS WITH RETARDS

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Abstract. The paper is devoted to study the existence of periodic solutions for retarded differential inclusions. The nonsmooth guiding potential method is used and topological degree theory for multivalued maps is applied.

The present paper concerns the existence of periodic solutions for differential inclusions with retarded arguments of the form

$$(\mathrm{Q}_F) \qquad \begin{cases} \dot{x}(t) \in F(t, x(t-\tau_1), \ldots, x(t-\tau_m)) & \text{for } t \in [0,T] \text{ a.e.} \\ x(t) = x(t+T) & \text{for every } t \in [-\tau, 0], \end{cases}$$

where $F : [0,T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ is a multivalued map. The solvability of problem (Q_F) is closely related to the existence of fixed points for the multivalued Poincaré operator $P_F : C([-\tau,0],\mathbb{R}^n) \multimap C([-\tau,0],\mathbb{R}^n)$, which associates with each function y the set $\{x(\cdot+T) \in C([-\tau,0],\mathbb{R}^n) : \dot{x}(t) \in F(t,x(t-\tau_1),\ldots,x(t-\tau_m))$ for $t\in[0,T]$ a.e. and $x(t)=y(t)$ for every $t \in [-\tau,0]\}$. Adopting the classical Liapunov–Krasnosel'skiĭ guiding potential method to retarded differential inclusions we are able to calculate the topological degree of the decomposable compact vector field $I-P_F$. If on some ball of $C([-\tau,0],\mathbb{R}^n)$ the degree of $I-P_F$ turns out to be different from zero, then problem (Q_F) has a solution.

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The idea of using the topological degree for the translation map along trajectories of ODE's in the study of periodic solutions comes from Krasnosel'skiĭ [16], [17]. The generalization of the Poincaré translation map to an admissible multivalued operator is due to Dylawerski and Górniewicz [6]. There are several papers concerning periodic solutions for various differential equations and inclusions making use of the topological degree theory (see e.g. [3], [5], [7], [8], [12], [21], [20]).

In Section 1 we prove (see Theorem 1.1) that the so called solution set map S_F associated to the boundary value problem

$$(\mathrm{C}_F) \qquad \begin{cases} \dot{x}(t) \in F(t, x(t-\tau_1), \ldots, x(t-\tau_m)) & \text{for } t \in [0,T] \text{ a.e.} \\ x|_{[-\tau,0]} = y \end{cases}$$

is R_δ-valued.

The acyclicity of S_F is an important property since it allows us to use the topological degree theory for the Poincaré operator P_F. The structure of the solution set for retarded and functional differential inclusions was investigated by Lasry–Robert [18], [19], Haddad [13], Haddad–Lasry [14] and recently Hu–Papageorgiou [15].

In Section 2 we remind the notion of the topological degree for the class of decomposable compact vector fields in Banach spaces. Necessary properties of the degree are gathered in Theorem 2.1. In this section we state the new definition of the guiding potential V for the right-hand side F of a retarded differential inclusion. In particular, Definition 3.2 includes the case of nontrivial delay differential problem: $\dot{x}(t) \in F(t, x(t-\tau))$. Since we assume that V is only locally Lipschitzean function, the notion of guiding potential is given in terms of Clarke generalized gradient. Nonsmooth guiding potentials have been recently introduced by de Blasi, Górniewicz and Pianigiani [3] to study periodic problems for differential inclusions of the type $\dot{x}(t) \in F(t, x(t))$, with F convex or nonconvex valued.

Section 3 contains Theorem 3.1, which ensures the existence of periodic solutions for problem (Q_F), with F an upper semicontinuous map taking nonempty compact convex values. This is the main result of the paper and should be considered as an extension, to the multivalued case, of the method presented by Dylawerski and Jodel in [7]. The proof is based on the construction of suitable homotopies reducing the calculation of the degree of the vector field $I - P_F$ defined on an infinite dimensional functional Banach space to comparision with the index of the guiding potential $V : \mathbb{R}^n \to \mathbb{R}$ associated with the multimap F. Assuming that the index of V is nonzero we establish a sufficient condition for finding periodic solutions of (Q_F).

It should be noted that the existence of periodic solutions for functional differential inclusions satisfying Nagumo type tangential condition has been established by Haddad–Lasry [14] (autonomous case) and Hu–Papageorgiou [15] (nonautonomous case).

We will use the following notations. Let E be a Banach space and $A \subset E$. Then $\operatorname{bd} A$ denotes the boundary of A and $\operatorname{co} A$ the convex hull of A. If $A \subset \mathbb{R}^n$ is a nonempty and bounded set then $|A|^+ = \sup\{\|a\| : a \in A\}$. A closed ball in $\mathbb{R}^n$ (resp. in E) with center x (resp. zero) and radius $r > 0$ is denoted by $B^n(x,r)$ (resp. K_r). Further, we set $B^n(r) = B^n(0,r)$ and $S^{n-1}(r) = \operatorname{bd} B^n(0,r)$. $\mathbb{Z}$ denotes the set of integers and $\langle \cdot\,, \cdot \rangle$ the inner product in $\mathbb{R}^n$.

By $f : X \to Y$ (resp. $F : X \multimap Y$) we denote a single valued (resp. multivalued) map from X to Y. Throughout this paper we will consider only multivalued maps (multimaps) having nonempty values. The set of all fixed points of the multivalued map $F : X \multimap X$ is denoted by $\operatorname{Fix}(F)$. The symbol $C(X,Y)$ stands for the space of all continuous functions from X to Y and $C_n[a,b]$ denotes the space $C([a,b],\mathbb{R}^n)$ equiped with the usual supremum norm.

We will say that a space X is *contractible*, if there is a continuous map (*homotopy*) $h : X \times [0,1] \to X$ and some point $x_0 \in X$ such that $h(x,0) = x$, $h(x,1) = x_0$ for any $x \in X$.

A nonempty compact space X is called an *R_δ-set*, if there is a decreasing sequence $\{X_n\}$ of compact contractible spaces X_n satisfying $X = \bigcap_{n=1}^{\infty} X_n$.

A metric space X is called an *absolute neighbourhood retract* ($X \in$ ANR), if for every metric space Y and any closed subset $A \subset Y$ and every $f \in C(A,X)$, there is an open neighbourhood U of A in Y and $\widetilde{f} \in C(U,X)$ such that $\widetilde{f}(x) = f(x)$ for any $x \in A$.

A multimap $F : X \multimap Y$ is called *upper semicontinuous* (u.s.c.), if $\{x \in X : F(x) \subset U\}$ is open in X for every open U in Y. If the image $F(X)$ is relatively compact in Y, then we say that $F : X \multimap Y$ is a *compact* multivalued map. A multimap $F : [a,b] \multimap \mathbb{R}^n$ is called *measurable*, if $\{t \in [a,b] : F(t) \subset A\}$ belongs to the Lebesgue σ-field of $[a,b]$ for every closed $A \subset \mathbb{R}^n$.

We will say that a multivalued map $F : [a,b] \times \mathbb{R}^k \multimap \mathbb{R}^n$ is *u-Carathéodory*, if it satisfies:

- (i) the multimap $t \mapsto F(t,x)$ is measurable for every fixed $x \in \mathbb{R}^k$,
- (ii) the multimap $x \mapsto F(t,x)$ is u.s.c. for $t \in [a,b]$ a.e.

Of course, any u.s.c. map $F : [a,b] \times \mathbb{R}^k \multimap \mathbb{R}^n$ is u-Carathéodory.

A multimap $F : [a,b] \times \mathbb{R}^k \multimap \mathbb{R}^n$ with compact values is called *μ-integrably bounded*, if $\mu : [a,b] \to [0,\infty)$ is a Lebesgue integrable function such that

$$|F(t,x)|^+ \le \mu(t) \quad \text{for every } (t,x) \in [a,b] \times \mathbb{R}^k.$$

We shall say that F is integrably bounded, if it is μ-integrably bounded for some μ. As usual $L_1([a,b],\mathbb{R}^n)$ stands for the Banach space of Lebesgue integrable maps with the norm $\|\cdot\|_1$.

Finally, a single valued function $f : [a,b] \times X \to Y$ is said to be *measurable-locally Lipschitzean* provided for every $x \in X$ the map $f(\,\cdot\,,x)$ is measurable and for each $x \in X$ there exists a neighbourhood U_x of x in X and a Lebesgue integrable map $L_x : [a,b] \to [0,\infty)$ such that

$$\|f(t,x_1) - f(t,x_2)\| \leq L_x(t)\|x_1 - x_2\| \quad \text{for every } t \in [a,b] \text{ and } x_1, x_2 \in U_x.$$

1. Regularity of the solution set

For a given $\tau_1, \ldots, \tau_m \geq 0$, let $\tau = \max\{\tau_i : i = 1, \ldots, m\}$. Assume that $F : [0,T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ and $y \in C_n[-\tau,0]$ are given. The goal of this section is to verify that under some suitable assumptions about the multivalued map F, all Carathéodory solutions of the following problem

$$(\mathrm{C}_F) \qquad \begin{cases} \dot{x}(t) \in F(t, x(t-\tau_1), \ldots, x(t-\tau_m)) & \text{for } t \in [0,T] \text{ a.e.} \\ x|_{[-\tau,0]} = y \end{cases}$$

form a set of R_δ-type. Below we give some preliminary results, which will allow us to show that the correspondence depending on the initial value condition for the considered inclusion (C_F) is an u.s.c. multivalued map with R_δ values.

Proposition 1.1. *Let $F : [0,T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ be an integrably bounded multimap. Suppose that F has a measurable-locally Lipschitzean selector. Then the set $S_F(y) = \{x \in C_n[-\tau,T] :\ x$ is a solution of problem $(\mathrm{C}_F)\}$ is nonempty and contractible for every $y \in C_n[-\tau,0]$.*

Proof. If $x \in C_n[-\tau,T]$ is given, then for every $t \in [0,T]$ the symbol x_t denotes the shift of x at t, i.e. the function in $C_n[-\tau,0]$ such that $x_t(s) = x(t+s)$ for each $s \in [-\tau,0]$. In particular the boundary condition of (C_F) could be rewritten in the form: $x_0 = y$.

Fix $y \in C_n[-\tau,0]$. Let $f : [0,T] \times \mathbb{R}^{mn} \to \mathbb{R}^n$ be a measurable-locally Lipschitzean selection of F. It is then easy to check that the following problem

$$(\mathrm{C}_{a,y}) \qquad \begin{cases} \dot{x}(t) = f(t, x(t-\tau_1), \ldots, x(t-\tau_m)) & \text{for } t \in [a,T] \text{ a.e.} \\ x_a = y \end{cases}$$

has exactly one solution $x = x(a,y) \in C_n[a-\tau,T]$ for every $a \in [0,T]$ and $y \in C_n[-\tau,0]$. Thanks to this we are able to define the map $h : S_F(y) \times [0,1] \to S_F(y)$ by

$$h(z,s)(t) = \begin{cases} z(t) & \text{for } t \in [-\tau, sT], \\ x(sT, z_{sT})(t) & \text{for } t \in [sT, T]\,, \end{cases} \tag{1.1}$$

where $x(sT, z_{sT})$ is a unique solution of the problem $(\mathrm{C}_{sT,z_{sT}})$. Since the mapping $(a, y) \mapsto x(a, y)$ is continuous, the formula (1.1) defines a homotopy contracting the solution set $S_F(y)$ to the unique point $x(0, y)$ in $S_F(y)$. This completes the proof. $\square$

Lemma 1.1. *Let X, Y be metric spaces. Let $F : Y \times X \multimap Y$ be a compact multimap with closed graph. Suppose that* $\mathrm{Fix}(F(\,\cdot\,, x)) \neq \phi$ *for every $x \in X$. Then the multimap $\Phi : X \multimap Y$ such that $\Phi(x) = \mathrm{Fix}(F(\,\cdot\,, x))$ is u.s.c.*

Proof. Suppose that Φ is not u.s.c in some point $x_0 \in X$. Let U be an open neighbourhood of $\Phi(x_0)$ such that for every $n \geq 1$ there exists x_n in the ball around x_0 with radius $1/n$ and $y_n \in \Phi(x_n)$ such that $y_n \notin U$. Since $(y_n)_{n=1}^{\infty}$ is a sequence of elements of a relatively compact set $F(Y \times X)$, there is a subsequence $(y_{k_n})_{n=1}^{\infty}$ converging to some point y_0. The closedness of the graph of F implies that $y_0 \in \Phi(x_0)$. On the other hand we have $y_{k_n} \notin U$ for every $n \geq 1$. So $y_0 \notin U$. From this a contradiction follows, completing the proof. $\square$

Lemma 1.2 below can be proved as in [10, Theorem 4.13].

Lemma 1.2. *Let X be a metric space. If $F : [0, T] \times X \multimap \mathbb{R}^n$ is a μ-integrably bounded u-Carathéodory multimap with compact convex values, then there exists a sequence $(F_k : [0, T] \times X \multimap \mathbb{R}^n)_{k=1}^{\infty}$ such that for every $k \geq 1$*

(i) *F_k is a μ-integrably bounded u-Carathéodory multivalued map with non-empty compact convex values,*
(ii) *$F_{k+1}(t, x) \subset F_k(t, x)$ for every $(t, x) \in [0, T] \times X$,*
(iii) *$F(t, x) = \bigcap_{k=1}^{\infty} F_k(t, x)$ for every $x \in X$ and for $t \in [0, T]$ a.e.*
(iv) *F_k has a measurable–locally Lipschitzean selection.*

Theorem 1.1. *Let $F : [0, T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ be a μ-integrably bounded u-Carathéodory multimap with convex compact values. Then the solution set map $S_F : C_n[-\tau, 0] \multimap C_n[-\tau, T]$ given by*

$$S_F(y) = \{x \in C_n[-\tau, T] : x \text{ is a solution of problem } (\mathrm{C}_F)\}$$

is u.s.c. with nonempty R_δ values. Moreover, the map $K_r \ni y \mapsto S_F(y)$ is compact (*for any $r > 0$*).

Proof. At first we will show that $S_F(y)$ is nonempty, R_δ-set for every $y \in C_n[-\tau, 0]$. Applying Lemma 1.2 to the map F we obtain a sequence $(F_k : [0, T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n)_{k=1}^{\infty}$ satisfying (i)–(iv). By virtue of Proposition 1.1, the solution set $S_{F_k}(y)$ of inclusion (C_{F_k}) is nonempty, contractible for every $y \in C_n[-\tau, 0]$ and $k \geq 1$. Moreover, $S_{F_k}(y)$ is compact. To see this, let $(x_n)_{n=1}^{\infty}$ be a sequence of elements from $S_{F_k}(y)$. Since the set $\{x_n(t) : n \geq 1\}$ is bounded for every $t \in [0, T]$

and $\|\dot{x}_n(t)\| \le \mu(t)$ for almost all $t \in [0,T]$ it follows from Theorem 4 ([2], p. 13) that there exists a subsequence (again denoted by) (x_n) converging uniformly on the interval $[0,T]$ to an absolutely continuous function x. Furthermore, the sequence of derivatives $(\dot{x}_n)$ converges weakly to $\dot{x}$ in $L_1([0,T],\mathbb{R}^n)$. By yx we will denote a function in $C_n[-\tau,T]$ given by the formula

$$yx(t) = \begin{cases} y(t) & \text{for } t \in [-\tau,0], \\ x(t) & \text{for } t \in [0,T], \end{cases}$$

which is well defined as $x(0) = y(0)$. Clearly, the subsequence (x_n) converges uniformly to yx. We will show that $yx \in S_{F_k}(y)$. From Mazur's Theorem it follows that $\dot{x}$ belongs to the strong closure of $\operatorname{co}\{\dot{x}_n : n \ge l\}$ for every $l \ge 1$. Thus there is a sequence (z_l) converging to $\dot{x}$ in the norm topology of L_1 such that $z_l \in \operatorname{co}\{\dot{x}_n : n \ge l\}$ for every $l \ge 1$. Further, there is a subsequence (again denoted by) (z_l) converging to $\dot{x}$ a.e. in $[0,T]$. Let I be a set of full measure in the segment $[0,T]$ satisfying

$$z_l(t) \xrightarrow[l\to\infty]{} \dot{x}(t) \quad \text{for every } t \in I, \tag{1.2}$$

$$\mathbb{R}^{mn} \ni (\xi_1,\ldots,\xi_m) \mapsto F_k(t,\xi_1,\ldots,\xi_m) \quad \text{is u.s.c. for every } t \in I \tag{1.3}$$

and

$$\dot{x}_n(t) \in F_k(t, x_n(t-\tau_1),\ldots,x_n(t-\tau_m)) \quad \text{for every } t \in I,\ n \ge 1. \tag{1.4}$$

Take an arbitrary $t \in I$ and $\varepsilon > 0$. From (1.3) we get $\delta > 0$ such that

$$F_k(t,\xi_1,\ldots,\xi_m) \subset F_k(t, yx(t-\tau_1),\ldots,yx(t-\tau_m)) + B^n(\varepsilon)$$

for every $(\xi_1,\ldots,\xi_m) \in B^{mn}((yx(t-\tau_1),\ldots,yx(t-\tau_m)),\delta)$. Since $(x_n(t-\tau_1),\ldots,x_n(t-\tau_m)) \xrightarrow[n\to\infty]{} (yx(t-\tau_1),\ldots,yx(t-\tau_m))$, there is $N \in \mathbb{N}$ such that

$$\dot{x}_n(t) \in F_k(t, yx(t-\tau_1),\ldots,yx(t-\tau_m)) + B^n(\varepsilon) \quad \text{(by (1.4))}$$

for every $n \ge N$. Observe that $(z_l)_{l=N}^{\infty} \subset \operatorname{co}\{\dot{x}_n : n \ge N\}$. By virtue of (1.2) it follows that

$$\dot{x}(t) \in F_k(t, yx(t-\tau_1),\ldots,yx(t-\tau_m)) + B^n(\varepsilon).$$

Since $\varepsilon > 0$ and $t \in I$ was arbitrary, we have $\dot{x}(t) \in F_k(t, yx(t-\tau_1),\ldots, yx(t-\tau_m))$ for almost all $t \in [0,T]$. Whence, $yx \in S_{F_k}(y)$.

Let us notice that $S_F(y) = \bigcap_{k=1}^{\infty} S_{F_k}(y)$, by (iii) and $S_{F_{k+1}}(y) \subset S_{F_k}(y)$, by (ii) for every $y \in C_n[-\tau,0]$ and $k \ge 1$. Therefore $S_F(y)$ appears as a countable intersection of the decreasing family $\{S_{F_k}(y)\}$ of nonempty, compact contractible spaces $S_{F_k}(y)$, i.e. $S_F(y)$ is a nonempty R_δ-set.

Now, we are able to prove that the map $K_r \ni y \mapsto S_F(y)$ is u.s.c. for a given radius $r > 0$. Define the operator $\Psi : C_n[-\tau, T] \times K_r \multimap C_n[-\tau, T]$, by the formula

$$\Psi(x, y) = \left\{ z : z(t) = y(0) + \int_0^t f(s)\, ds \text{ for every } t \in [0, T],\ z|_{[-\tau,0]} = y \right\},$$

where $f(s) \in F(s, x(s-\tau_1), \ldots, x(s-\tau_m))$ a.e. in $[0, T]$. Observe that the function $[0, T] \ni t \mapsto (x(t - \tau_1), \ldots, x(t - \tau_m))$ is continuous. In view of Theorem 7 ([1], p. 124) the map $F(\,\cdot\,, x(\,\cdot\, - \tau_1), \ldots, x(\,\cdot\, - \tau_m))$ has a measurable selection. Since F is integrably bounded, this selection is integrable. Thus Ψ is a well defined multimap.

It is easy to see that

$$S_F(y) = \operatorname{Fix}(\Psi(\,\cdot\,, y)) \quad \text{for every } y \in K_r. \tag{1.5}$$

It left to check, whether the map Ψ fulfiles assumptions of Lemma 1.1. The compactness of Ψ easily follows from the classical Arzelá–Ascoli Theorem. On the other hand the graph of Ψ is closed. Indeed, take a sequence $((z_n, x_n, y_n))_{n=1}^{\infty}$ such that $z_n \in \Psi(x_n, y_n)$ for every $n \geq 1$, converging to some $(z, x, y) \in C_n[-\tau, T] \times C_n[-\tau, T] \times K_r$. Observe that functions z_n are absolutely continuous on $[0, T]$ and

$$\|z_n(t)\| \leq \|y_n(0)\| + \int_0^t \|f_n(s)\|\, ds \leq r + \|\mu\|_1$$

for every $t \in [0, T]$ and $n \geq 1$. Furthermore, $\|\dot{z}_n(t)\| \leq \|f_n(t)\| \leq \mu(t)$ for almost all $t \in [0, T]$. Again, by Theorem 4 ([2], p. 13), there is a subsequence (z_n) such that $(\dot{z}_n)$ converges weakly to z in $L_1([0, T], \mathbb{R}^n)$. Similiar argumentation as in the proof of the compactness of the set $S_{F_k}(y)$ implies that $\dot{z}(t) \in F(t, x(t - \tau_1), \ldots, x(t - \tau_m))$ a.e. in $[0, T]$. Since

$$z(t) = y(0) + \int_0^t \dot{z}(s)\, ds \quad \text{for every } t \in [0, T]$$

and $z|_{[-\tau,0]} = y$, we have $z \in \Psi(x, y)$.

By virtue of Lemma 1.1 the map $K_r \ni y \mapsto \operatorname{Fix}(\Psi(\,\cdot\,, y))$ is u.s.c. Consequently, the upper semicontinuity of S_F follows from (1.5). The compactness of $K_r \ni y \mapsto S_F(y)$ is obvious in view of Arzelá–Ascoli Theorem. This completes the proof. □

Corollary 1.1. *Let $F : [0, T] \times \mathbb{R}^{mn} \times \Lambda \multimap \mathbb{R}^n$, Λ a metric space, be a map with nonempty compact convex values satisfying:*

(i) $t \mapsto F(t, x, \lambda)$ *is measurable, for every* $(x, \lambda) \in \mathbb{R}^{mn} \times \Lambda$,
(ii) $(x, \lambda) \mapsto F(t, x, \lambda)$ *is u.s.c. for* $t \in [0, T]$ *a.e.*

(iii) $|F(t,x,\lambda)|^+ \leq \mu(t)$ *for every* $(t,x,\lambda) \in [0,T] \times \mathbb{R}^{mm} \times \Lambda$, *where* $\mu : [0,T] \to [0,+\infty)$ *is an integrable function.*

Then the operator $H : C_n[-\tau,0] \times \Lambda \multimap C_n[-\tau,T]$ *defined by*

$$H(y,\lambda) = S_{F(\,\cdot\,,\,\cdot\,,\lambda)}(y)$$

is u.s.c. with nonempty R_δ *values. Moreover, the map* $K_r \times \Lambda \ni (y,\lambda) \mapsto S_{F(\,\cdot\,,\,\cdot\,,\lambda)}(y)$ *is compact* (*for any* $r > 0$).

2. Topological degree and guiding potentials

In this section we recall the notion of the topological degree for multivalued maps, which will be applied in the next section to establishing the existence of periodic trajectories of a retarded inclusion.

Let us consider a Banach space E, not necessarily finite dimensional, and a closed ball K_r in E.

For any $X \in \mathrm{ANR}$ we denote by $J(K_r,X)$ the class of u.s.c. multivalued maps $F : K_r \multimap X$ with compact, R_δ-values.

A compact map $\Phi : K_r \multimap E$ is called *decomposable* ($\Phi \in D(K_r,E)$) if there exists a space $X \in \mathrm{ANR}$ and two maps $F \in J(K_r,X)$, $f \in C(X,E)$ such that $\Phi = f \circ F$. The diagram $D_\Phi : K_r \overset{F}{\multimap} X \overset{f}{\longrightarrow} E$ associated to the map Φ is called the *decomposition* of Φ.

It is known (see [9], [11]) that using approximation methods for multivalued maps it is possible to define the topological degree for the class

$$FD(K_r,E) = \{I - \Phi : \Phi \in D(K_r,E),\ (I-\Phi)(\mathrm{bd}\,K_r) \subset E \setminus \{0\}\}$$

of compact vector fields (without singular points on the boundary of K_r) associated to decomposable maps.

Remark 2.1. Let us notice that, if we set

$$D_0(B^n(r),\mathbb{R}^n) = \{\Phi \in D(B^n(r),\mathbb{R}^n) : \Phi(S^{n-1}(r)) \subset \mathbb{R}^n \setminus \{0\}\},$$

then $D_0(B^n(r),\mathbb{R}^n) = FD(B^n(r),\mathbb{R}^n)$.

By $i_0, i_1 : X \to X \times [0,1]$ we denote the maps given by the formulae $i_0(x) = (x,0)$ and $i_1(x) = (x,1)$ for every $x \in K_r$ and any space X.

Definition 2.1. Let $I - \Phi, I - \Psi \in FD(K_r,E)$ be two vector fields with Φ and Ψ having decompositions of the form

$$D_\Phi : K_r \overset{F_1}{\multimap} X \overset{f_1}{\longrightarrow} E, \quad D_\Psi : K_r \overset{F_2}{\multimap} Y \overset{f_2}{\longrightarrow} E.$$

We shall say that $I-\Phi$ and $I-\Psi$ are *homotopic* in $FD(K_r,E)$ if there exist maps $\Xi \in D(K_r \times [0,1],E)$ (with $D_\Xi : K_r \times [0,1] \overset{H}{\multimap} Z \overset{h}{\longrightarrow} E$) and $g_1 \in C(X,Z)$,

$g_2 \in C(Y,Z)$ such that $x \notin \Xi(x,\lambda)$ for every $(x,\lambda) \in \operatorname{bd} K_r \times [0,1]$ and the following diagram is commutative:

$$\begin{array}{ccccc}
K_r & \xrightarrow{F_1}\!\!\circ & X & & \\
{\scriptstyle i_0}\big\downarrow & & \big\downarrow{\scriptstyle g_1} & \searrow^{f_1} & \\
K_r \times [0,1] & \xrightarrow{H}\!\!\circ & Z & \xrightarrow{h} & E \\
{\scriptstyle i_1}\big\uparrow & & \big\uparrow{\scriptstyle g_2} & \nearrow_{f_2} & \\
K_r & \xrightarrow[F_2]{}\!\!\circ & Y & &
\end{array}$$

Below we formulate a result which summarizes some useful properties of the degree.

Theorem 2.1. *The topological degree function* $\operatorname{Deg} : FD(K_r,E) \to \mathbb{Z}$ *satisfies:*

(i) (Existence) *If* $\operatorname{Deg}(I-\Phi,K_r) \neq 0$, *where* $I-\Phi \in FD(K_r,E)$, *then there exists* $y \in K_r$ *such that* $y \in \Phi(y)$.

(ii) (Localization) *If* $I-\Phi \in FD(K_r,E)$ *is such that* $0 \notin (I-\Phi)(K_r \setminus \operatorname{int} K_{r_0})$ *for some* $r > r_0 > 0$, *then* $\operatorname{Deg}(I-\Phi,K_r) = \operatorname{Deg}(I-\Phi,K_{r_0})$.

(iii) (Homotopy) *If* $I-\Phi$, $I-\Psi$ *are homotopic in* $FD(K_r,E)$, *then* $\operatorname{Deg}(I-\Phi,K_r) = \operatorname{Deg}(I-\Psi,K_r)$.

(iv) (Contractivity) *Let* $I-\Phi \in D(K_r,E)$ *and* $D_\Phi : K_r \xrightarrow{F}\!\!\circ\, X \xrightarrow{f} E$ *be the decomposition of* Φ. *If there is a finite dimensional subspace* E_1 *of* E *such that* $f(X) \subset E_1$ *then* $\operatorname{Deg}(I-\Phi,K_r) = \operatorname{Deg}(I-\Phi|_{E_1},K_r \cap E_1)$.

(v) *Let* E^n *be an n-dimensional space,* $K_r \subset E^n$ *and* $\Theta : \mathbb{R}^n \to E^n$ *be a linear isometry. If* $I-\Phi \in FD(K_r,E^n)$, *then* $I-\Theta^{-1}\circ\Phi\circ\Theta \in FD(B^n(r),\mathbb{R}^n)$ *and* $\operatorname{Deg}(I-\Theta^{-1}\circ\Phi\circ\Theta,B^n(r)) = \operatorname{Deg}(I-\Phi,K_r)$.

(vi) *If* $\Phi \in D_0(B^n(r),\mathbb{R}^n)$, *then* $\operatorname{Deg}(\Phi,B^n(r)) = (-1)^{n+1}\operatorname{Deg}(-\Phi,B^n(r))$.

(vii) *Let* $F \in J(K_r,X)$ *and* $h \in C(X\times[0,1],E)$. *Suppose that* $h(F(K_r)\times[0,1])$ *is a relatively compact set and* $0 \notin (I-h_\lambda\circ F)(x)$ *for every* $(x,\lambda) \in \operatorname{bd} K_r \times [0,1]$, *where* $h_\lambda(x) = h(x,\lambda)$. *Then* $\operatorname{Deg}(I-h_0\circ F,K_r) = \operatorname{Deg}(I-h_1\circ F,K_r)$.

(viii) *Let* $I-\Phi$, $I-\Psi \in FD(K_r,E)$ *be two vector fields with* Φ *and* Ψ *having decompositions of the form*

$$D_\Phi : K_r \xrightarrow{F_1}\!\!\circ\, X \xrightarrow{f_1} E, \quad D_\Psi : K_r \xrightarrow{F_2}\!\!\circ\, Y \xrightarrow{f_2} E.$$

If there exists a map $g \in C(X,Y)$ *such that the diagram*

$$\begin{array}{ccc}
K_r & \xrightarrow{F_1}\!\!\circ & X \\
{\scriptstyle F_2}\big\downarrow\!\circ & \swarrow^{g} & \big\downarrow{\scriptstyle f_1} \\
Y & \xrightarrow[f_2]{} & E
\end{array}$$

is commutative, then $\mathrm{Deg}(I-\Phi,K_r)=\mathrm{Deg}(I-\Psi,K_r)$.

(ix) *Let* $I-\Phi$, $I-\Psi\in FD(K_r,E)$ *be two vector fields with* Φ *and* Ψ *having decompositions of the form*

$$D_\Phi : K_r \overset{F_1}{\multimap} X_1 \overset{f_1}{\longrightarrow} E, \quad D_\Psi : K_r \overset{F_2}{\multimap} X_2 \overset{f_2}{\longrightarrow} E.$$

Suppose that

$$0\notin \lambda(I-\Phi)(x)+(1-\lambda)(I-\Psi)(x) \tag{2.1}$$

for every $(x,\lambda)\in \mathrm{bd}\,K_r\times[0,1]$. *Then* $\mathrm{Deg}(I-\Phi,K_r)=\mathrm{Deg}(I-\Psi,K_r)$.

Proof. Properties (i)–(v) can be proved as in [9], [11], while (vi) is a variant of a well known property of the ordinary Brouwer degree and follows directly from the construction of the degree in $FD(B^n(r),\mathbb{R}^n)$.

In order to verify (vii), consider the diagram

$$\begin{array}{ccccc}
K_r & \overset{F}{\longrightarrow} & X & & \\
{\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle i_0} & \searrow{\scriptstyle h_0} & \\
K_r\times[0,1] & \overset{F\times I}{\multimap} & Z & \overset{h}{\longrightarrow} & E \\
{\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle i_1} & \nearrow{\scriptstyle h_1} & \\
K_r & \underset{F}{\multimap} & X & &
\end{array}$$

where $(F\times I)(x,\lambda)=F(x)\times\{\lambda\}$. Observe that $X\times[0,1]\in\mathrm{ANR}$ and $F\times I\in J(K_r\times[0,1],X\times[0,1])$.Thus the composition $h\circ(F\times I)$ is a decomposable map satisfying $x\notin(h\circ(F\times I))(x,\lambda)$ for every $(x,\lambda)\in\mathrm{bd}\,K_r\times[0,1]$. Since the above diagram is commutative the maps $I-h_0\circ F$, $I-h_1\circ F\in FD(K_r,E)$ are homotopic in the sense of Definition 2.1, proving (vii).

To see (viii), let $F:K_r\times[0,1]\multimap Y$ be a map such that $F(x,\lambda)=F_2(x)$. Then the following diagram

$$\begin{array}{ccccc}
K_r & \overset{F_1}{\multimap} & X & & \\
{\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle g} & \searrow{\scriptstyle f_1} & \\
K_r\times[0,1] & \overset{H}{\multimap} & Y & \overset{f_2}{\longrightarrow} & E \\
{\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle I} & \nearrow{\scriptstyle f_2} & \\
K_r & \underset{F_2}{\multimap} & Y & &
\end{array}$$

is commutative. Obviously, $f_2\circ F$ is a decomposable map ($f_2\circ F\in D(K_r\times[0,1],E)$) and $x\notin(f_2\circ F)(x,\lambda)$ for every $(x,\lambda)\in\mathrm{bd}\,K_r\times[0,1]$. Therefore $I-\Phi$ and $I-\Psi$ are homotopic in $FD(K_r,E)$, and (viii) follows.

According to the last property let us consider the following two commutative diagrams

$$\begin{array}{ccc} K_r & \overset{G}{\multimap} & X_1 \times X_2 \\ {\scriptstyle F_i}\big\downarrow\circ & {\scriptstyle \pi_i}\swarrow & \big\downarrow{\scriptstyle g_i} \\ X_i & \underset{f_i}{\longrightarrow} & E \end{array}$$

where $G(x) = F_1(x) \times F_2(x)$, $g_i(x_1, x_2) = f_i(x_i)$, $\pi_i(x_1, x_2) = x_i$ and $i = 1, 2$. Observe that $I - g_1 \circ G$ and $I - g_2 \circ G$ are in $FD(K_r, E)$. By virtue of (viii) we have

$$\begin{aligned} \operatorname{Deg}(I - \Phi, K_r) &= \operatorname{Deg}(I - g_1 \circ G, K_r), \\ \operatorname{Deg}(I - \Psi, K_r) &= \operatorname{Deg}(I - g_2 \circ G, K_r). \end{aligned} \tag{2.2}$$

Define a continuous path $h : X_1 \times X_2 \times [0, 1] \to E$, connecting g_2 with g_1, by the formula $h(x_1, x_2, \lambda) = \lambda g_1(x_1, x_2) + (1 - \lambda) g_2(x_1, x_2)$. Then we have

$$h_\lambda(x_1, x_2) = \lambda f_1(x_1) + (1 - \lambda) f_2(x_2) \in \lambda\Phi(x) + (1 - \lambda)\Psi(x)$$

for every $(x, \lambda) \in \operatorname{bd} K_r \times [0, 1]$ and $(x_1, x_2) \in G(x)$. Hence, in view of (2.1), it follows that $0 \notin (I - h_\lambda \circ G)(\operatorname{bd} K_r)$ for every $\lambda \in [0, 1]$. Applying property (vii) one obtains $\operatorname{Deg}(I - g_1 \circ G, K_r) = \operatorname{Deg}(I - g_2 \circ G, K_r)$. Comparing this equality with (2.2) we get (ix). This completes the proof. $\square$

Let us recall the notion of Clarke generalized gradient. If $V : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitzean function, then the *generalized gradient* $\partial V(x_0)$ of V at x_0 is the set given by

$$\partial V(x_0) = \left\{ y \in \mathbb{R}^n : \limsup_{x \to x_0, t \to 0^+} \frac{V(x + tv) - V(x)}{t} \geq \langle y, v\rangle \ \text{ for every } v \in \mathbb{R}^n \right\}. \tag{2.3}$$

Proposition 2.1 ([4], pp. 27, 29). *The multivalued map $\partial V : \mathbb{R}^n \multimap \mathbb{R}^n$ defined by* (2.3) *is u.s.c. with nonempty compact convex values.*

The symbol $\langle A, B\rangle^-$ we use below stands for the *lower inner product* of nonempty compact subsets of $\mathbb{R}^n$, i.e.

$$\langle A, B\rangle^- = \inf\{\langle a, b\rangle : a \in A, b \in B\}.$$

Definition 2.2. A locally Lipschitzean function $V : \mathbb{R}^n \to \mathbb{R}$ is called a nonsingular potential, provided there exists a nonzero radius r_0 such that

$$\langle \partial V(x), \partial V(x)\rangle^- > 0 \quad \text{for every } \|x\| \geq r_0. \tag{2.4}$$

Observe that the condition: $0 \notin \partial V(x)$ for $\|x\| \geq r_0$, is weaker than (2.4). If we assume the continuous differentiability of the potential V, then $\partial V(x)$ reduces to the singleton $\{\operatorname{grad} V(x)\}$ (see [4]) and Definition 2.2 receives the following form:

A C^1-map $V : \mathbb{R}^n \to \mathbb{R}$ is called a nonsingular potential, if for some $r_0 > 0$, V satisfies $\operatorname{grad} V(x) \neq 0$ for every $\|x\| \geq r_0$.

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a nonsingular potential. Then $\partial V \in D_0(B^n(r), \mathbb{R}^n)$ for every $r \geq r_0$. In view of Theorem 2.1(ii), the topological degree of ∂V is independent of the choice of a radius r. Therefore the number

$$\operatorname{Ind}(V) = \operatorname{Deg}(\partial V, B^n(r)) \quad (r \geq r_0),$$

called the *index* of the nonsingular potential V, is well defined. When $V \in C^1$, then the index of V is simply the Brouwer degree of $\operatorname{grad} V$. For examples of nonsingular potentials with nonzero index we refer reader to [17]. One of them is contained in the following

Proposition 2.2. *If* $V : \mathbb{R}^n \to \mathbb{R}$ *is a nonsingular potential satisfying*

$$\lim_{\|x\| \to \infty} V(x) = \infty,$$

then $\operatorname{Ind}(V) = 1$.

The classical method of guiding C^1 functions, introduced by Liapunov, was developed by Krasnosel'skii, Mawhin and other authors to studying the periodic problems for ODE's and differential inclusions of the form

$$\dot{x}(t) \in F(t, x(t)), \quad x(0) = x(T).$$

In comparison with earlier papers (see [12], [22]) the guiding potentials employed in [3] were supposed to be only locally Lipschitzean. Following this conception we admit Definitions 2.3, used subsequently in the study of periodic problem for retarded differential inclusions.

Definition 2.3. Let $F : [0,T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ be a μ-integrably bounded multimap with compact values and $M = \max\{\|\mu\|_1, T\}$. A nonsingular potential $V : \mathbb{R}^n \to \mathbb{R}$ is called a guiding potential for F, if there is $r_0 > 0$ such that for every $(t, x, x_1, \ldots, x_m) \in (0,T) \times \mathbb{R}^n \times \mathbb{R}^{mn}$, with $\|x\| \geq r_0$ and $x_i \in B^n(x, M)$ for $i = 1, \ldots, m$, we have

$$\langle \partial V(x), F(t, x_1, \ldots, x_m) \rangle^- \geq 0. \tag{2.5}$$

In the case when V is a nonsingular potential of C^1 class, the condition (2.5) is equivalent to

$$\langle \operatorname{grad} V(x), F(t, x_1, \ldots, x_m) \rangle^- \geq 0.$$

Example. Let $F : [0,T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ be a μ-integrably bounded multimap with compact values. Suppose that there exist positive constants c, r such that for every $(t, x_1, \ldots, x_m) \in (0, T) \times \mathbb{R}^{mn}$ with $\|x_1\|, \ldots, \|x_m\| \geq r$ and for every $y \in F(t, x_1, \ldots, x_m)$ there exists $i \in \{1, \ldots, m\}$ such that

$$\langle y, x_i \rangle \geq c \cdot \|y\| \, \|x_i\|.$$

Then the C^1 nonsingular potential $V : \mathbb{R}^n \to \mathbb{R}$ given by $V(x) = \|x\|^2/2$ is a guiding potential for F. Indeed, set $r_0 = \max\{r + M, M/(1 - \sin(\arccos c))\}$. It is not difficult to see that with this choice of r_0 the statement of Definition 2.3 is satisfied. Moreover, $\mathrm{Ind}(V) = 1$, by Proposition 2.2.

3. Periodic problem

In the present section we discuss the periodic problem for retarded differential inclusion of the form

$$(\mathrm{Q}_F) \qquad \begin{cases} \dot{x}(t) \in F(t, x(t - \tau_1), \ldots, x(t - \tau_m)) & \text{for } t \in [0, T] \text{ a.e.} \\ x(t) = x(t + T) & \text{for every } t \in [-\tau, 0], \end{cases}$$

where $F : [0, T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ is a multivalued map and solutions are understood in the sense of Carathéodory.

Consider the following diagram

$$C_n[-\tau, 0] \xrightarrow{S_F} C_n[-\tau, T] \xrightarrow{S} C_n[-\tau, 0],$$

where S_F is the solution set map associated to the problem

$$(\mathrm{C}_F) \qquad \begin{cases} \dot{x}(t) \in F(t, x(t - \tau_1), \ldots, x(t - \tau_m)) & \text{for } t \in [0, T] \text{ a.e.} \\ x|_{[-\tau, 0]} = y \end{cases}$$

and the function S is given by

$$S(x)(t) = x(t + T)$$

for every $t \in [-\tau, 0]$. In the study of problem (Q_F) an important role will be played by a multivalued map $P_F : C_n[-\tau, 0] \multimap C_n[-\tau, 0]$ defined by $P_F = S \circ S_F$ and called the *Poincaré operator* for the problem (C_F).

To establish an existence theorem for the periodic problem (Q_F) we will show that the degree of a compact vector field $I - P_F$ associated to the Poincaré operator for the boundary problem (C_F) is nonzero on some ball K_r in $C_n[-\tau, 0]$. This idea is illustrated in the following

Proposition 3.1. *Let $F : [0, T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ be an integrably bounded u-Carathéodory multimap with compact convex values and P_F be the Poincaré operator for problem (C_F). If $0 \notin (I - P_F)(\mathrm{bd}\, K_r)$ for some $r > 0$ and $\mathrm{Deg}(I - P_F, K_r) \neq 0$, then the periodic problem (Q_F) has a solution.*

Proof. By Theorem 1.1, the solution set map $S_F : K_r \multimap C_n[-\tau, T]$ associated to the Cauchy problem (C_F) is compact, u.s.c. with R_δ-values. The function $S : C_n[-\tau, T] \to C_n[-\tau, 0]$ is continuous. Therefore the Poincaré operator $P_F = S \circ S_F$ is decomposable and the compact vector field $I - P_F$ belongs to the class $FD(K_r, C_n[-\tau, 0])$. If $\mathrm{Deg}(I - P_F, K_r) \neq 0$, then by Theorem 2.1(i) there exist $y \in K_r$ and $x \in S_F(y)$ such that $y = S(x)$. Obviously, x is the solution of (Q_F), completing the proof. □

In order to check that $\mathrm{Deg}(I - P_F, K_r) \neq 0$, we will use the method of guiding potentials defined in the previous section. At first we list some auxiliary lemmas. The following revokes the notion of a generalized Jacobian of a vector-valued function (see paragraph 2.6 in [4] for details).

Lemma 3.1. *Let $F : [0, T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ be an integrably bounded u.s.c. map with nonempty compact convex values. Let $x : [-\tau, T] \to \mathbb{R}^n$ be any solution of the problem (C_F). Then, for every $t \in (0, T)$, we have*

$$\partial x(t) \subset F(t, x(t - \tau_1), \ldots, x(t - \tau_m)),$$

where $\partial x(t)$ stands for a generalized Jacobian of x at point t.

Proof. Let $t_0 \in (0, T)$ and $\varepsilon > 0$ be arbitrary. Since F is u.s.c and x is continuous, there is $\delta > 0$ such that

$$F(t, x(t - \tau_1), \ldots , x(t - \tau_m)) \subset F(t_0, x(t_0 - \tau_1), \ldots , x(t_0 - \tau_m)) + B^n(\varepsilon) \tag{3.1}$$

for every $t \in B(t_0, \delta) \cap (0, T)$. In particular, x is locally Lipschitzean. As a consequence of Proposition 2.6.4 in [4] it follows that

$$\partial x(t_0) = \mathrm{co}\left\{ \lim_{k\to\infty} \dot{x}(t_k) : t_k \xrightarrow[k\to\infty]{} t_0,\ t_k \notin (\Omega_x \cup S) \right\},$$

where $S \subset \mathbb{R}$ is a set of Lebesgue measure zero and Ω_x is a set of points, where x is not differentiable. Put $S := \{t \in [0, T] : \dot{x}(t) \notin F(t, x(t - \tau_1), \ldots, x(t - \tau_m))\}$. Now from (3.1), we obtain

$$\partial x(t_0) \subset F(t_0, x(t_0 - \tau_1), \ldots, x(t_0 - \tau_m)) + B^n(\varepsilon).$$

As $\varepsilon > 0$ was arbitrary, the proof is complete. □

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a nonsingular potential. The generalized gradient ∂V of V can be unbounded, in general. Therefore we quote the following observation, made in [3].

Lemma 3.2. *Let $M_k = \sup\{|\partial V(x)|^+ : x \in B^n(k)\}$ and $\eta : \mathbb{R}^n \to \mathbb{R}$ be a function defined by*

$$\eta(x) = 1 + (\|x\| - k)M_{k+2} + (k + 1 - \|x\|)M_{k+1}$$

for $k \leq \|x\| \leq k+1$, $k = 0,1,\ldots$ *Then* η *is continuous and a multimap* $W : \mathbb{R}^n \multimap \mathbb{R}^n$ *given by*

$$W(x) = \frac{\partial V(x)}{\eta(x)} \tag{3.2}$$

is u.s.c. with nonempty compact convex values, satisfying $|W(x)|^+ \leq 1$ *for every* $x \in \mathbb{R}^n$.

In the sequel we will also use the following lemma proved in [3].

Lemma 3.3. *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a nonsingular potential and let* $W(x) = \partial V(x)/\eta(x)$, $x \in \mathbb{R}^n$. *Then for each* $r > r_0$ *there exists* $t_r \in (0,T]$ *such that, for every* $(x_0,\lambda) \in S^{n-1}(r) \times [0,1]$ *and any solution* $x : [0,T] \to \mathbb{R}^n$ *of the following problem*

$$(\mathrm{C}_W) \qquad \begin{cases} \dot{x}(t) \in W(x(t)) & \textit{for } t \in [0,T] \textit{ a.e.} \\ x(0) = x_0, \end{cases}$$

we have

$$0 \notin \lambda(x(t) - x_0) + (1-\lambda)\partial V(x_0) \quad \text{for every} \quad t \in (0,t_r].$$

Now, we formulate the main result of the paper. The calculation of $\mathrm{Deg}(I - P_F, K_r)$ carried out in the proof of this theorem rests on some homotopy arguments, which will allow us to compare this degree with the index of a guiding potential.

Theorem 3.1. *Let* $F : [0,T] \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ *be a* μ*-integrably bounded u.s.c. multimap with compact convex values and* $V : \mathbb{R}^n \to \mathbb{R}$ *be a guiding potential for* F *with* $\mathrm{Ind}(V) \neq 0$. *Then the periodic problem* (Q_F) *has a solution.*

Proof. Let $r_0 > 0$ be such that (2.4) and (2.5) are fulfilled. Take $r > r_0 + \max\{\|\mu\|_1, T\}$.

Step 1. Consider the family (C_λ) of boundary value problems

$$\begin{cases} \dot{x}(t) \in (1-\lambda)W(x(t)) + \lambda F(t, x(t-\tau_1),\ldots,x(t-\tau_m)) & \text{for } t \in [0,T] \text{ a.e.} \\ x|_{[-\tau,0]} = y, \end{cases}$$

and let $S_\lambda : C_n[-\tau,0] \multimap C_n[-\tau,T]$ be a multivalued map such that

$$S_\lambda(y) = \{x \in C_n[-\tau,T] : x \text{ is a solution of problem } (\mathrm{C}_\lambda)\}.$$

Now, define the multivalued homotopy $H : K_r \times [0,1] \multimap C_n[-\tau,T]$ by $H(y,\lambda) = S_\lambda(y)$. In view of Corollary 1.1, the map H is compact, u.s.c. with R_δ-values. Thus the superposition $S \circ H$ belongs to $D(K_r \times [0,1], C_n[-\tau,0])$. We claim that

$$0 \notin (I - S \circ H)(x,\lambda) \quad \text{for every } (x,\lambda) \in \mathrm{bd}\, K_r \times [0,1]. \tag{3.3}$$

Suppose that $0 \notin (I - S \circ H(\,\cdot\,, 1))(\operatorname{bd} K_r)$, otherwise (Q_F) has a solution and there is nothing to prove. Assume on the contrary that there exist $\lambda \in [0,1)$ and a function $y \in \operatorname{bd} K_r$ such that $y \in S(H(y,\lambda))$. Then there is a solution x of the problem (C_λ) satisfying $x(t+T) = y(t)$ for every $t \in [-\tau, 0]$. Thus $x(t) = x(t+T)$ for each $t \in [-\tau, 0]$ and $\|x(t_0)\| = r$ for some $t_0 \in [0,T]$. Since $\|x(t) - x(s)\| \leq (1-\lambda)T + \lambda\|\mu\|_1$ for every $t, s \in [-\tau, T]$, we have

$$\|x(t)\| \geq \|x(t_0)\| - (1-\lambda)T - \lambda\|\mu\|_1 \geq r - \max\{\|\mu\|_1, T\} \geq r_0 \tag{3.4}$$

for every $t \in [-\tau, T]$ and $\|x(t) - x(t-\tau_i)\| \leq \max\{\|\mu\|_1, T\}$ for any $t \in [0,T]$, $i = 1, \ldots, m$. From Definition 2.3 it follows that

$$\langle\, \partial V(x(t)), F(t, x(t-\tau_1), \ldots, x(t-\tau_m))\,\rangle^- \geq 0 \tag{3.5}$$

for every $t \in (0,T)$.

The function $V \circ x$ is continuous and $V \circ x(0) = V \circ x(T)$, hence it attains a global extremum at some $t_\star \in (0,T)$. Since x is Lipschitzean near point $t_\star$ (see Lemma 3.1), we have

$$0 \in \partial(V \circ x)(t_\star), \tag{3.6}$$

by Proposition 2.3.2 in [4]. Applying a Jacobian chain rule (Theorem 2.6.6, p. 72 in [4]) we get

$$\partial(V \circ x)(t_\star) \subset \operatorname{co}\langle\, \partial V(x(t_\star)), \partial x(t_\star)\rangle,$$

where $\langle\, \partial V(x(t_\star)), \partial x(t_\star)\,\rangle = \{\langle \zeta, \xi\rangle : \zeta \in \partial V(x(t_\star)), \xi \in \partial x(t_\star)\}$. The u.s.c. multimap $(1-\lambda)W + \lambda F$ is integrably bounded and has compact convex values. Hence, by Lemma 3.1,

$$\partial x(t_\star) \subset (1-\lambda)W(x(t_\star)) + \lambda F(t_\star, x(t_\star - \tau_1), \ldots, x(t_\star - \tau_m)). \tag{3.7}$$

Using some properties of the lower inner product (see Proposition 3.1, p. 224 in [3]), we get

$$\begin{aligned}
\langle\, \partial V(x(t_\star)), \partial x(t_\star)\,\rangle^- &\geq \langle\, \partial V(x(t_\star)), (1-\lambda)W(x(t_\star)) \\
&\qquad + \lambda F(t_\star, x(t_\star - \tau_1), \ldots, x(t_\star - \tau_m))\,\rangle^- \quad \text{(by (3.7))} \\
&\geq (1-\lambda)\langle\, \partial V(x(t_\star)), W(x(t_\star))\,\rangle^- \\
&\qquad + \lambda\langle\, \partial V(x(t_\star)), F(t_\star, x(t_\star - \tau_1), \ldots, x(t_\star - \tau_m))\,\rangle^- \\
&\geq \frac{1-\lambda}{\eta(x(t_\star))}\,\langle\, \partial V(x(t_\star)), \partial V(x(t_\star))\,\rangle^- \quad \text{(by (3.5))}.
\end{aligned}$$

From (2.4) (by (3.4)) it follows that the last quantity is strictly positive, which yields a contradiction with (3.6). Hence (3.3) is verified.

If we denote by $S_W : C_n[-\tau,0] \multimap C_n[-\tau,T]$ the solution set map S_0 and by $P_W : C_n[-\tau,0] \multimap C_n[-\tau,0]$ the composition $S \circ S_W$, then P_W is the Poincaré operator associated to the following problem

$$(\mathrm{C}'_W) \qquad \begin{cases} \dot{x}(t) \in W(x(t)) & \text{for } t \in [0,T] \text{ a.e.} \\ x|_{[-\tau,0]} = y. \end{cases}$$

Observe that vector fields $I - P_W = I - S \circ H_0$ and $I - P_F = I - S \circ H_1$ are homotopic in $FD(K_r, C_n[-\tau,0])$ (see Definition 2.1). By virtue of Theorem 2.1(iii), it follows that

$$\mathrm{Deg}(I - P_F, K_r) = \mathrm{Deg}(I - P_W, K_r). \tag{3.8}$$

Step 2. Let $\psi : C_n[-\tau,0] \to C_n[-\tau,0]$ be given by the formula $\psi(y)(t) = y(0)$ and $\Psi = \psi \circ P_W$. Define a homotopy $h : C_n[-\tau,T] \times [0,1] \to C_n[-\tau,0]$ by $h(x,\lambda) = (1-\lambda)S(x) + \lambda(\psi \circ S)(x)$. Clearly h is continuous. Let us remind that the multifunction S_W is compact and belongs to the class $J(K_r, C_n[-\tau,T])$. We assert that

$$0 \notin (I - h_\lambda \circ S_W)(x) \quad \text{for all } (x,\lambda) \in \mathrm{bd}\, K_r \times [0,1]. \tag{3.9}$$

Suppose there is $\lambda \in [0,1]$ and $y \in \mathrm{bd}\, K_r$ satisfying $y \in h(S_W(y),\lambda)$. Then we get $x : [-\tau,T] \to \mathbb{R}^n$ which is a solution of (C'_W), with an initial condition given by y. Moreover, $y(t) = (1-\lambda)x(t+T) + \lambda x(T)$ for every $t \in [-\tau,0]$. Let $t_0 \in [-\tau,0]$ be such that $\|y(t_0)\| = r$. Obviously, there is a natural number k, with $t_0 + kT \in [0,T]$. Since $y(t_0) = (1-\lambda)^k x(t_0+kT) + (1-\lambda)^{k-1}\lambda x(T) + \ldots + (1-\lambda)\lambda x(T) + \lambda x(T)$, we have

$$\begin{aligned} \|y(t_0) - x(t)\| &\le (1-\lambda)^k \|x(t_0+kT) - x(t)\| + (1-\lambda)^{k-1}\lambda\|x(T) - x(t)\| \\ &\qquad + \ldots + (1-\lambda)\lambda\|x(T) - x(t)\| + \lambda\|x(T) - x(t)\| \\ &\le (1-\lambda)^k \int_0^T \|\dot{x}(s)\|\, ds + \ldots + \lambda \int_0^T \|\dot{x}(s)\|\, ds \\ &\le (1-\lambda)^k T + (1-\lambda)^{k-1}\lambda T + \ldots + \lambda T = T \end{aligned}$$

for every $t \in [0,T]$. Thus

$$\|x(t)\| \ge \|y(t_0)\| - T \ge r_0 \quad \text{for every } t \in [0,T]. \tag{3.10}$$

Let us remark that the statement of Lemma 3.1 for autonomous differential inclusions without retards is valid, too. Thus, using this lemma for the solution x, we get

$$\partial x(t) \subset W(x(t)) \quad \text{for every } t \in (0,T). \tag{3.11}$$

Analogously, like in Step 1, there is $t_\star \in (0,T)$, as $V \circ x(0) = V \circ x(T)$, such that

$$0 \in \mathrm{co}\,\langle\, \partial V(x(t_\star)), \partial x(t_\star)\,\rangle.$$

On the other hand, we have

$$\begin{aligned}\langle\, \partial V(x(t_\star)), \partial x(t_\star)\,\rangle^- &\geq \langle\, \partial V(x(t_\star)), W(x(t_\star))\,\rangle^- \quad \text{(by (3.11))}\\ &= \frac{1}{\eta(x(t_\star))}\,\langle\, \partial V(x(t_\star)), \partial V(x(t_\star))\rangle^- > 0\end{aligned}$$

(by (3.10) and (2.4)). From this a contradiction follows immediately and (3.9) is verified.

Now, from Theorem 2.1(vii) we obtain

$$\begin{aligned}\text{(3.12)}\qquad \operatorname{Deg}(I - P_W, K_r) &= \operatorname{Deg}(I - h_0 \circ S_W, K_r)\\ &= \operatorname{Deg}(I - h_1 \circ S_W, K_r) = \operatorname{Deg}(I - \Psi, K_r).\end{aligned}$$

Step 3. Consider the n-dimensional subspace $E^n = \{y \in C_n[-\tau,0] : y = \text{const}\}$ of $C_n[-\tau,0]$. The map $\Psi \in D(K_r, C_n[-\tau,0])$ has a decomposition of the form $D_\Psi : K_r \xrightarrow{S_W} C_n[-\tau,T] \xrightarrow{\psi\circ S} C_n[-\tau,0]$. Let us notice that $(\psi \circ S)(C_n[-\tau,T]) \subset E^n$. As a consequence of the contractivity property (Theorem 2.1(iv)) we get

$$\text{(3.13)}\qquad \operatorname{Deg}(I - \Psi, K_r) = \operatorname{Deg}(I - \Psi|_{E^n}, K_r \cap E^n).$$

Step 4. Let $\widetilde{P}_W : \mathbb{R}^n \multimap \mathbb{R}^n$ be the Poincaré operator for the problem (C_W), i.e. $\widetilde{P}_W = \mathrm{ev}_T \circ \widetilde{S}_W$, where $\widetilde{S}_W : \mathbb{R}^n \multimap C_n[0,T]$ is the solution set map for (C_W) and ev_T is the evaluation at point T. If we denote by $\Theta : E^n \to \mathbb{R}^n$ a natural isometry, then $\Psi|_{E^n} = \Theta^{-1} \circ \widetilde{P}_W \circ \Theta$. In view of Theorem 2.1(v) it follows that

$$\begin{aligned}\text{(3.14)}\qquad \operatorname{Deg}(I - \Psi|_{E^n}, K_r \cap E^n) &= \operatorname{Deg}(I - \Theta \circ \Psi|_{E^n} \circ \Theta^{-1}, B^n(r))\\ &= \operatorname{Deg}(I - \widetilde{P}_W, B^n(r)).\end{aligned}$$

Step 5. Let t_r be the point in the statement of Lemma 3.3. Define the maps $k : C_n[0,T] \times [0,1] \to \mathbb{R}^n$, $\widetilde{P}_W^{t_r} : \mathbb{R}^n \multimap \mathbb{R}^n$ by $k(x,\lambda) = x((1-\lambda)T + \lambda t_r)$ and $\widetilde{P}_W^{t_r} = \mathrm{ev}_{t_r} \circ \widetilde{S}_W$, where ev_{t_r} is the evaluation at point t_r. It is easy to prove that

$$\text{(3.15)}\qquad 0 \notin (I - k_\lambda \circ \widetilde{S}_W)(x) \quad \text{for every } (x,\lambda) \in S^{n-1}(r) \times [0,1].$$

Indeed, in the contrary case, there is $(x_0,\lambda) \in S^{n-1}(r) \times [0,1]$ and a solution $x : [0,T] \to \mathbb{R}^n$ of (C_W) such that $x(0) = x((1-\lambda)T + \lambda t_r)$. Thus $V \circ x(0) = V \circ x(t_\lambda)$ for some $t_\lambda \in (0,T]$. The function $V \circ x$ is continuous in $[0,t_\lambda]$, hence it has a global extremum at some point $t_\star \in (0,t_\lambda)$. Observe that $\|x(t_\star)\| \geq r_0$, as $\|x(0)\| = r$. By the same reason as in Step 2, we have $0 \in \operatorname{co}\langle\, \partial V(x(t_\star)), \partial x(t_\star)\,\rangle$ and $\langle\, \partial V(x(t_\star)), \partial x(t_\star)\,\rangle^- > 0$. The contradiction is evident and (3.15) follows.

The maps k and $\widetilde{S}_W$ satisfy the assumptions of Theorem 2.1(vii). Therefore we get

$$\begin{aligned}(3.16)\qquad \operatorname{Deg}(I-\widetilde{P}_W, B^n(r)) &= \operatorname{Deg}(I-k_0\circ \widetilde{S}_W, B^n(r))\\ &= \operatorname{Deg}(I-k_1\circ \widetilde{S}_W, B^n(r))\\ &= \operatorname{Deg}(I-\widetilde{P}_W^{t_r}, B^n(r)).\end{aligned}$$

Step 6. With the choice of r we have guaranteed that $\partial V\in FD(B^n(r),\mathbb{R}^n)$. From (3.15) it follows that $\widetilde{P}_W^{t_r}-I$ is in the class $FD(B^n(r),\mathbb{R}^n)$ too. In view of Lemma 3.3 it is clear that

$$0\notin \lambda(\widetilde{P}_W^{t_r}-I)(x)+(1-\lambda)\partial V(x)\quad \text{for every}\quad (x,\lambda)\in S^{n-1}(r)\times[0,1].$$

Applying properties (ix) and (vi) of the degree (Theorem 2.1), we gather

$$(3.17)\quad \operatorname{Deg}(\partial V, B^n(r)) = \operatorname{Deg}(\widetilde{P}_W^{t_r}-I, B^n(r)) = (-1)^{n+1}\operatorname{Deg}(I-\widetilde{P}_W^{t_r}, B^n(r)).$$

Remind that $\operatorname{Deg}(\partial V, B^n(r)) = \operatorname{Ind}(V)$ and the index of the nonsingular potential V is nonzero. Thus $\operatorname{Deg}(I-\widetilde{P}_W^{t_r}, B^n(r))\neq 0$.

Combining calculations in above steps (compare (3.8), (3.12)–(3.14), (3.16) and (3.17)) one obtains $\operatorname{Deg}(I-P_F,K_r)\neq 0$. By Proposition 3.1, the periodic problem (Q_F) has a solution. This completes the proof. □

Corollary 3.1. *Let $F:[0,T]\times\mathbb{R}^{mn}\multimap\mathbb{R}^n$ be a bounded u-Carathéodory multimap with compact convex values and $V:\mathbb{R}^n\to\mathbb{R}$ be a guiding potential of C^1 class for F with* $\operatorname{Ind}(V)\neq 0$. *Then the periodic problem* (Q_F) *has a solution.*

Proof. Let us remind that Theorem 1.1 characterizes the solution set of problem (C_F) with the right-hand side F of u-Carathéodory type. Therefore the proof becomes strictly analogous to that of Theorem 1.1. The only difference is that by the current assumptions of F we can't apply Lemma 3.1 to the map $(1-\lambda)W+\lambda F$. Thus we don't know, whether the condition (3.7) in step one of the previous proof is true or not. However, it is sufficient to know that the generalized Jacobian $\partial x(t_\star)$ satisfies

$$(3.18)\qquad \langle \operatorname{grad} V(x(t_\star)), \partial x(t_\star)\rangle^- \geq \frac{1-\lambda}{\eta(x(t_\star))}\langle \operatorname{grad} V(x(t_\star)), \operatorname{grad} V(x(t_\star))\rangle.$$

To show this, take a sequence $t_k\to t_\star$ such that $t_k\in I_x=\{s\in(0,T):\dot{x}(s)\in (1-\lambda)W(x(s))+\lambda F(s,x(s-\tau_1),\dots,x(s-\tau_m))\}$ and $\lim \dot{x}(t_k)$ exists. Then, for each of the point $t_k\in I_x$, we have

$$\langle \operatorname{grad} V(x(t_k)), \dot{x}(t_k)\rangle \geq \frac{1-\lambda}{\eta(x(t_k))}\langle \operatorname{grad} V(x(t_k)), \operatorname{grad} V(x(t_k))\rangle.$$

The way to estimate the inner product $\langle \operatorname{grad} V(x(t_k)), \dot{x}(t_k)\rangle$ remains the same as for the term $\langle \partial V(x(t_\star)), \partial x(t_\star)\rangle^-$ in Step 1 of the proof of Theorem 3.1.

Using the continuous differentiability of the guiding potential V and passing to the limit one obtains

$$\langle \operatorname{grad} V(x(t_\star)), \lim \dot{x}(t_k) \rangle \geq \frac{1-\lambda}{\eta(x(t_\star))} \langle \operatorname{grad} V(x(t_\star)), \operatorname{grad} V(x(t_\star)) \rangle.$$

Since $\partial x(t_\star) = \operatorname{co}\{\lim \dot{x}(t_k) : t_k \to t_\star,\ t_k \in I_x\}$, the last inequality justifies (3.18) and completes the proof. □

Corollary 3.2. *Let $F : \mathbb{R} \times \mathbb{R}^{mn} \multimap \mathbb{R}^n$ be an integrably bounded u.s.c. and T-periodic with respect to the first variable multimap with compact convex values. If $V : \mathbb{R}^n \to \mathbb{R}$ is a guiding potential for $F|_{[0,T]\times\mathbb{R}^{mn}}$ with $\operatorname{Ind}(V) \neq 0$, then there exists a T-periodic Carathéodory solution on $\mathbb{R}$ for the inclusion*

$$\dot{x}(t) \in F(t, x(t-\tau_1), \ldots, x(t-\tau_m)).$$

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