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THE BORSUK–ULAM PROPERTY FOR CYCLIC GROUPS

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Abstract. An orthogonal representation V of a group G is said to have the Borsuk–Ulam property if the existence of an equivariant map $f : S(W) \to S(V)$ from a sphere of representation W into a sphere of representation V implies that $\dim W \leq \dim V$. It is known that a sufficient condition for V to have the Borsuk–Ulam property is the nontriviality of its Euler class $\mathbf{e}(V) \in H^*(BG;\mathcal{R})$. Our purpose is to show that $\mathbf{e}(V) \neq 0$ is also necessary if G is a cyclic group of odd and double odd order. For a finite group G with periodic cohomology an estimate for G-category of a G-space X is also derived.

1. The Euler class of cohomology sphere

Let V be an n-dimensional orthogonal representation of a compact Lie group G. Assume that V is $\mathcal{R}$-orientable, i.e. the vector bundle $V \subset EG \times_G V \to BG$ is orientable over a ring $\mathcal{R}$, or equivalently, the action of G on $H^{n-1}(S(V);\mathcal{R}) \cong H^{n-1}(S(V);\mathbb{Z}) \otimes \mathcal{R}$ is trivial, where $S(V)$ stands for a unit sphere in V (see [14], [15]).

By the Euler class of V over $\mathcal{R}$, denoted $\mathbf{e}(V)$, we call the Euler class of the vector bundle $EG \times_G V \to BG$ in $H^*(BG;\mathcal{R})$. It is then an element of $H^n(BG;\mathcal{R})$.

Definition 1.1. We say that V has the Borsuk–Ulam property if whenever there is a G-equivariant map $f : S(W) \to S(V)$, W an orthogonal representation

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of G, then $\dim W \leq \dim V$. Otherwise, we say that V does not have the Borsuk–Ulam property.

It is shown in [13] that if V is $\mathcal{R}$-orientable and $\mathbf{e}(V) \neq 0$ then V has the Borsuk–Ulam property. We will study the converse problem, that of whether the condition $\mathbf{e}(V) = 0$ implies that V does not have the Borsuk–Ulam property.

Suppose that G is a nontrivial, compact, connected Lie group and put $\mathcal{R} = \mathbb{Q}$. Then

$$\mathbf{e}(V) \neq 0 \Leftrightarrow V^T = \{0\},$$

where $T \subset G$ is the maximal torus of G. Moreover, $\mathbf{e}(V)$ is equal to the multiple of all weights of V (see [10], or [13] for more details). In general, even in the case when G is finite it is difficult to derive $\mathbf{e}(V)$. However, there exists a simple formula for $\mathbf{e}(V)$ if G has periodic cohomology, in particular, if G is a finite cyclic group C_k of order k.

Assume that $G = C_k$, and choose $\mathcal{R} = \mathbb{Z}_k$, the ring of integers modulo k. It is well known (cf. [5], [17]), that $H^i(C_k; \mathbb{Z}_k) = \mathbb{Z}_k$, $0 \leq i < \infty$, and a periodicity is given by multiplication by the element $u = \mathbf{e}(V^1) \in H^2(C_k, \mathbb{Z}_k)$, where V^1 is the standard linear complex representation of C_k given by the embedding $C_k \subset S^1$ (the generator $g \in C_k$ is identified with $\exp(2\pi\sqrt{-1}k^{-1}) \in S^1$). Denote by V^i, $1 \leq i \leq k/2$, the i-th tensor power of V^1 (over $\mathbb{C}$) and by V^0 the 1-dimensional (real) trivial representation of G. If $k = 2m$, then for $i = m = k/2$, we denote by $V^m_{\mathbb{R}}$ a 1-dimensional real representation of C_k given by the epimorphism $C_k \to C_2 \simeq \{-1, 1\} = O(1)$.

We shall use the following

Fact 1.2. *Every real orthogonal representation of C_k is of the form:*

$$\begin{aligned} V &= \bigoplus l_i V^i, && 0 \leq i \leq (k-1)/2, \quad \textit{if } k \textit{ is odd},\\ V &= \bigoplus l_i V^i \oplus l_m V^m_{\mathbb{R}}, && 0 \leq i \leq m-1, \qquad \textit{if } k = 2m. \end{aligned}$$

Moreover, for $k = 2m$, $k \neq 2$, V is orientable if and only if $V = \bigoplus_{i=0}^m l_i V^i$, where $V^m := 2V^m_{\mathbb{R}}$. Every representation $V = l_0 V^0 \oplus l_1 V^1_{\mathbb{R}}$ of C_2 is $\mathbb{Z}_2$-orientable.

Since $\mathbf{e}(V \oplus W) = \mathbf{e}(V) \cdot \mathbf{e}(W)$ and $\mathbf{e}(V^i) = i \cdot \mathbf{e}(V^1) = i \cdot u \in H^2(C_k; \mathbb{Z}_k)$ (cf. [13] and [16]), we get $\mathbf{e}(V) = \prod i^{l_i} \cdot u^r$, for $V = \bigoplus l_i V^i$, $0 \leq i \leq [k/2]$ and $r = \sum l_i$. The last means that $\mathbf{e}(V) \neq 0$ if and only if the integer

$$\mathbf{h}(V) = \prod i^{l_i}, \quad 0 \leq i \leq [k/2],$$

(with the convention $0^0 = 1$, $0^r = 0$ for $r > 0$) is not divisible by k. In particular $\mathbf{e}(V) = 0$ if $l_0 \neq 0$. Our main theorems state that for cyclic groups of odd and double odd order $\mathbf{e}(V)$ is the only obstruction for V to have the Borsuk–Ulam property.

Theorem 1.3. *Let V be $\mathbb{Z}_k$-orientable, orthogonal representation of the group $G = C_k$, with k being an odd number. Then V has the Borsuk–Ulam property if and only if* $\mathbf{e}(V) \neq 0$.

Proof. It is sufficient to show that if $\mathbf{e}(V) = 0$ then there exist an orthogonal representation W satisfying $\dim W > \dim V$, and a G-equivariant map $f : S(W) \to S(V)$.

We assume that $V^G = \{0\}$, otherwise $S(W)$ can be mapped into a point. Let V be of dimension n. Consider the vector bundle $\xi = EG \times_G V$ over BG. We claim that if $\mathbf{e}(V) = 0$ then the sphere bundle $S(\xi) = EG \times_G S(V)$ of ξ restricted to the $(n+1)$-skeleton $BG^{(n+1)}$ of BG has a nonzero section. Indeed, $\pi_i(S(V)) = 0$ if $i < n-1$, and therefore $S(\xi)_{|BG^{(n-1)}}$ admits a section since all obstructions are in zero groups $H^{i+1}(BG; \pi_i(S(V)))$, $i \leq n-2$. Furthermore, by its geometric interpretation $\mathbf{e}(V)$ is the only obstruction to extending such a section over the n-skeleton of BG. This yields that there is a section on $BG^{(n)}$, since $\mathbf{e}(V) = 0$.

Moreover, since V is $\mathbb{Z}_k$-orientable and $k \neq 2$ the dimension of V is even. This implies $H^{n+1}(BG; \pi_n(S(V))) = 0$, because either $n = 2$ and $\pi_2(S^1) = 0$ or $n \geq 4$ and $H^{n+1}(BG; \pi_n(S(V))) = H^{n+1}(BG; \mathbb{Z}_2) = 0$. Thus the section can be extended over $BG^{(n+1)}$. Since G acts freely on EG the sections of the fibration $EG^{(n+1)} \times_G S(V)$ are in one-to-one correspondence with G-mappings from EG^{n+1} to $S(V)$ (cf. [4]). Hence, there is a G-map $f : EG^{(n+1)} \to S(V)$. Put $W = (n/2+1)V^1$ so that, W is a representation of C_k of (real) dimension $n+2$. Let $\phi : S(W) \to EG$ be a G-map into the universal space. Note that EG and $S(V)$ are G-CW complexes, thus ϕ can be replaced, up to a G-homotopy, by a G-cellular map $\overline{\phi} : S(W) \to EG^{(n+1)}$, and the composition $f\overline{\phi}$ gives the required map. The proof is complete. □

Theorem 1.4. *Let $G = C_k$ be a cyclic group of order $k = 2 \cdot \mathrm{odd}$. Let V be an orthogonal, $\mathbb{Z}_k$-orientable representation of G. Then V has the Borsuk–Ulam property if and only if* $\mathbf{e}(V) \neq 0$.

We begin with the following

Proposition 1.5. *Let V be a $\mathbb{Z}_k$-orientable, orthogonal representation of a cyclic group C_k $(k \neq 2)$ such that $\mathbf{h}(V) \equiv 0 \pmod{2k}$. Then V does not have the Borsuk–Ulam property.*

More precisely, for any cyclic group $G = C_k$ and $V = \bigoplus m_i V^i$ there exists a G-equivariant map $f : S((r+1)V^1) \to S(V)$, for $r = \sum m_i$ provided $\prod i^{m_i}$ is divisible by $2k$.

Proof. Denote by W^1 the standard linear (complex) representation of C_{2k} and by W^i its i-th tensor power (over $\mathbb{C}$). Let W be an orthogonal representation of C_{2k} defined as a direct sum $\bigoplus m_i W^i$ where m_i are taken from the splitting of V. Obviously, $\mathbf{h}(W) = \mathbf{h}(V)$ and $\mathbf{e}(W) = 0$, since $2k$ divides $\mathbf{h}(W)$. Consequently, there exists a C_{2k}-equivariant map

$$\widetilde{f} : S(rW^1) * C_{2k} \to S(W), \quad r = \sum m_i,$$

where $A * B$ denotes the join of A and B (cf. [16]). For a fixed generator g of C_{2k} we denote by $\widetilde{f}_s$ the restriction of $\widetilde{f}$ to the space $S(rW^1) * \{g^s, g^{s+1}\}$ which is homeomorphic to a $2r$-dimensional sphere. Since W is $\mathbb{Z}_k$-orientable, $\widetilde{f}_s$ represents the same element in the homotopy group $\pi_{2r}(S^{2r-1}) \cong \mathbb{Z}_2$ for every $s = 0, \dots, 2k-1$. Now, the map

$$\overline{f} : S(rW^1) * \{g^0, g^2, \dots, g^{2k-2}\} \to S(W)$$

given by the restriction of $\widetilde{f}$ can be considered as a C_k-equivariant map

$$h : S(rV^1) * C_k \to S(V)$$

with an action of C_k induced by the standard inclusion $C_k \subset C_{2k}$. Let γ be a fixed generator of C_k and $\overline{f}_s$ be the restriction of $\overline{f}$ to the space $S(rV^1) * \{\gamma^s, \gamma^{s+1}\}$. Note, that the map $\overline{f}_s$ represents the sum of homotopy classes of maps $\widetilde{f}_{2s}$ and $\widetilde{f}_{2s+1}$ in the group $\pi_{2r}(S^{2r-1})$. Since $\pi_{2r}(S^{2r-1}) = \mathbb{Z}_2$ and $\widetilde{f}_{2s}$ is homotopic to $\widetilde{f}_{2s+1}$, $\overline{f}_s$ is homotopically trivial. This gives us an extension of the map $\overline{f}$ to a C_k-equivariant map

$$f : S((r+1)V^1) \to S(V),$$

and the proof is complete. □

Corollary 1.6. *Let V be a $\mathbb{Z}_k$-orientable, orthogonal representation of C_k, $k \neq 2$. If the Euler class $\mathbf{e}(V) = 0$ then:*

(a) *the representation $V \oplus V$ does not have the Borsuk–Ulam property,*
(b) *the representation $V \oplus V^{2t}$ does not have the Borsuk–Ulam property for $t = 0, 1, 2, \dots$.*

Corollary 1.7. *If $V = \bigoplus m_i V^i$ is any $\mathbb{Z}_k$-orientable representation of C_k $(k \neq 2)$ with the Euler class $\mathbf{e}(V)$ equal to zero then the representation $V' = m_{i_0} V^{2i_0} \bigoplus_{i \neq i_0} m_i V^i$, with $m_{i_0} \neq 0$, does not have the Borsuk–Ulam property.*

Theorem 1.4 is a direct consequence of the following

Proposition 1.8. *Let $V = \bigoplus m_i V^i$ be a representation of a cyclic group C_k, with $k = 2^d \cdot \mathrm{odd}$. Assume there is i_0 divisible by 2^d with $m_{i_0} \neq 0$. Then $\mathbf{e}(V) = 0$ implies V does not have the Borsuk–Ulam property.*

Proof. If $\mathbf{h}(V)$ is divisible by 2^{d+1} then the above result follows directly from Proposition 1.5. Assume then, that $\mathbf{h}(V)$ is divisible by 2^d and is not divisible by 2^{d+1}. It follows that $m_{i_0} = 1$. Obviously, V^{i_0} is isomorphic to V^{i_0+k}

and that is why the summand V^{i_0} can be replaced by V^{i_0+k} in the direct sum $V = \bigoplus m_i V^i$.

Since $k = 2^d \cdot \text{odd}$ and i_0 is also of the form $2^d \cdot \text{odd}$, $i_0 + k$ is divisible by 2^{d+1}. Now again by Proposition 1.5 we have the desired result. $\square$

REMARK 1.9. It has been proved in [3] that for $G = C_p \times C_q = C_{pq}$, where p, q distinct primes, and any orthogonal representation V of G with $\dim V^{C_p} \geq 1$ and $\dim V^{C_q} \geq 1$ there exists a G-map $f : S(V) \to S(V^{C_p} \oplus V^{C_q})$. Of course, that result allows to construct a pair of representations U, W of $G = C_k$, $k = pq$, with $U^G = W^G = \{0\}$ and $\dim U > \dim W$ such that there is a G-map $f : S(U) \to S(W)$. Nevertheless, our result gives a necessary and sufficient condition.

PROBLEM 1.10. For which group G and representation V the condition $\mathbf{e}(V) = 0$ implies that V does not have the Borsuk–Ulam property?

2. An estimate of equivariant category

Closely related to the problem of equivariant mapping into spheres is the computation of equivariant category, $\operatorname{cat}_G(X)$, of a G-space X (cf. [6]–[8], [12]). This is by definition the smallest natural number m (or ∞) such that there exists a covering of X consisting of m G-invariant open subsets $\mathcal{U}_1, \ldots, \mathcal{U}_m$ each of which can be equivariantly deformed to an orbit Gx_i inside X.

Let X be a G-space. We say that an orbit type $(G/H) = (G/G_x)$, $x \in X$, is minimal in X if there is no $y \in X$ such that $H \subset G_y$ and $H \neq G_y$. By $\alpha = \alpha(X)$ we denote the number of connected components of $\bigcup X^{(H)}/G = \bigcup X_{(H)}/G$, where (H) runs over all minimal orbit types of X (cf. [12, Definition 1.2]).

PROPOSITION 2.1. *Let G be a finite group, $\{e\} \neq H \subset G$ its subgroup. Suppose that X is a connected G-space and $\alpha = \alpha(X)$ is taken with respect to the action of H on X. Then*

$$\operatorname{cat}_G(X) \geq \frac{1}{\alpha} \operatorname{cat}_H(X).$$

PROOF. Let $\{\mathcal{U}_i\}_1^m$, $m = \operatorname{cat}_G(X)$, be a G-covering of X with $\operatorname{cat}_G(\overline{\mathcal{U}}_i, X) = 1$. We shall have established the proposition if we prove that

$$\operatorname{cat}_H(\overline{\mathcal{U}}_i, X) = \operatorname{cat}_H(G/G_{x_i}, X) \leq \alpha.$$

If X is a G-ANR then this follows from Theorem 1.10 of [12].

In the general case, since G is a finite group

$$Gx_i = \bigcup_{j=1}^{n_i} Hx_{i,j} \simeq \bigcup_{j=1}^{n_i} H/H_{i,j}$$

as an H-space. Let us choose a point x^*_β, $1 \le \beta \le \alpha$, in every connected component of $X_{(H)}/G$. By Lemma 1.11 of [12], the inclusion $\iota_{i,j} : H/H_{i,j} \subset X$ is H-homotopic to a H-equivariant map $\phi_{i,j} : H/H_{i,j} \to H/H_{x_\beta}$, for some $1 \le \beta(i,j) \le \alpha$, which proves the required inequality and consequently Proposition 2.1 □

Lemma 2.2. *Let p be a prime number. Suppose that X is a $(n-1)$-dimensional cohomology sphere over $\mathbb{Z}$, $n \ge 2$. Assume that a p-group G acts on X without fixed points. Then for every minimal orbit type (H) the set $X^{(H)}/G$ is connected.*

Proof. Since (H) is minimal $X^{(H)}/G = X_{(H)}/G = X_H/N(H)$ (cf. [4]). It is sufficient to prove that X_H is connected. From P. Smith theorem (cf. [4]) it follows that

$$X^H \simeq_{\mathbb{Z}_p} S^{n_1-1}, \quad n_1 \le n.$$

Let $\{e\} \subset H_1 \subset \ldots \subset H \subset \overline{H} \subset \ldots \subset G$ be a resolving tower of G with factors isomorphic to C_p. Since H is minimal, $X^G = \emptyset$ and $H \ne \overline{H}$, we obtain

$$\emptyset = X^{\overline{H}} = (X^H)^{N(\overline{H},H)} = (X^H)^{C_p}.$$

This leads to a contradiction if $X_H = X^H \simeq_{\mathbb{Z}_p} S^0$ and p is odd. If $p = 2$, then $N(\overline{H},H) = \mathbb{Z}_2$ has to permute two connected components of $X^H = X_H \simeq_{\mathbb{Z}_2} S^0$. Indeed, any such component $X^H_0 \simeq_{\mathbb{Z}_2} *$, and $(X^H_0)^{C_2} \simeq_{\mathbb{Z}_2} *$ by the Smith theory, which shows that $X^{\overline{H}}$ is nonempty if $N(\overline{H},H)$ preserves a component X^H_0 of X^H. This proves that $X^{(H)}/G$ is connected. □

The following proposition is an immediate consequence of Proposition 2.1 and Lemma 2.2.

Proposition 2.3. *Let G_p be a p-subgroup of a finite group G and X be a G-space which is an $(n-1)$-cohomology sphere over $\mathbb{Z}_p$. Let α_p be a number of distinct minimal orbit types of the action of G_p on X. Suppose that $X^{G_p} = \emptyset$. Then*

$$\operatorname{cat}_G(X) \ge \frac{1}{\alpha_p}\operatorname{cat}_{G_p}(X).$$

In particular, $\operatorname{cat}_G(X) \ge \operatorname{cat}_H(X)$ if H is a cyclic p-group.

Combining the above with the main result of [2] we get the following generalization of Bartsch's result.

Theorem 2.4. *Let G be a finite group with periodic cohomology such that its 2-Sylow group is cyclic. Suppose that $X = S(V)$ is the sphere of a $\mathbb{Z}$-orientable orthogonal representation of G of dimension n. Assume that $X^H = \emptyset$ for some p-subgroup $H \subset G$. Then*

$$\operatorname{cat}_G(X) \ge n/p^{r-1} \quad \textit{where } |H| = p^r.$$

In particular, if $\mathbf{e}(V) \neq 0$ *in* $H^n(G;\mathbb{Z}) \cong \mathbb{Z}/|G|\mathbb{Z}$*, then*

$$\mathrm{cat}_G(X) \geq n/p^{r-1}$$

for every divisor p *of* $|G|$, $p^r = |G_p|$, *such that* $\mathbf{e}(V) \not\equiv 0 (\mathrm{mod}\, p^r)$.

Proof. By Theorem IV.9.7 of [5], G has periodic cohomology if and only if every its Sylow p-subgroup is cyclic (if p is odd), or cyclic and generalized quaternionic if $p = 2$. For $H = C^{p^r}$ the condition $V^H = \{0\}$ implies $\mathrm{cat}_H(X) \geq n/p^{r-1}$, by the main result of [2]. By Lemma 2.3 $\alpha_H = 1$, and consequently $\mathrm{cat}_G(X) \geq n/p^{r-1}$ as follows from Proposition 2.3 . This shows the first part of statement.

If $\mathbf{e}(V) \not\equiv 0 \ (\mathrm{mod}\, p^r)$ then $\mathbf{e}(V)_p = \mathrm{res}^G_H(\mathbf{e}(V))$, with $H = G_p$ is also different from 0 (cf. [5], [9], [17]), and consequently $V^H = \{0\}$, which reduces it to the first part of Theorem. □

Problem 2.5. Does $\mathbf{e}(V) \neq 0$ (in $H^*(G;\mathbb{Z})$) imply that $\mathrm{cat}_G(S(V)) \geq \dim V$ for an orthogonal representation V of G?

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