Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 00, 2000, 73–92

RELATIVE VERSIONS OF THE MULTIVALUED LEFSCHETZ AND NIELSEN THEOREMS AND THEIR APPLICATION TO ADMISSIBLE SEMI-FLOWS

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Abstract. The relative Lefschetz and Nielsen fixed-point theorems are generalized for compact absorbing contractions on ANR-spaces and nilmanifolds. The nontrivial Lefschetz number implies the existence of a fixed-point in the closure of the complementary domain. The relative Nielsen numbers improve the lower estimate of the number of coincidences on the total space or indicate the location of fixed-points on the complement. Nontrivial applications of these topological invariants (under homotopy) are given to admissible semi-flows and differential inclusions.

1. Introduction

In the theory of ordinary differential equations and, more generally, differential inclusions, the existence and multiplicity results for solutions with prescribed properties (i.e. under some constraints) belong to the main interest. If some related subdomains are (positively) flow-invariant, then a natural question arises,

2000 *Mathematics Subject Classification.* 34A60, 47H04, 47H10, 55M20, 58C06.

Key words and phrases. Relative Lefschetz and Nielsen numbers, proper pairs of spaces, fixed-points and coincidences, CAC-maps, ANR-spaces, nilmanifolds, admissible semi-flows, differential inclusions.

[1]Supported by the Council of Czech Government (J14/98:153100011) and by the Grant Agency of Czech Republic no. 201/00/0768.

[2]Supported by the Polish KBN Grant No. 2 P03A 02416.

[3]Supported by the Polish KBN Grant No. 2 P03A 03315.

whether or not these domains, their complements or the closures of their complements contain fixed-points, somehow associated to solutions. If so, then another question follows, namely how many fixed-points are there. Sometimes, the existence, additivity and excision properties of the fixed-point index can be applied to this goal, of course, provided it is well defined. However, if e.g. fixed-points are allowed on the boundaries, then the invariance under homotopy of fixed-point index cannot be employed in order to simplify the situation. Frequently, simply connected domains involve subdomains which can contain several essential fixed-point classes, but the standard Nielsen number on the total domain (equal 1) informs, in this case, only about the sole existence.

On the other hand, C. Bowszyc [7] introduced in 1968 a relative Lefschetz number for a compact map of pairs of invariant ANR-spaces (one being involved in another) whose nontriviality implies the existence of a fixed-point in the closure of the complement. For this topological invariant, defined in terms of the relative singular homology on the pair over the rationals $\mathbb{Q}$, the fixed-points on the boundaries are allowed. Besides another, it can serve to make the more precise location of a fixed-point. The result of C. Bowszyc has been generalized in various ways (see e.g. [16], [22], [30], [31]). Nevertheless, the similar idea of admissible index pairs, which is essential in the Conley index theory, based on the celebrated Ważewski topological principle, can be recognized also there (cf. [21], [24], [28], [29] and the references therein).

In 1986, H. Schirmer [25] introduced the relative Nielsen number for a lower bound of the number of fixed-points on the total space X, for compact maps on pairs of spaces $A \subset X$, $F : (X, A) \to (X, A)$, which can make better lower estimate than the standard Nielsen number. Later on, other relative Nielsen numbers were defined, which provide lower bounds for either the number of fixed-points on $X \setminus A$ or on $\overline{X \setminus A}$, in order to study the location of fixed-points. The essential idea underlying the definitions, allowing us to make the estimates on the total space or on the complement, uses the concept of (weakly) common essential fixed-point classes (for more details and the computation of these topological invariants, see [9]–[12], [17], [26], [27], [32]–[36] and the references therein).

Although, in the single-valued case, the relative Lefschetz and Nielsen numbers have been already several times applied to dynamical systems and ordinary differential equations (see e.g. [30], [31] and [27]), the application of only nonrelative multivalued Lefschetz and Nielsen-type theorems in this direction is very rare (see [2]–[5] and [21]). To eliminate this handicap, our applications here are just oriented in that way.

Hence, the paper is organized as follows. At first, the notion of compact absorbing contractions (CAC-maps), including only a certain amount of compactness, which are suitable for definitions of the relative Lefschetz and Nielsen fixed-point theorems, is established. Then the fixed-point index for CAC-maps is developed. Finally, the relative theorems are applied to admissible semi-flows and differential inclusions for obtaining existence and multiplicity criteria concerning, for example, periodic trajectories. The obtained results bring new information also in the single-valued case and we believe that they deserve a further interest into the future. On this basis, we would like to develope and apply, besides another, the multivalued Conley index theory (cf. [15], [21]), elsewhere.

2. Compact absorbing contractions

All topological spaces are assumed to be metric. A space X is called an *absolute neighbourhood retract* (*absolute retract*) if, for any space Y and its closed subset $B \subset Y$, any continuous map $f : B \to X$ has a continuous extension over a neighbourhood U of B in Y (over Y) $\widetilde{f} : U \to X$ ($\widetilde{f} : Y \to X$), where $\widetilde{f}(x) = f(x)$, for every $x \in B$. We let:

$$X \in \mathrm{ANR} \Leftrightarrow X \text{ is an absolute neighbourhood retract,}$$
$$X \in \mathrm{AR} \Leftrightarrow X \text{ is an absolute retract.}$$

Of course, if $X \in \mathrm{AR}$, then $X \in \mathrm{ANR}$.

A continuous map $p : Y \to X$ is called *Vietoris*, when:

(i) p is onto, i.e. $p(Y) = X$,
(ii) p is proper, i.e. $p^{-1}(K)$ is compact, for every compact $K \subset Y$,
(iii) $p^{-1}(x)$ is an acyclic set, for every $x \in Y$, where acyclicity is understood in the sense of Čech homology functor with compact carriers and coefficients in the field $\mathbb{Q}$ of rationals (for details, see [15]).

In what follows, the symbol $p : Y \Rightarrow X$ (or $Y \overset{p}{\Longrightarrow} X$) is reserved for Vietoris maps.

For given two pairs of metric spaces (X, A) and (Y, B), by a map $f : (X, A) \to (Y, B)$, we understand a continuous map from X to Y such that $f(A) \subset B$.

If $f : (X, A) \to (Y, B)$ is a map of pairs, then we denote by $f_X : X \to Y$ and $f_A : A \to B$ the induced mappings by f, i.e. $f_X(x) = f(x)$ and $f_A(x) = f(x)$, for every $x \in X$ and $x \in A$, respectively.

A map $p : (Y, B) \to (X, A)$ is called a *Vietoris* map of pairs if $B = p^{-1}(A)$ and both $p_Y : Y \Rightarrow X$ and $p_B : B \Rightarrow A$ are Vietoris maps.

Similarly as above, we reserve the symbol $p : (Y, B) \Rightarrow (X, A)$ for Vietoris maps of pairs.

In what follows, by a multivalued map $\varphi : X \rightsquigarrow Y$ we understand a map such that, for every $x \in X$, the values $\varphi(x)$ are compact nonempty subsets of Y. A map $\varphi : X \rightsquigarrow Y$ is called *u.s.c.* (*l.s.c.*) if, for every open $U \subset Y$, the set:

$$\{x \in X \mid \varphi(x) \subset U\} \quad (\{x \in X \mid \varphi(x) \cap U \neq \emptyset\})$$

is open. An u.s.c. map $\varphi : X \multimap Y$ is called *compact* if there exists a compact subset $K \subset Y$ such that

$$\varphi(X) = \bigcup_{x\in X} \varphi(x) \subset K.$$

φ is called *locally compact* if, for every $x \in X$, there exists an open neighbourhood U_x of x in X such that the restriction $\widetilde{\varphi} : U_x \multimap Y$ of φ to U_x is a compact map.

Assume that we have a diagram $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ in which q is a continuous map. The above diagram induces a multivalued map $\varphi = \varphi(p, q) : X \multimap Y$ by the following formula:

$$\varphi(x) = q(p^{-1}(x)) \quad \text{for every } x \in X.$$

It is easy to see (cf. [15]) that $\varphi(p, q)$ is an u.s.c. map. Moreover, $\varphi(p, q)$ is compact, whenever q is compact.

A pair (p, q) determines a multivalued map $\varphi(p, q) = qp^{-1}$, although this presentation is not unique. We shall sometimes overuse the notation and call (p, q) a multivalued map, where it leads to no misunderstanding. The (p, q) as above with p Vietoris will be called *admissible* map (cf. [6]). Note that the class of admissible maps is quite large, in particular, it contains acyclic mappings and their compositions.

The following two notations are equivalent:

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y \quad \text{and} \quad (p, q) : X \multimap Y.$$

Consider a multivalued map $(p, q) : X \multimap X$. Denoting the sets of *coincidence points* and *fixed-points* as

$$\begin{aligned} C(p, q) &= \{z \in \Gamma \mid p(z) = q(z)\}, \\ \operatorname{Fix}(p, q) &= \{x \in X \mid x \in q(p^{-1}(x))\}, \end{aligned}$$

we have:

Proposition 2.1. $p(C(p, q)) = \operatorname{Fix}(p, q)$. *In particular,* $C(p, q) \neq \emptyset$ *if and only if* $\operatorname{Fix}(p, q) \neq \emptyset$.

Define the composition of pairs $X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$ and $Y \overset{p'}{\Longleftarrow} \Gamma' \overset{q'}{\longrightarrow} Z$ as a pair $X \overset{\overline{p}}{\Longleftarrow} \Gamma \overset{\overline{q}}{\longrightarrow} Z$, where $\overline{\Gamma} = \{(u, u') \in \Gamma \times \Gamma' \mid q(u) = p'(u')\}$, $\overline{p}(u, u') = p(u)$, $\overline{q}(u, u') = q'(u')$. Notice that the composition of the corresponding multivalued maps are equal $\phi(\overline{p}, \overline{q}) = \phi(p', q') \circ \phi(p, q)$.

In this light, for a given admissible map $(p, q) : X \multimap X$, by $(p, q)^n : X \multimap X$ we denote its n-th iterate, i.e.

$$(p, q)^n = \underbrace{(p, q) \circ \ldots \circ (p, q)}_{n\text{-times}}, \quad n = 1, 2, \ldots$$

Obviously, $(p, q)^1 = (p, q)$.

Definition 2.2 (cf. [14]). An admissible map $(p,q) : X \rightsquigarrow X$ is called a *compact absorbing contraction* (written $(p,q) \in \mathrm{CAC}(X)$) if:

(2.2.1) there exists an open subset $U \subset X$ such that the restriction $\widetilde{(p,q)} : U \rightsquigarrow U$ of (p,q) to the pair (U,U) is a compact map,

(2.2.2) for every $x \in X$, there exists $n = n(x)$ such that $(p,q)^n(x) \subset U$, where $\widetilde{(p,q)} = (\widetilde{p},\widetilde{q})$ and $\widetilde{p} : p^{-1}(U) \to X$, $\widetilde{p}(x) = p(x)$, $\widetilde{q} : p^{-1}(U) \to U$, $\widetilde{q}(x) = q(x)$, for every $x \in p^{-1}(U)$.

Let us note that the class of compact absorbing contractions contains compact, eventually compact, asymptotically compact and compact attraction mappings, provided they are locally compact (for more details see [14] or [15]).

We consider also multivalued mappings of pairs. Namely, a diagram

$$(X,A) \overset{p}{\Longleftarrow} (\Gamma,\Gamma_0) \xrightarrow{q} (Y,B)$$

is called an *admissible map of pairs*. We shall denote it as follows:

$$(p,q) : (X,A) \rightsquigarrow (Y,B).$$

Then $(p,q)_X = (p_\Gamma, q_\Gamma) : X \rightsquigarrow Y$ and $(p,q)_A = (p_{\Gamma_0}, q_{\Gamma_0}) : A \rightsquigarrow B$ are induced mappings by (p,q).

Definition 2.3. An admissible map $(p,q) : (X,A) \rightsquigarrow (Y,B)$ is *compact* (*compact absorbing contraction*) if both $(p,q)_X$ and $(p,q)_A$ are compact (compact absorbing contractions).

3. The fixed-point index for compact absorbing contractions

A general fixed-point index for admissible maps is studied in [18]. Below, we shall present a slight generalization and application of results obtained in [18]. Let $\mathcal{M}$ denote the class of all triples $(X, W, (p,q))$, where $X \in \mathrm{ANR}$, W is open in X, $(p,q) \in \mathrm{CAC}$ and $\mathrm{Fix}\,(p,q) \cap \partial W = \emptyset$, where ∂W denotes the boundary of W in X. The aim of this section is to generalize the fixed-point index over $\mathcal{M}$.

Consider a triple $(X, W, (p,q))$. Since $(p,q) \in \mathrm{CAC}(X)$, there exists an open $U \subset X$ satisfying conditions (2.2.1) and (2.2.2). Let us observe that $\mathrm{Fix}\,(p,q)$ is a compact subset of U and $U \in \mathrm{ANR}$ as an open subset of $X \in \mathrm{ANR}$. Consequently, the triple $(U, U \cap W, \widetilde{(p,q)})$ is in B in the sense of W. Kryszewski (see [18]).

Therefore, the fixed-point index $\mathrm{ind}\,(U, U \cap W, \widetilde{(p,q)})$ of the triple $(U, U \cap W, \widetilde{(p,q)})$ is well-defined, according to [18]. Note that $\widetilde{(p,q)} : U \rightsquigarrow U$ is compact admissible.

We define the fixed-point index $\mathrm{Ind}\,(X, W, (p,q))$ of the triple $(X, W, (p,q))$ as follows:

$$\mathrm{Ind}\,(X, W, (p,q)) = \mathrm{ind}\,(U, U \cap W, \widetilde{(p,q)}). \tag{3.1}$$

The above definition (3.1) is correct, i.e. it does not depend on the choice of U. In fact, it follows immediately from the additivity property (or, more precisely, from the localization or excision properties of the fixed-point index in B (see [18] or [15])).

The fixed-point index Ind defined in (3.1) satisfies all the usual properties (see again [18] or [15]). Below, we list the properties which are necessary in Section 4.

(3.2) (Excision) If $(X, W, (p,q)) \in \mathcal{M}$ and $\operatorname{Fix}(p,q) \subset W$, then
$$\operatorname{Ind}(X, W, (p,q)) = \operatorname{Ind}(X, X, (p,q)).$$

(3.3) (Contraction) If $(p,q)(W) \subset A$, $A \in \mathrm{ANR}$ and the restriction $\overline{(p,q)} : A \rightsquigarrow A$, $\overline{(p,q)}(x) = (p,q)(x)$, for every $x \in A$, is a compact absorbing contraction, then
$$\operatorname{Ind}(X, W, (p,q)) = \operatorname{Ind}(A, A \cap W, \overline{(p,q)}).$$

(3.4) (Normalization) $\operatorname{Ind}(X, X, (p,q)) = \Lambda(p,q)$, where $\Lambda(p,q)$ denotes the generalized Lefschetz number of (p,q) (cf. [14], [15] or [18]).

We left to the reader the formulations of further properties of the fixed-point index defined in (3.1).

4. The relative Lefschetz fixed-point theorem

Let us recall that in [14] the following theorem is proved.

Theorem 4.1. *If $X \in \mathrm{ANR}$ and $(p,q) \in \mathrm{CAC}(X)$, then the generalized Lefschetz number $\Lambda(p,q)$ of (p,q) is well-defined and $\Lambda(p,q) \neq 0$ implies that* $\operatorname{Fix}(p,q) \neq \emptyset$.

Observe that (4.1) is the most general formulation of the Lefschetz fixed-point theorem for multivalued mappings on ANRs.

We shall use the following proposition:

Proposition 4.2 ([15], [16]). *Let $\varphi : (X, A) \rightsquigarrow (X, A)$ be an admissible map. If any two maps of (p,q), $(p,q)_X$, $(p,q)_A$ are the Lefschetz maps (subsequently, the generalized Lefschetz number is well-defined), then so is the third and in that case*
$$\Lambda(p,q) = \Lambda((p,q)_X) - \Lambda((p,q)_A).$$

Now, we are able to prove the following generalization of (4.1).

Theorem 4.3 (The Lefschetz fixed-point theorem for pairs of spaces). *Let $X, A \in \mathrm{ANR}$ and $(p,q) : (X, A) \rightsquigarrow (X, A)$ be an admissible compact absorbing contraction mapping on pairs. Then*

(4.3.1) *the generalized Lefschetz number $\Lambda(p,q)$ of (p,q) is well-defined (although it does not follow from* (4.2)*, it can be proved similarly), and*

(4.3.2) $\Lambda(p,q) \neq 0$ *implies* $\mathrm{Fix}\,(p,q) \cap \overline{X \setminus A} \neq \emptyset$, *where* $\overline{X \setminus A}$ *denotes the closure of* $X \setminus A$ *in* X.

PROOF. At first, in view of (4.2) and (4.1), we see that (p,q) is a Lefschetz map and

$$\Lambda(p,q) = \Lambda((p,q)_X) - \Lambda((p,q)_A). \tag{4.4}$$

To prove the second part of our theorem, assume that $\Lambda(p,q) \neq 0$ and (p,q) has no fixed-points in $\overline{X \setminus A}$, i.e. $\mathrm{Fix}\,(p,q) \subset X \backslash (\overline{X \setminus A})$. Letting $W = X \backslash (\overline{X \setminus A}) = \mathrm{int} A$, then W is an open subset of X and, moreover, $W \subset A$. Therefore, from (3.2) and (3.4), we get

$$\mathrm{Ind}\,(X, W, (p,q)_X) = \mathrm{Ind}\,(X, X, (p,q)_X) = \Lambda((p,q)_X). \tag{4.5}$$

Analogously, since $W = \mathrm{Int}_X A \subset A$, we obtain:

$$\mathrm{Ind}\,(A, W, (p,q)_A) = \mathrm{Ind}\,(A, A, (p,q)_A) = \Lambda(p,q)_A). \tag{4.6}$$

Now, using the contraction property of the fixed-point index, we get:

$$\mathrm{Ind}\,(X, W, (p,q)_X) = \mathrm{Ind}\,(A, W, (p,q)_A).$$

From this, when taking into account (4.4), (4.5) and (4.6), we finally obtain that $\Lambda((p,q)) = 0$, and the proof is complete. $\square$

5. Relative Nielsen fixed-point theorem

By a multivalued map we mean again a pair of (single-valued) maps $X \xLeftarrow{p} \Gamma \xrightarrow{q} X$. In [4], [5], a Nielsen-type number $N(p,q)$ is defined. This is a homotopy invariant and a lower bound of the cardinality of coincidences (see [4] for details). Here, we consider multivalued maps between pairs of spaces and we generalize the Nielsen theory into this situation. In the case of single-valued mapping (i.e. $qp^{-1}(x)$ is a singleton, for each $x \in X$), we get the relative Nielsen number introduced by H. Schirmer in 1986 [25].

We assume that the considered spaces are ANRs. Let us recall that X (as an ANR) admits a universal covering $p_X : \widetilde{X} \to X$.

We need the following assumptions:

(A) For any $x \in X$, the restriction $q|_{p^{-1}(x)} : p^{-1}(x) \to X$ admits a *lift* $\widetilde{q}$ to the universal covering space i.e. $p_X \widetilde{q} = q$.

(B) There exists a normal subgroup $H \subset \pi_1 X$ of a finite index, satisfying $\widetilde{q}_! \widetilde{p}^!(H) \subset H$. Here, $\widetilde{q}_! \widetilde{p}^!$ denotes a homomorphism of the fundamental group induced by the admissible map $X \xLeftarrow{p} \Gamma \xrightarrow{q} X$, satisfying (A). If (p,q) represents a single-valued map (i.e. $q(p^{-1}(x))$ is a singleton, for each $x \in X$), then $\widetilde{q}_! \widetilde{p}^!$ coincides with the induced homomorphism $f_\#$.

It is proved in [14] that, for a CAC-mapping, the generalized Lefschetz number $\Lambda(p,q)$ is well-defined which is a homotopy invariant and $\Lambda(p,q) \neq 0$ implies a coincidence. On the other hand, it is proved in [5] (see also [4]) that (A) and (B) imply a Nielsen number $N_H(p,q)$, a homotopy invariant satisfying $N_H(p,q) \leq \#\,\mathrm{C}\,(p,q)$, where $\mathrm{C}\,(p,q)$ denotes the coincidence set of the pair (p,q).

Let $A \subset X$ be a closed connected ANR, satisfying $q(p^{-1}(A)) \subset A$. Denote $\Gamma_A = p^{-1}(A)$ and consider the restriction $A \overset{p_|}{\Longleftarrow} \Gamma_A \overset{q_|}{\longrightarrow} A$. If $U \subset X$ is an open subset in the definition of a CAC-mapping, for (p,q), then $U \cap A$ can be associated to the CAC-mapping of $(p_|, q_|)$. Let $H_0 = i_{\#}^{-1}(H) \subset \pi_1 A$. Since the induced homomorphism $i_\# : (\pi_1 A)/H_0 \to (\pi_1 X)/H$ is mono, H_0 is also a normal subgroup of a finite order. Hence, $A \overset{p_|}{\Longleftarrow} \Gamma_A \overset{q_|}{\longrightarrow} A$ (where $p_|$, $q_|$ denote the natural restrictions) also satisfies the assumptions (A) and (B).

Let us note that the diagram

$$\begin{tikzcd} A & \Gamma_A \arrow[l, "p_|"'] \arrow[r, "q_|"] \arrow[d, "i_\Gamma"'] & A \arrow[d, "i"'] \\ X & \Gamma \arrow[l, "p"'] \arrow[r, "q"] & X \end{tikzcd}$$

where the vertical lines are natural inclusions, is commutative.

Let $p_X : \widetilde{X} \to X$, $p_A : \widetilde{A} \to A$ be fixed coverings corresponding to the subgroups H, H_0, respectively. In view of the results [4], there exist lifts $\widetilde{q}$, $\widetilde{q}_A$ making the diagrams

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p_X"'] & \widetilde{\Gamma} \arrow[l, "\widetilde{p}"'] \arrow[r, "\widetilde{q}"] \arrow[d, "p_\Gamma"'] & \widetilde{X}_H \arrow[d, "p_X"'] \\ X & \Gamma \arrow[l, "p"'] \arrow[r, "q"] & X \end{tikzcd} \qquad \begin{tikzcd} \widetilde{A} \arrow[d, "p_A"'] & \widetilde{\Gamma}_A \arrow[l, "\widetilde{p}_A"'] \arrow[r, "\widetilde{q}_A"] \arrow[d, "p_{\Gamma_A}"'] & \widetilde{A} \arrow[d, "p_A"'] \\ A & \Gamma_A \arrow[l, "p_|"'] \arrow[r, "q_|"] & A \end{tikzcd}$$

commutative, where

$$\begin{aligned} \widetilde{\Gamma} &= \{(\widetilde{x}, z) \in \widetilde{X} \times \Gamma;\ p_X(\widetilde{x}) = p(z)\}, \qquad && \widetilde{p}(\widetilde{x}, z) = \widetilde{x}, \quad && p_\Gamma(\widetilde{x}, z) = z, \\ \widetilde{\Gamma}_A &= \{(\widetilde{a}, z) \in \widetilde{A} \times \Gamma_A;\ p_A(\widetilde{a}) = p_A(z)\}, && \widetilde{p}_A(\widetilde{a}, z) = \widetilde{a}, && p_{\Gamma_A}(\widetilde{a}, z) = z. \end{aligned}$$

Let $O_X = \{\alpha : \widetilde{X} \to \widetilde{X};\ p_X \alpha = p_X\}$, $O_A = \{\alpha : \widetilde{A} \to \widetilde{A};\ p_A \alpha = p_A\}$ denote the groups of the covering transformations. Recall that $O_X \approx \pi_1 X$, $O_A \approx \pi_1 A$.

LEMMA 5.1. *There exist maps $\widetilde{i} : \widetilde{A} \to \widetilde{X}$, $\widetilde{i}_\Gamma : \widetilde{\Gamma}_A \to \widetilde{\Gamma}$ making the following diagrams commutative.*

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\widetilde{i}"] \arrow[d, "p_A"'] & \widetilde{X} \arrow[d, "p_X"'] \\ A \arrow[r, "i"] & X \end{tikzcd} \qquad \begin{tikzcd} \widetilde{\Gamma}_A \arrow[r, "\widetilde{i}_\Gamma"] \arrow[d, "p_{\Gamma_A}"'] & \widetilde{\Gamma} \arrow[d, "p_\Gamma"'] \\ \Gamma_A \arrow[r, "i_\Gamma"] & \Gamma \end{tikzcd}$$

Let us fix such maps $\widetilde{i}$, $\widetilde{i}_\Gamma$. Then, for any $\alpha_0 \in O_A$, there is exactly one $\alpha \in O_A$ making the following diagram

$$\begin{tikzcd}
\widetilde{A} \arrow[dd, "p_A"'] \arrow[dr, "\widetilde{i}"'] & & \widetilde{\Gamma}_A \arrow[ll, "\widetilde{p}_A"'] \arrow[rr, "\alpha_0\widetilde{q}_A"] \arrow[dd, "p_{\Gamma_A}"] \arrow[dr, "\widetilde{i}_\Gamma"] & & \widetilde{A} \arrow[dd, "p_A"] \arrow[dr, "\widetilde{i}"] & \\
& \widetilde{X} \arrow[dd, "p_X"'] & & \widetilde{\Gamma} \arrow[ll, "\widetilde{p}"'] \arrow[rr, "\alpha\widetilde{q}"] \arrow[dd, "p_\Gamma"'] & & \widetilde{X} \arrow[dd, "p_X"] \\
A \arrow[dr, "i"'] & & \Gamma_A \arrow[ll, "\widetilde{p}_|"'] \arrow[rr, "\widetilde{q}_|"] \arrow[dr, "i_\Gamma"'] & & A \arrow[dr, "i"'] & \\
& X & & \Gamma \arrow[ll, "p"] \arrow[rr, "q"'] & & X
\end{tikzcd}$$

commutative.

Proof. Since $(ip_A)_\#(\pi_1\widetilde{A}) = i_\#(H_0) \subset H = \mathrm{im}\,(p_X)_\#$, there exists a lift $\widetilde{i}$ making the first diagram commutative. We define $\widetilde{i}_\Gamma : \widetilde{\Gamma}_A \to \widetilde{\Gamma}$ by putting $\widetilde{i}_\Gamma(\widetilde{a}, z) = (\widetilde{i}(\widetilde{a}), i_\Gamma(z))$, and the second diagram commutes.

Now, we consider the big diagram. The commutativity of all squares, but the deck ones, follow from the already discussed diagrams. It remains to check the commutativity of the deck squares. One can easily check that $\widetilde{p}\,\widetilde{i}_\Gamma = \widetilde{i}\,\widetilde{p}_A$. Then we can examine the right deck square. At first, we notice that $p_X(\widetilde{q}\,\widetilde{i}_\Gamma) = p_X(\widetilde{i}\alpha_0 q_A)$. Thus, $\widetilde{q}\,\widetilde{i}_\Gamma$, $\widetilde{i}\alpha_0 q_A$ are lifts of the same map $qi_\Gamma = iq_|$, by which there is exactly one $\alpha \in O_X$ such that $\alpha\widetilde{q}\,\widetilde{i}_\Gamma = \widetilde{i}\alpha_0 q_A$. $\square$

Remark 5.2. We can replace the lifts $\widetilde{q}$, $\widetilde{q}_A$ by those for which the above big diagram commutes (i.e. it commutes for α_0 and α being identities). Then the mapping $O_A \ni \alpha_0 \to \alpha \in O_X$ defined in the Lemma (5.1) coincides with the homomorphism $\pi_1 A \to \pi_1 X$, induced by the inclusion $A \to X$. So, we denote it by $i_\#$.

Let us recall that a pair (p, q) satisfying (A) induces a homomorphism $\widetilde{q}_!\widetilde{p}^! : O_X \to O_X$, corresponding, in the single-valued case ($\rho = qp^{-1}$), to $\rho : \pi_1 X \to \pi_1 X$. We define the action of O_X on itself $\gamma \circ \alpha = \gamma\alpha\widetilde{q}_!\widetilde{p}^!(\gamma^{-1})$). By an analogy with the single-valued case, we define the quotient set, the *set of Reidemeister classes,* and we denote it by $\mathcal{R}_H(p, q)$. We define the *Nielsen class* corresponding to a class $[\alpha] \in \mathcal{R}(p, q)$, as $p_\Gamma(C(\widetilde{p}, \alpha\widetilde{q}))$. This splits $\mathrm{C}\,(p, q)$ into disjoint Nielsen classes and defines the natural injection $\eta : \mathcal{N}_H(p, q) \to \mathcal{R}_H(p, q)$. If $\mathrm{L}\,(\widetilde{p}, \alpha\widetilde{q}) \neq 0$, then $\mathrm{C}\,(\widetilde{p}, \alpha\widetilde{q}) \neq \emptyset$, and the Nielsen class corresponding to $[\alpha] \in \mathcal{R}_H(p, q)$ is called *essential.* We define the *H-Nielsen number* as the number of essential classes and denote it by $N(p, q)$.

Lemma 5.3. *The homomorphism* $i_\#$ *in Remark* 5.2 *induces a map of the Reidemeister sets* $\mathcal{R}(i) : \mathcal{R}_{H_0}(p_|, q_|) \to \mathcal{R}_H(p,q)$.

Proof. It is enough to show the commutativity of the diagram

$$\begin{array}{ccc} O_A \times O_A & \xrightarrow{\ \circ\ } & O_A \\ {\scriptstyle i_\# \times i_\#}\downarrow & & {\scriptstyle i_\#}\downarrow \\ O_X \times O_X & \xrightarrow{\ \circ\ } & O_X \end{array}$$

where the horizontal lines are given by the Reidemeister action

$$(\gamma, \alpha) \to \gamma \circ \alpha = \gamma\alpha(\widetilde{q}_! \widetilde{p}^! \gamma)^{-1}.$$

In fact, $i_\#$ is a homomorphism which implies $i_\#(\gamma \circ \alpha) = i_\#(\gamma\alpha(\widetilde{q}_!\widetilde{p}^!\gamma)^{-1}) = i_\#(\gamma) \cdot i_\#\alpha \cdot i_\#(\widetilde{q}_!\widetilde{p}^!\gamma)^{-1}) = i_\#(\gamma) \cdot i_\#\alpha \cdot (\widetilde{q}_!\widetilde{p}^!(i_\#\gamma))^{-1}) = i_\#\gamma \circ i_\#\alpha$. $\square$

Lemma 5.4. *The following diagram commutes*

$$\begin{array}{ccc} \mathcal{N}_{H_0}(p_|, q_|) & \xrightarrow{\mathcal{N}(i)} & \mathcal{N}_H(p,q) \\ {\scriptstyle \eta}\downarrow & & {\scriptstyle \eta}\downarrow \\ \mathcal{R}_{H_0}(p_|, q_|) & \xrightarrow{\mathcal{R}(i)} & \mathcal{R}_H(p,q) \end{array}$$

Let $S_H(p,q;A) \subset \mathcal{R}_H(p,q)$ denote the set of essential Reidemeister classes which contain no essential class from $\mathcal{R}_{H_0}(p_|, q_|)$.

Theorem 5.5. *Under the assumptions* (CAC), (A) *and* (B), *the pair* (p,q) *has at least* $N_H(p,q) + (\#S_H(p,q;A))$ *coincidences.*

Proof. We choose a point $z_1, \ldots, z_k \in C(f_|, g_|)$, from each essential class of $f_|, g_|$, and a point $w_1, \ldots, w_l \in C(f,g)$ from, each one in $S_H(p,q;A)$. It remains to show that $i(z) \neq w$, for any $z = z_1, \ldots, z_k$, $w = w_1, \ldots, w_l$. Suppose the contrary. Then, by Lemma 5.4, the essential Nielsen class (of $p_|, q_|$) containing z is involved in the class (of p,q) containing w. Since the last class belongs to $S_H(p,q;A)$, we get a contradiction. $\square$

Remark 5.6. Since any essential Reidemeister class corresponds always to a nonempty Nielsen class, in the definition of $S_H(p,q;A)$, the name of Reidemeister can be replaced by Nielsen: $S_H(p,q;A)$ becomes the set of Nielsen classes (from $\mathcal{N}_H(p,q)$) which contains no essential Nielsen class (from $\mathcal{N}_{H_0}(p_|, q_|)$).

The following theorem gives a lower bound for the number of coincidences of the pair (p,q) lying outside Γ_A (cf. the surplus Nielsen number in [35]). Let $SN_H(p,q;A)$ be the cardinality of the set of essential classes in $\mathcal{R}_H(p,q) \setminus \operatorname{im} \mathcal{R}(i)$.

Theorem 5.7. *Under the assumptions* (CAC), (A) *and* (B), *the pair* (p,q) *has at least* $SN_H(p,q;A)$ *coincidences in* $\Gamma\setminus\Gamma_A$.

Proof. We can observe that each essential class from $\mathcal{R}_H(p,q)\setminus\operatorname{im}\mathcal{R}(i)$ is non-empty and disjoint from Γ_A. □

6. Applications to multivalued semi-flows

Following e.g. [28], [29] (in the single-valued case), we give at first suitable definitions of (periodic) [semi-] processes and proper pairs.

Assume that X is a metrizable space and $\Phi : D \rightsquigarrow X$ is a CAC-mapping, where $D\subset\mathbb{R}\times X\times\mathbb{R}$ $[D\subset\mathbb{R}\times X\times\mathbb{R}_0^+]$ is an open set. Denoting the mapping $\Phi(\sigma,\,\cdot\,,t)$ by $\Phi_{(\sigma,t)}$, we can give the definition of admissible (multivalued) [semi-] processes as follows.

Definition 6.1. Φ is called a *generalized local* [*semi-*] *process* (on the space X) if the following conditions are satisfied:

(i) for all $\sigma\in\mathbb{R}$, $x\in X : \{t\in\mathbb{R}\ [t\geq 0]\mid(\sigma,t,x)\in D\}$ is an interval,
(ii) for all $\sigma\in\mathbb{R}:\ \Phi_{(\sigma,0)}=\mathrm{id}$,
(iii) for all $\sigma,t,s\in\mathbb{R}$ [for all $\sigma\in\mathbb{R},\ t\geq 0, s\geq 0$] : $\Phi_{(\sigma,s+t)}\subset[=]\Phi_{(\sigma+s,t)}\circ\Phi_{(\sigma,s)}$.

In the case $D=\mathbb{R}\times X\times\mathbb{R}$ $[\mathbb{R}\times X\times\mathbb{R}_0^+]$, we call Φ a *generalized* (*global*) [*semi-*] *process*.

For $(\sigma,x)\in\mathbb{R}\times X$, under the assumption that

$$\Phi_{(\sigma,t)}(x)=\{y_{(\sigma,t)}(x)\in X\mid y_{(\sigma,\,\cdot\,)}(x)\text{ is a continuous function with }y_{(\sigma,0)}(x)=x\text{ and }(\sigma,x,t)\in D\},$$

the set

$$\{(\sigma+t,y_{(\sigma,t)}(x))\in\mathbb{R}\times X\mid y_{(\sigma,t)}(x)\subset\Phi_{(\sigma,t)}\text{ is a single-valued continuous selection and }(\sigma,x,t)\in D\}$$

is called the *set of trajectories of* (σ,x) *in* Φ.

If $T>0$ is an integer and Φ still fulfills

(iv) $$\text{for all }\sigma,\ t\in\mathbb{R}\ [\text{for all }\sigma\in\mathbb{R},\ t\geq 0]:\Phi_{(\sigma,t)}=\Phi_{(\sigma+T,t)},$$

we call Φ a *T-periodic local* (or *global*) *generalized* [*semi-*] *process*.

For $\sigma=0$ a generalized local (or global) [semi-] process $\Phi_{(0,t)}$ is called a *generalized local* (or *global*) *generalized* [*semi-*] *dynamical system*.

A local generalized process Φ on X determines a *local generalized* [*semi-*] *flow* Φ^* on $\mathbb{R}\times X$ by

$$\Phi_t^*(\sigma,x)=(\sigma+t,\Phi_{(\sigma,t)}(x)).$$

Denoting

$$Z(t) = \{x \in X \mid (t,x) \in Z\},$$

for every $Z \subset \mathbb{R} \times X$ and $\sigma \in \mathbb{R}$, we call the *set* Z to be T*-periodic* if, for every $t \in \mathbb{R}$,

$$Z(\sigma) \equiv Z(\sigma + T).$$

In this case, we put

$$Z = \left\{ \left(\exp\left(\frac{2\pi i \sigma}{T}\right), x \right) \in S^1 \times Z(\sigma) \right\},$$

i.e. there is a circumference by identifying σ and $\sigma + T$.

Assuming that (A,B) is a pair of subsets of $\mathbb{R} \times X$, where $A \subset B$, we can give

Definition 6.2. (A,B) is called a *proper pair* if

(i) $A(\sigma)$ and $B(\sigma)$ are ANR-spaces, for each $\sigma \in \mathbb{R}$,

(ii) there is a generalized [semi-] process Φ on X such that A and B are [positively-] invariant under the flow Φ^*, defined as above.

If still

(iii) A and B are T-periodic, then we speak about a T*-periodic proper pair*.

Now, consider the system

$$y' \in F(t,y), \tag{6.3}$$

where $F : \mathbb{R} \times \Omega \rightsquigarrow \mathbb{R}^n$ is an (upper) Carathéodory multivalued function, which is essentially bounded in t and linearly bounded in y, and $\Omega \subset \mathbb{R}^n$ is an open subset (possibly, the whole space $\mathbb{R}^n$). Denoting by $y(t) = y(\sigma,x,t)$, $[t \geq \sigma]$, its (Carathéodory-like) solution, satisfying $y(\sigma) = x$, the *generalized* [*semi-*] *process* ϕ *generated by* (6.3) takes the form $[t \geq 0]$

$$\varphi_{(\sigma,t)}(x) = \varphi(\sigma,x,t) = \{y(\sigma,x,t+\sigma)\}.$$

In particular, $\varphi_{(0,t)}(x) = \{y(0,x,t)\}$ or

$$\varphi_{(\sigma,t)}(x) = \{y(\sigma,x,t+\sigma)\} = \varphi_{(\sigma+T,t)}(x) = \{y(\sigma+T,x,t+\sigma+T)\},$$

provided $F(t,y) \equiv F(t+T,y)$, represent a *generalized* [*semi-*] *dynamical system* or a *generalized* T*-periodic* [*semi-*] *process generalized by* (6.3), respectively.

It follows from the following investigations that if (A,B), where $B \subset A \subset \mathbb{R} \times X$, is a proper pair for a generalized [semi-] process $\varphi : (A(\sigma), B(\sigma)) \rightsquigarrow (A(\sigma), B(\sigma))$, then the generalized Lefschetz number $\Lambda(\varphi)$ is well-defined, for every $\sigma \in \mathbb{R}$, satisfying $\Lambda(\varphi) = \Lambda(\varphi_{A(\sigma)}) - \Lambda(\varphi_{B(\sigma)})$, where $\varphi_{A(\sigma)} : A(\sigma) \rightsquigarrow A(\sigma)$ and $\varphi_{B(\sigma)} : B(\sigma) \rightsquigarrow B(\sigma)$ are particular [semi-] flows on $A(\sigma)$ and $B(\sigma)$, $\sigma \in \mathbb{R}$.

Moreover, for every $\sigma \in \mathbb{R}$, $t \in \mathbb{R}$ $[t \in \mathbb{R}_0^+]$, $\varphi_{(\sigma,t)} \simeq \varphi_{(\sigma,0)} = \mathrm{id}$, a homotopy is given by $\varphi_{(\sigma,kt)}$, $k \in [0,1]$.

In particular, for a T-periodic proper pair (A,B), $T > 0$

$$\varphi_0 \simeq \varphi_T : (A(0), B(0)) \rightsquigarrow (A(t), B(T)),$$

by which (in view of the invariance under homotopy)

$$\Lambda(\varphi_0^*) = \Lambda(\varphi_T) = \Lambda(\varphi_0) = \chi(A(0)) - \chi(B(0)),$$

where χ stands for the Euler–Poincaré characteristic.

Thus, if $\chi(A(0)) \neq \chi(B(0))$, then there exists a fixed-point $z^* \in \{0\} \times \overline{A(0) \setminus B(0)}$ of the semi-flow $\varphi_T^* : \{0\} \times A(0) \rightsquigarrow \{T\} \times A(T)$, i.e. $z^* \in \varphi_T^*(z^*)$, and subsequently also a T-periodic trajectory of $(0, z^*)$ is φ_A.

Hence, we can give the first statement of this section.

Theorem 6.4. *If (A,B), where $B \subset A \subset \mathbb{R} \times X$, is a T-periodic proper pair for the T-periodic semi-flow $\varphi^* : (A,B) \rightsquigarrow (A,B)$, determined by some T-periodic semi-process φ on $X \times X$, then there exists a T-periodic trajectory of some point $(0,z^*) \in \{0\} \times \overline{A(0) \setminus B(0)}$ in $\varphi_{A(0)} : A(0) \rightsquigarrow A(T)$, provided*

$$\varphi_{A(0)}(x) = \{y_{(0,t)}(x) \in A(t) \mid y_{(0,\,\cdot\,)}(x) \text{ is a continuous function with } y_{(0,0)}(x) = x \text{ and } (x,t) \in A(t) \times [0,T]\}.$$

For the differential system (6.3), where $F(t,y) \equiv F(t+T,y)$, a T-periodic semi-process can be generated by means of the associated *Poincaré translation operator* along the trajectories of (6.3) at the time kT, $k \in [0,1]$, defined as follows:

$$\Phi_{kT} = \{y(x,kT) \mid y(x,\,\cdot\,) \text{ is a solution of (6.3) with } y(x,0) = x\}. \tag{6.5}$$

It is known (see e.g. [1], [15]) that Φ is admissible. More precisely, it is a compact composition $\Phi = \psi_2 \circ \psi_1$ of an R_δ-mapping ψ_1 and a continuous (single-valued) evaluation mapping ψ_2, which is homotopic (in the same class of maps) to identity.

Therefore, if A_1, B_1, where $B_1 \subset A_1 \subset \Omega \subset \mathbb{R}^n$, are compact ENR-spaces such that

$$\Phi_{kT}(A_1) \subset A_1 \text{ and } \Phi_{kT}(B_1) \subset B_1 \quad \text{for all k } \in [0,1], \tag{6.6}$$

then Φ_{kT} becomes a CAC (in fact, compact) -mapping. We can even put

$$\Phi_{kT}^*(0,x) = (kT, \Phi_{(0,kT)}(x)) = (kT, \Phi_{kT}(x)),$$

in order to demonstrate the correspondence to a T–periodic generalized semi–flow in the above sense.

After all, assuming (6.6) and

$$\chi(A_1) \neq \chi(B_1), \tag{6.7}$$

system (6.3) admits a T-periodic solution $y(t)$ with $y(0) \in \overline{A_1 \setminus B_1}$.

Condition (6.6) can be expressed more explicitly by means of locally Lipschitzean bounding functions. For this, we use the following lemma, which is stated without the proof, because it is only a slight modification of the well-known results (see e.g. [13], [19], [23], and the references therein).

Lemma 6.8. *Let $V_u(t,y) \equiv V_u(t+T,y) \in C([0,T] \times \Omega, \mathbb{R})$ be a family of (bounding) functions and $c \in \mathbb{R}$. Set $M = [V_u \leq c] = \{y \in \Omega \mid V_u(t,y) \leq c\}$; the set $[V_u > c]$ is defined analogously. Assume that, for each $u \in \partial M$ and $t \in [0,T]$, there exists $\varepsilon > 0$ such that V_u is locally Lipschitzean in $[V_u > c] \cap B(u,\varepsilon)$, uniformly w.r.t. $t \in [0,T]$, and*

$$\limsup_{h \to 0^+} \frac{1}{h}[V_u(t+h, y+hf) - V_u(t,y)] \leq 0,$$

for every $y \in [V_u > c] \cap B(u,\varepsilon)$ and $f \in F(t,y)$. Then M is positively flow-invariant for (6.3), *i.e.* $\Phi_{kT}(M) \subset M$, $k \in [0,1]$, *where Φ is the associated translation operator in* (6.5).

We are in position to give the second theorem of this section.

Theorem 6.9. *Let A_1 and B_1, where $B_1 \subset A_1 \subset \Omega \subset \mathbb{R}^n$, be compact* ENR*-spaces such that* (6.7) *holds. Assume the existence of a family of bounding functions $V_u(t,u)$, $W_u(t,y)$ and constants c_1, c_2 satisfying the conditions of Lemma* 6.8, *for $A_1 = [V_u \leq c_1]$ and $B_1 = [W_u \leq c_2]$. Then the system* (6.3), *where $F(t,y) \equiv F(t+T,y)$, admits a T-periodic solution $y(t)$ with $y(0) \in \overline{A_1 \setminus B_1}$.*

Remark 6.10. Instead of Lemma 6.8, we could employ other criteria (see e.g. [13]) ensuring the strong positive flow-invariance of A_1, B_1, when the boundaries ∂A_1 and ∂B_1 are not reached by a solution from the interior of A_1 and B_1. Then, the additivity, excision and existence properties of the fixed-point index could be also applied for the same aim (see e.g. [15]). However, since fixed-points of the translation operator in (6.5) are allowed in Theorem 6.9, for any $k \in [0,1]$, on the boundaries ∂A_1 and ∂B_1, Theorem 6.9 can be regarded as a nontrivial example of an application of the relative Lefschetz number. The same is all the better true for Theorem 6.4.

Now, we would like to make a nontrivial application of the relative Nielsen number. If Ω is a nilmanifold (for its definition and more details, see e.g. [20], [33] and the references therein), for example, a torus or an open disk with one hole in $\mathbb{R}^2$, then the well-known result of D. Anosov asserts that, for a single-valued self-map $f : \Omega \to \Omega$, we have $N(f) = |\Lambda(f)|$, where N and Λ stands for the (well-defined) Nielsen and Lefschetz numbers, respectively. If Ω is an AR-space (not necessarily a nilmanifold), then $N(f) = \Lambda(f) = 1$.

Furthermore, if Ω_1, ω_2 are compact ANR-spaces such that $\Omega_2 \subset \Omega_1$ (Ω_2 is assumed to be closed) and $f : (\Omega_1, \Omega_2) \to (\Omega_1, \Omega_2)$, then (cf. e.g. Definitions 2.2. and 2.3 in [27])

$$N(f; \Omega_1, \Omega_2) = N(f_{\Omega_1}) + N(f_{\Omega_2}) - N(f_{\Omega_1}, f_{\Omega_2}),$$

where $N(f_{\Omega_1}, f_{\Omega_2})$ is the relative Nielsen number of the map $f : (\Omega_1, \Omega_2) \to (\Omega_1, \Omega_2)$, $N(f_{\Omega_1})$ is the Nielsen number of $f|_{\Omega_1} : \Omega_1 \to \Omega_1$, $N(f_{\Omega_2})$ is the one of $f|_{\Omega_2} : \Omega_2 \to \Omega_2$ and $N(f_{\Omega_1}, f_{\Omega_2})$ is the number of essential common fixed-point classes of $f|_{\Omega_1}$ and $f|_{\Omega_2}$. Let us note that, under suitable assumptions, P. Wong generalized in [33] D. Anosov's result to $N(f; \Omega_1, \Omega_2)$, for compact manifolds.

Therefore, if $(M_1, M_2) = (\mathbb{R}\times\Omega_1, \mathbb{R}\times\Omega_2)$, where $\Omega_2 \subset \Omega_1 \subset X$ are connected nilmanifolds or AR-spaces (Ω_2-closed) having (if they are not compact) finitely generated abelian fundamental groups ($\Rightarrow$ B-property; see [5]), is a proper pair for the semi-flow $\varphi^* : (M_1, M_2) \rightsquigarrow (M_1, M_2)$, determined by a semi-process $\varphi = \psi_2 \circ \psi_1$ on $X \times X$, where ψ_1 is a R_δ-mapping and ψ_2 is a continuous (single-valued) mapping ($\Rightarrow$ A-property; see [5]), then (in view of the obvious invariance under homotopy) we arrive at (see the foregoing section and cf. [4], [5])

$$\begin{aligned} N_H(p,q) + (\sharp S_H(p,q;\Omega_2)) &= N_H(f \circ \varphi_{(0,T)}; \Omega_1, \Omega_2) = N(f \circ \mathrm{id}; \Omega_1, \Omega_2) \\ N(f; \Omega_1, \Omega_2) &= N(f_{\Omega_1}) + N(f_{\Omega_2}) - N(f_{\Omega_1}, f_{\Omega_2}) \\ &\qquad |\Lambda(f_{\Omega_1})| + |\Lambda(f_{\Omega_2})| - N(f_{\Omega_1}, f_{\Omega_2}), \end{aligned}$$

where

$$\Omega_1 \xleftarrow{p} \Gamma_{f\circ\varphi_{(0,T)}} \xrightarrow{q} \Omega_1 \quad \text{and} \quad \Omega_2 \xleftarrow{p} \Gamma_{f\circ\varphi_{(0,T)}} \xrightarrow{q} \Omega_2.$$

In many situations $N(f; \Omega_1, \Omega_2) = |\Lambda(f_{\Omega_1})|$ when $\Lambda(f_{\Omega_1}) \neq 0$, according to P. Wong's result in [33], by which "only" $N(f; \Omega_1, \Omega_2) = N(f_{\Omega_1})$. Thus, a nontrivial situation can be mostly expected, when $\Lambda(f_{\Omega_1}) = 0$. In our (more general, but not so explicit) situation, there exist at least

$$|\Lambda(f_{\Omega_1})| + |\Lambda(f_{\Omega_2})| - N(f_{\Omega_1}, f_{\Omega_2}) (\geq |\Lambda(f_{\Omega_2})|)$$

coincidences of the pair (p, q) associated to the mapping $f \circ \varphi_{(0,T)} : \Omega_1 \rightsquigarrow \Omega_1$

Moreover, there is a one-to-one correspondence between the coincidences and the trajectories $(t, y_{(0,t)}(x^*)) \in [0,T] \times \Omega_1$ in $\varphi_{(0,t)}$, satisfying $y_{(0,0)}(x^*) = f(y_{(0,T)}(x^*))$, where x^* is a fixed-point of $f \circ \varphi_{(0,T)}$, provided

$$\begin{aligned} \varphi_{(0,t)}(x) = \bigg\{ y_{(0,t)}(x) \in \Omega_1 \,\bigg|\, & y_{(0,\,\cdot\,)}(x) \subset \psi_1 \text{ is a continuous selection,} \\ & \psi_1 = \bigcup y_{(0,\,\cdot\,)}(x) \text{ and } \varphi = \psi_2 \circ \psi_1 \bigg\}. \end{aligned}$$

If the homotopy endomorphisms

$$(f_{\Omega_i})_* : H_*(\Omega_i, \mathbb{Q}) \to H_*(\Omega_1, \mathbb{Q}), \quad i = 1, 2,$$

are known for nilmanifolds Ω_i (for AR-spaces Ω_i, we have $\Lambda(f_{\Omega_i}) = 1$), then still (cf. [20])

$$|\Lambda(f_{\Omega_i})| = \bigg| \sum_n (-1)^n \operatorname{tr}(f_{\Omega_i})_{n*} \bigg|, \quad i = 1, 2. \tag{6.11}$$

We can summarize our investigation about the relative Nielsen number (on the total space) as follows.

Theorem 6.12. *Assume that* $(M_1, M_2) = (\mathbb{R} \times \Omega_1, \mathbb{R} \times \Omega_2)$ *is a proper pair for the semi-flow* $\varphi^* : (M_1, M_2) \rightsquigarrow (M_1, M_2)$*, determined by some generalized semi-process* $\varphi = \psi_2 \circ \psi_1$ *on* $X \times X$*, where* $\Omega_2 \subset \Omega_1 \subset X$ *are connected nilmanifolds or* AR*-spaces* (Ω_2 *is closed*)*, having finitely generated abelian fundamental groups,* ψ_1 *is an* R_δ*-mapping and* ψ_2 *is a continuous* (*single-valued*) *mapping such that* $\varphi_{(0,t)}(x) = \{y_{(0,t)}(x) \in \Omega_1 \mid y_{(0,\,\cdot\,)}(x) \subset \psi_1$ *is a continuous selection,* $\psi_1 = \bigcup y_{(0,\,\cdot\,)}(x)$ *and* $\varphi = \psi_2 \circ \psi_1\}$*. Furthermore, let* $f : (\Omega_1, \Omega_2) \to (\Omega_1, \Omega_2)$ *be a continuous* (*single-valued*) *mapping. Then there exist at least* $(|\Lambda(f_{\Omega_1})| + |\Lambda(f_{\Omega_2})| - N(f_{\Omega_1}, f_{\Omega_2}))$ *trajectories* $(t, y_{(0,t)}(x^*)) \in [0, T] \times \Omega_1$ *in* $\varphi_{(0,t)}$*, satisfying* $y_{(0,0)}(x^*) = f(y_{(0,T)}(x^*))$*, for some* (*one or more*) $x^* \in \Omega_1$*. For nilmanifolds* Ω_i*, the generalized Lefschetz number* $|\Lambda(f_{\Omega_i})|$*,* $i = 1, 2$*, can be computed by means of* (6.11) *and* $N(f_{\Omega_1}, f_{\Omega_2})$ *denotes the number of essential common fixed-point classes of* $f_{\Omega_1} = f|_{\Omega_1}$ *and* $f_{\Omega_2} = f|_{\Omega_2}$.

We have already pointed out that the translation operator in (6.5), associated to (6.3), generates a generalized semi-process $\Phi_{(\sigma,kT)}(x)$ as well as the generalized semi-flow $\Phi^*(\sigma, t) = (\sigma + kT, \Phi_{(\sigma,kT)}(x))$, which are exactly of the type as above. Moreover, there is a one-to-one correspondence between the coincidences and the solutions of (6.3). Hence, Theorems 6.9 and 6.12 can be still specified in the following way.

Theorem 6.13. *Let* A_1 *and* B_1*, where* $B_1 \subset A_1 \subset \Omega \subset \mathbb{R}^n$*, be bounded connected either nilmanifolds and* ENR*-spaces or* AR*-spaces;* B_1*-closed and* A_1*-having a finitely generated abelian fundamental group, when it is not closed. Furthermore, let* $f : (A_1, B_1) \to (A_1, B_1)$ *be a continuos* (*single-valued*) *mapping such that* $f|_{\overline{A_1}} : \overline{A_1} \to A_1$*, when* A_1 *is assumed to be open. At last, assume the existence of a family of bounding functions* $V_u(t, u), W_u(t, u)$ *and constants* c_1, c_2*, satisfying the conditions of Lemma* 6.8*, for* $A_1 = [V_u \le c_1]$ *and* $B_1 = [W_u \le c_1]$*. Then system* (6.3) *admits at least* $(|\Lambda(f_{A_1})| + |\Lambda(f_{A_2})| - N(f_{A_1}, f_{A_2}))$ *solutions* $y(t)$*, with* $y(0) = f(y(T))$*,* $y(0) \in A_1$*. For nilmanifolds* A_i*, the generalized Lefschetz number* $\Lambda(f_{A_i})$*,* $i = 1, 2$*, can be computed by means of* (6.11) *and* $N(f_{A_1}, f_{A_2}))$ *denotes the number of essential common fixed-point classes of* $f_{A_1} = f|_{A_1}$ *and* $f_{A_2} = f|_{A_2}$.

Remark 6.14. If A_1 is assumed to be open, then another possibility consists in application of a different criterion of a flow-invariance of A_1, under the translation operators, namely the one ensuring a strong positive flow-invariance of $\overline{A_1}$, as mentioned in Remark 6.10 Then we need not assume (rather unnaturally) that $f|_{\overline{A_1}} : \overline{A_1} \to A_1$. If A_1 is an AR-space, then $|\Lambda(f_{A_1})| \geq N(f_{A_1}, f_{A_2})$, by which the lower estimate of solutions $y(t)$ of (6.3) with $y(0) = f(y(T))$ reduces either to 1 or to $|\Lambda(f_{A_2})|$.

Remark 6.15. In fact, in Theorems 6.12 and 6.13, the same amount of trajectories $(t + \sigma, y)$ and solutions y exists, for every $\sigma \in \mathbb{R}$, satisfying

$$y_{(\sigma,0)}(x_\sigma^*) = f(y_{(\sigma,T)}(x_\sigma^*)) \quad \text{and} \quad y(\sigma) = f(y(\sigma + T)),$$

respectively. Therefore, only the notion of local semi-dynamical systems was appropriate for conclusions in Theorems 6.12 and 6.13. On the other hand, if e.g. $f = \mathrm{id}$, then we can get at least $(|\chi(\Omega_1)| + |\chi(\Omega_2)| - N(f_{\Omega_1}, f_{\Omega_2}))$ T-periodic trajectories or $(|\chi(A_1)| + |\chi(A_2)| - N(f_{A_1}, F_{A_2}))$ T-periodic solutions, for every $\sigma \in \mathbb{R}$, under the suitable T-periodicity assumptions, as in Theorems 6.4 and 6.9. For $f = -\mathrm{id}$, much more interesting multiplicity results can be obtained (cf. [4]), under suitable restrictions, for anti-periodic trajectories or solutions.

As we could see, the relative Lefschetz number was concerned to the existence of a fixed-point on the closure of the complement, while the relative Nielsen number to the lower estimate of the number of coincidences on the total space. We conclude, therefore, this section by the application of a special relative Nielsen number (introduced for the first time in [34]), estimating from below the number of coincidences just on the complement (see Theorem 5.7 and, in the single-valued case, cf. the surplus number in [35]).

If Ω_1, Ω_2 are ANR-spaces such that $\Omega_2 \subset \Omega_1$ (Ω_2 is assumed again to be closed connected), then

$$(SN(f; \Omega_2) \geq) N(f; \Omega_1 \setminus \Omega_2) = N(f_{\Omega_1}) - E(f_{\Omega_1}, f_{\Omega_2})$$

is the so called *Nielsen number of essential weakly common fixed-point classes* of f_{Ω_1} and f_{Ω_2}, i.e. if there exists a path α from a point x_0 of an essential fixed-point class to a point in Ω_2 so that α is homotopic to $f \circ \alpha$, under a homotopy of the form $(I, 0, 1) \to (\Omega_1, x_0, \Omega_2)$, $I = [0, 1]$; for more details (including the computation of $N(f; \Omega_1 \setminus \Omega_2)$), see [26], [27], [33]–[36].

Replacing the relative Nielsen number (on the total space) by the above Nielsen number on the complement; Theorems 6.12 and 6.13 can be reformulated as follows.

Corollary 6.16. *Under the assumption of Theorem* 6.12*, there exist at least* $|\Lambda(f_{\Omega_1})| - E(f_{\Omega_1}, f_{\Omega_2})$ *trajectories* $(t, y_{(0,t)}(x^*)) \in [0,T] \times \Omega_1 \setminus \Omega_2$ *in* $\varphi_{(0,t)}$*, satisfying* $y_{(0,0)}(x^*) = f(y_{(0,T)}(x^*))$*, for some* (*one or more*) $x^* \in \Omega_1 \setminus \Omega_2$*. The number of essential weakly common fixed-point classes* $E(f_{\Omega_1}, f_{\Omega_2})$ *of* f_{Ω_1} *and* f_{Ω_2} *is defined as above and, for a nilmanifold* Ω_1*,* $\Lambda(f_{\Omega_1})$ *can be computed by means of* (6.11)*.*

Corollary 6.17. *Under the assumption of Theorem* 6.13*, there exist at least* $|\Lambda(f_{A_1})| - E(f_{A_1}, f_{A_2})$ *solutions* $y(t)$ *of* (6.3) *satisfying* $y(0) = f(y(T))$*,* $y(0) \in A_1 \setminus A_2$*, where* $E(f_{A_1}, f_{A_2})$ *stands for the number of essential weakly common fixed-point classes of* f_{A_1} *and* f_{A_2}*, and* $\Lambda(f_{A_1})$ *can be computed, for a nilmanifold* A_1*, by means of* (6.11)*.*

Let us still add two concluding notes. In the single-valued case, Ω_2 need not be connected (cf. [27]). Moreover, if Ω_1 or Ω_2 is particularly the 2-dimensional disk containing a finite number of fixed-points on the boundary $\partial\Omega_1$ or $\partial\Omega_2$, then, under certain special additional restrictions, some further fixed-points can be implied in the interior $\mathrm{int}\Omega_1$ or $\mathrm{int}\Omega_2$ (see e.g. [6], [8]).

If we would have used for our applications (in the single-valued case) continuos flows $\Phi_{(\sigma,t)}$, where t can take also negative values, then, for every $\sigma, t \in \mathbb{R}$,

$$\Phi \simeq \widehat{\Phi} : (A(\sigma), B(\sigma)) \to (A(\sigma+t), B(\sigma+t)),$$

a homotopy is given by

$$\widehat{\Phi}_{(\sigma+(1-s)t,st)} \circ \Phi_{(\sigma+t,-st)} \circ \Phi_{(\sigma,t)} \quad \text{for } s \in [0,1].$$

Nevertheless, the semi-flows seem to be more appropriate for our applications, in the single-valued case, as well.

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Manuscript received May 22, 2000

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