

STABILITY OF TRAVELLING-WAVE SOLUTIONS FOR REACTION-DIFFUSION-CONVECTION SYSTEMS

ELAINE C. M. CROOKS

ABSTRACT. We are concerned with the asymptotic behaviour of classical solutions of systems of the form

$$(1) \quad \begin{cases} u_t = Au_{xx} + f(u, u_x) & \text{for } x \in \mathbb{R}, t > 0, u(x, t) \in \mathbb{R}^N, \\ u(x, 0) = \varphi(x), \end{cases}$$

where A is a positive-definite diagonal matrix and f is a “bistable” nonlinearity satisfying conditions which guarantee the existence of a comparison principle for (1). Suppose that (1) has a travelling-front solution w with velocity c , that connects two stable equilibria of f . (There are hypotheses on f under which such a front is known to exist [5].) We show that if φ is bounded, uniformly continuously differentiable and such that $\|w(x) - \varphi(x)\|$ is small when $|x|$ is large, then there exists $\chi \in \mathbb{R}$ such that

$$(2) \quad \|u(\cdot, t) - w(\cdot + \chi - ct)\|_{BUC^1} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Our approach extends an idea developed by Roquejoffre, Terman and Volpert in the convectionless case, where f is independent of u_x . First φ is assumed to be increasing in x , and (2) proved via a homotopy argument. Then we deduce the result for arbitrary φ by showing that there is an increasing function in the ω -limit set of φ .

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1. Introduction

This paper is concerned with the asymptotic behaviour of classical solutions of the system

$$(3) \quad u_t = Au_{xx} + f(u, u_x), \quad x \in \mathbb{R}, t > 0, u(x, t) \in \mathbb{R}^N,$$

$$(4) \quad u(x, 0) = \varphi(x), \quad x \in \mathbb{R},$$

under the following hypotheses:

- (a) A is a positive-definite diagonal $N \times N$ matrix, $f : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a continuously-differentiable function such that
- (f1) $f_i(q, p) = \tilde{f}_i(q_1, \dots, q_N, p_i)$ (the i -th component of f does not depend on p_j for $j \neq i$),
- (f2) $\frac{\partial f_i}{\partial q_j}(q, p) > 0$, $i \neq j$, $i, j = 1, \dots, N$, $(q, p) \in \mathbb{R}^N \times \mathbb{R}^N$,
- (f3) $f(E^-, 0) = f(E^+, 0) = 0$, where $E^- < E^+$, $E^\pm \in \mathbb{R}^N$ and all the eigenvalues of $d_q f[E^\pm, 0]$ lie in the open left-half complex plane (bistability condition),
- (f4) there exists $\gamma \in (1, 2)$ and an increasing function $\mu : [0, \infty) \rightarrow [0, \infty)$ such that for each $p, q \in \mathbb{R}^N$,

$$\|f(q, p)\| \leq \mu(\|q\|)(1 + \|p\|^\gamma) \quad (\|\cdot\| \text{denotes a norm on } \mathbb{R}^N),$$

and

- (TW) there exists a monotone travelling-wave solution $w(x - ct)$ of (3) such that $w(x) \rightarrow E^\pm$ as $x \rightarrow \pm\infty$, and $w'(x) > 0$ is bounded independently of x . (In fact, these properties of w together with the above hypotheses on f ensure that $w'(x) \rightarrow 0$ at an exponential rate as $|x| \rightarrow \infty$. See the remark following the proof of Lemma 2.5.)

Note that [5] proves the existence of a wave w satisfying (TW) under hypotheses similar, though not identical, to (a), (f1)–(f4), together with an assumption on the nonexistence of stable equilibria of f between $(E^-, 0)$ and $(E^+, 0)$. Such equilibria could prevent the existence of a front connecting E^- to E^+ (see [7]). For the *scalar* bistable equation (3), in the convectionless case when $f \in \mathbb{R}$ and is independent of u_x , convergence to a travelling-front solution w from initial data φ is comprehensively treated in [7]. Stability of fronts for bistable convectionless systems is developed in [14] and [13]. Here we extend this work to nonlinearities dependent on u_x .

Throughout, $\mathbf{e} = (1, \dots, 1)$ and $d_q f[q, p]$, $d_p f[q, p]$ denote the partial Fréchet derivatives of f at $(q, p) \in \mathbb{R}^N \times \mathbb{R}^N$ with respect to the first and second arguments of f , respectively. If $q^\pm \in \mathbb{R}^N$, then $q^- < (\leq) q^+$ if $q_i^- < (\leq) q_i^+$ for each $i \in \{1, \dots, N\}$; $[q^-, q^+]$ denotes the set of $q \in \mathbb{R}^N$ such that $q^- \leq q \leq q^+$. For Υ a subset of a real or complex vector space, $k \in \mathbb{N} \cup \{\infty\}$, $\mathfrak{C}^k(\mathbb{R}, \Upsilon) = BUC^k(\mathbb{R}, \Upsilon)$,

the space of functions $g : \mathbb{R} \rightarrow \Upsilon$ such that g and the derivatives of g of order less than or equal to k are bounded and uniformly continuous on \mathbb{R} . For brevity, we write $\mathfrak{C}^k = \mathfrak{C}^k(\mathbb{R}, \mathbb{R}^N)$ and $\tilde{\mathfrak{C}}^k = \mathfrak{C}^k(\mathbb{R}, \mathbb{C}^N)$.

Known results yield, under hypotheses (a), (f1)–(f4), that there exists $\varepsilon > 0$ such that system (3)–(4) with initial data $\varphi \in \mathfrak{C}^1(\mathbb{R}, [E^- - \varepsilon \mathbf{e}, E^+ + \varepsilon \mathbf{e}])$ has a unique classical solution u^φ that exists for all time and depends continuously in \mathfrak{C}^1 on the initial data φ . See the Appendix for references. We will prove that if $\varphi \in \mathfrak{C}^1$ is such that $\|w(x) - \varphi(x)\|$ is small when $|x|$ is large, then u^φ converges to a shift of the travelling wave w , in the sense that there exists $\chi \in \mathbb{R}$, depending on φ , such that

$$(5) \quad \|u^\varphi(\cdot, t) - w(\cdot + \chi - ct)\|_{\mathfrak{C}^1} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Let $v(x, t) = u(x + ct, t)$, where u is a solution of (3). Then

$$(6) \quad v_t = Av_{xx} + cv_x + f(v, v_x).$$

Note that w is a stationary solution of (6) and that $v(x, 0) = u(x, 0)$ for all $x \in \mathbb{R}$. We seek $\chi \in \mathbb{R}$ such that

$$(7) \quad \|v^\varphi(\cdot, t) - w(\cdot + \chi)\|_{\mathfrak{C}^1} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

(v^φ will denote the unique classical solution of (6) with initial data $\varphi \in \mathfrak{C}^1(\mathbb{R}, [E^- - \varepsilon \mathbf{e}, E^+ + \varepsilon \mathbf{e}])$ throughout.)

To prove (7), it will first be shown, in Theorem 3.1, that w is “locally” stable in \mathfrak{C}^1 ; that is, given initial data φ which is a sufficiently small \mathfrak{C}^1 -perturbation of w , the corresponding solution v^φ of (6) converges in \mathfrak{C}^1 to a translate of w as $t \rightarrow \infty$. This is a consequence of the fact that the spectrum of the linearisation of (6) about w is in a sector in the open left-half plane, except for a simple eigenvalue at zero caused by the translation invariance of (6). For $g \in \mathfrak{C}^2$ define

$$(8) \quad \begin{aligned} \mathcal{L}g(x) &= Ag''(x) + \{c + d_p f[w(x), w'(x)]\}g'(x) + d_q f[w(x), w'(x)]g(x) \\ &= Ag''(x) + C(x)g'(x) + B(x)g(x), \end{aligned}$$

say; $B, C : \mathbb{R} \rightarrow M^{N \times N}$ are uniformly continuous $N \times N$ -matrix-valued functions of x . Consider \mathcal{L} as an operator acting in \mathfrak{C} , with domain \mathfrak{C}^2 . We abuse notation slightly by also using the symbol \mathcal{L} for the complexification of \mathcal{L} when appropriate. The spectrum of \mathcal{L} is analysed in Section 2. Section 3 is devoted to proving local stability of w in \mathfrak{C}^1 , following a method in [8].

The main convergence result, Theorem 5.4, is proved in two steps. First, in Section 4, $\varphi \in \mathfrak{C}^1$ is assumed to be increasing, and convergent to E^\pm at $\pm\infty$ respectively. Our approach derives from that of [14]. A function φ^* is constructed from φ and the wave w so that the solution v^{φ^*} of (6) corresponding to initial data φ^* satisfies (7). The corresponding result for v^φ is then deduced

using a homotopy argument. Section 5 concludes the paper by showing that for more general initial data φ , close to w at infinity, there is an increasing function in the ω -limit set of φ . This last step is motivated by [13]. Note that the main convergence Theorem 5.4 implies uniqueness of travelling-front solutions of (3) within a certain class (see Corollary 5.5 for details).

In an Appendix, we state some useful known results for (6) – namely a comparison principle, local/global existence theorems and *a priori* bounds. Some wave-dependent sub- and super-solutions, useful in the stability analysis of w , are also given. This material will often be referred to in the body of the paper.

2. Properties of \mathcal{L}

Let Y, W be complex Banach spaces and let $L(Y, W)$ denote the space of bounded linear operators from Y into W . A linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Y \rightarrow Y$ is said to be *sectorial in Y* if it is a closed densely-defined operator such that for some $\omega \in \mathbb{R}, \theta \in (\pi/2, \pi), M > 0$,

$$\Sigma = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\} \subset \rho(\mathcal{A}), \quad \text{the resolvent set of } \mathcal{A},$$

and

$$\|(\lambda I - \mathcal{A})^{-1}\|_{L(Y, Y)} \leq \frac{M}{|\lambda - \omega|} \quad \text{for all } \lambda \in \Sigma,$$

(see [11, p. 33]). If \mathcal{A} is sectorial in Y , then \mathcal{A} is the infinitesimal generator of an analytic semigroup $e^{t\mathcal{A}}$ in the Banach space Y .

LEMMA 2.1. *The operator $\mathcal{L} : \mathfrak{C}^2 \subset \mathfrak{C} \rightarrow \mathfrak{C}$ defined in (8) is sectorial in \mathfrak{C} .*

PROOF. In (8), the matrices A and $C(\cdot)$ are diagonal. It follows from the scalar-valued-equation analysis of [11, Corollary 3.1.9(ii)] that the operator $\mathcal{T} : \mathfrak{C}^2 \subset \mathfrak{C} \rightarrow \mathfrak{C}$ defined by $\mathcal{T}g = Ag'' + C(\cdot)g'$ is sectorial.

Define $\mathcal{S} : \mathfrak{C} \rightarrow \mathfrak{C}$ by $\mathcal{S}g = B(\cdot)g$. Clearly $\mathcal{S} \in L(\mathfrak{C}, \mathfrak{C})$. So [11, Proposition 2.4.1] yields that $\mathcal{L} = \mathcal{T} + \mathcal{S}, \mathcal{L} : \mathfrak{C}^2 \subset \mathfrak{C} \rightarrow \mathfrak{C}$ is sectorial. \square

For $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Y \rightarrow Y$ and Y^\sharp be a Banach space with $\mathcal{D}(\mathcal{A}) \subset Y^\sharp$ and $Y^\sharp \hookrightarrow Y$, where \hookrightarrow denotes continuous embedding, let the *part* of \mathcal{A} in Y^\sharp [11] be \mathcal{A}^\sharp , where

$$\mathcal{D}(\mathcal{A}^\sharp) = \{g \in \mathcal{D}(\mathcal{A}) : Ag \in Y^\sharp\} \subset Y^\sharp \quad \text{and} \quad \mathcal{A}^\sharp g = Ag \quad \text{for each } g \in \mathcal{D}(\mathcal{A}^\sharp).$$

LEMMA 2.2. *The part of \mathcal{L} in \mathfrak{C}^1 is sectorial in \mathfrak{C}^1 .*

PROOF. Define $\mathcal{M} : \mathfrak{C}^2 \subset \mathfrak{C} \rightarrow \mathfrak{C}$ by $\mathcal{M}g = Ag''$. The proof of Lemma 2.1 shows that both \mathcal{M} and \mathcal{L} are sectorial in \mathfrak{C} . Let $\mu_0 \in \mathbb{R}$ be such that if $\mu \in \mathbb{C}$ and $\text{Real } \mu \geq \mu_0$, then given $f \in \mathfrak{C}$, $(\mathcal{L} - \mu I)g = f$ and $(\mathcal{M} - \mu I)h = \tilde{f}$ are solvable for g and h , respectively. Then, keeping in mind that functions in $\tilde{\mathfrak{C}}$ are

vector-valued, an argument similar to that in the proof of [11, Proposition 3.1.18] yields the existence of $K > 0$, independent of $\mu \in \mathbb{C}$ with $\text{Real } \mu \geq \mu_0$, such that

$$\|\mu(\mu I - \mathcal{L})^{-1}\|_{L(\tilde{\mathcal{C}}^1, \tilde{\mathcal{C}}^1)} < K \quad \text{if } \text{Real } \mu \geq \mu_0.$$

The result follows from [11, Proposition 2.1.11]. \square

We turn now to the spectral analysis of \mathcal{L} . Denote the spectrum of \mathcal{L} by $\sigma(\mathcal{L})$ and the essential spectrum by $\sigma_{\text{ess}}(\mathcal{L})$. (Here, as in [8], the essential spectrum of \mathcal{L} is the complement, in $\sigma(\mathcal{L})$, of the set of those eigenvalues of finite (algebraic) multiplicity which are isolated points of $\sigma(\mathcal{L})$.) Of crucial importance is the following lemma concerning the eigenvalues of the ‘‘asymptotic form of \mathcal{L} at infinity’’. It makes critical use of the bistability condition (f3). We define

$$\begin{aligned} C^\pm &= \lim_{x \rightarrow \pm\infty} C(x) = cI + d_p f[E^\pm, 0] \quad \text{and} \\ B^\pm &= \lim_{x \rightarrow \pm\infty} B(x) = d_q f[E^\pm, 0]. \end{aligned}$$

LEMMA 2.3. *Suppose that there exist $\tau \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^N$ such that*

$$(10) \quad (-\tau^2 A + i\tau C^+ + B^+)z = \lambda z.$$

Then $\text{Real } \lambda < 0$. The same conclusion holds if C^+, B^+ are replaced by C^-, B^- in (10).

PROOF. By condition (f3), all the eigenvalues of B^\pm lie in the open left-half complex plane. By condition (f1), C^\pm are diagonal and, by condition (f2), B^\pm each have positive off-diagonal elements. So the result follows immediately from [14, Chapter 5, Lemma 4.1]. \square

LEMMA 2.4. *$\sigma_{\text{ess}}(\mathcal{L}) \neq \emptyset$, and there exists $\beta > 0$ such that if $\lambda \in \sigma_{\text{ess}}(\mathcal{L})$ then $\text{Real } \lambda < -\beta$.*

PROOF. Let

$$(11) \quad S^\pm = \{\lambda \in \mathbb{C} : \det(-\tau^2 A + i\tau C^\pm + B^\pm - \lambda I) = 0 \text{ for some } \tau \in \mathbb{R}\}.$$

Then Lemma 2.3 shows that

$$\lambda \in S^+ \cup S^- \Rightarrow \text{Real } \lambda < 0.$$

[8, Chapter 5, Theorem A.2] yields that S^\pm each consists of a finite number of algebraic curves parametrised by a real number σ , which are asymptotically parabolic: $\lambda(\sigma) = -\sigma^2 \alpha + O(\sigma)$ as $\sigma \rightarrow \infty$, where α is on the diagonal of A .

¹An eigenvalue λ_0 which is an isolated point of the spectrum is said to have finite (algebraic) multiplicity if $\mathcal{P}\mathcal{E}$ is finite-dimensional, where \mathcal{P} is the linear operator defined by $\mathcal{P} = 1/(2/\pi i) \int_{\partial\Omega} (\xi I - \mathcal{L})^{-1} d\xi$, Ω being a ball in \mathbb{C} , centre λ_0 , such that $\sigma(\mathcal{L}) \cap \bar{\Omega} = \{\lambda_0\}$ ([9]).

Moreover, $S^+ \cup S^- \subset \sigma_{\text{ess}}(\mathcal{L})$ and $\sigma_{\text{ess}}(\mathcal{L}) \subset \Lambda$, where $\mathbb{C} \setminus \Lambda$ is the component of $\mathbb{C} \setminus (S^+ \cup S^-)$ which contains the right-half plane.

Since S^\pm are contained in the open left-half plane, Λ is also. Moreover, S^\pm each consist of a finite number of algebraic curves parametrised by σ , the real parts of which tend to $-\infty$ as $\sigma \rightarrow \pm\infty$. Whence Λ is bounded away from the imaginary axis. The result follows. \square

We next show, using Lemma 2.3, that the bistability condition (f3) implies that bounded solutions of certain equations must decay at infinity.

LEMMA 2.5. *Suppose that there exist $\lambda \in \mathbb{C}$, $\text{Re } \lambda \geq 0$ and $g \in \tilde{\mathfrak{C}}^2$ such that $\mathcal{L}g = \lambda g + \psi_0$, where $\psi_0 \in \mathfrak{C}$ is such that $\psi_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Then $\|g(x)\| \rightarrow 0$ as $|x| \rightarrow \infty$. If $\psi_0 \equiv 0$, then there exist $M, \omega > 0$ such that $\|g(x)\| \leq Me^{-\omega|x|}$ for all $x \in \mathbb{R}$.*

PROOF. Define

$$\hat{h} = \begin{pmatrix} g \\ g' \end{pmatrix}, \quad M^+ = \begin{pmatrix} 0 & I \\ -A^{-1}\{B^+ - \lambda I\} & -A^{-1}C^+ \end{pmatrix},$$

$$\hat{r}(x) = \hat{H}(x)\hat{h}(x) + \begin{pmatrix} 0 \\ \psi_0(x) \end{pmatrix},$$

where

$$\hat{H}(x) = \begin{pmatrix} 0 & 0 \\ -A^{-1}\{B(x) - B^+\} & -A^{-1}\{C(x) - C^+\} \end{pmatrix}.$$

Then $\hat{h}'(x) = M^+\hat{h}(x) + \hat{r}(x)$, $x \in \mathbb{R}$, where \hat{h} is bounded on \mathbb{R} , and $\hat{r}(x) \rightarrow 0$ as $x \rightarrow \infty$. By Lemma 2.3, M^+ has no purely imaginary eigenvalues. So, as in the proof of [4, Chapter 13, Theorem 4.1], there exist $K, \alpha, \sigma > 0$, a real nonsingular matrix $P \in M^{2N \times 2N}$ and operators $U_1(t), U_2(t)$ such that

$$(12) \quad \|U_1(x)\| \leq Ke^{-\alpha x}, \quad x \geq 0 \quad \text{and} \quad \|U_2(x)\| \leq Ke^{\sigma x}, \quad x \leq 0,$$

and $h = P\hat{h}$, $r = P\hat{r}$ satisfy

$$h(x) = U_1(x)h(0) + U_2(x)k + \int_0^x U_1(x-s)r(s) ds - \int_x^\infty U_2(x-s)r(s) ds$$

where

$$k = h(0) + \int_0^\infty U_2(-s)r(s) ds.$$

Estimates (12) together with the facts that \hat{h} is bounded and $\hat{r} \rightarrow 0$ as $x \rightarrow \infty$ yield that $h(x) \rightarrow 0$ as $x \rightarrow \infty$. The exponential decay in the case $\psi_0 \equiv 0$ follows from the proof of [4, Theorem 4.1]. \square

REMARK. Clearly, due to translation invariance, $\mathcal{L}w' = 0$, where w' is the derivative of the travelling wave w . By hypothesis (TW), w' is bounded on \mathbb{R} , so Lemma 2.5 yields that w' decays exponentially to zero at $\pm\infty$. Also, by (TW),

$w'(x) > 0$ for all $x \in \mathbb{R}$. Thus $\mathcal{L}u = 0$ has a positive solution which decays exponentially to zero at infinity. Further, Lemma 2.4 shows that zero is not in the essential spectrum of \mathcal{L} , so it must be an isolated point of the spectrum and an eigenvalue of finite multiplicity.

LEMMA 2.6.

(i) For $\lambda \in \mathbb{C} \setminus \{0\}$ with $\text{Real } \lambda \geq 0$, there are no non-zero solutions of the equation

$$(13) \quad \mathcal{L}g = \lambda g, \quad g \in \tilde{\mathfrak{C}}^2.$$

(ii) Let $g \in \tilde{\mathfrak{C}}^2$ be a solution of $\mathcal{L}g = 0$. Then there exists $k \in \mathbb{R}$ such that $g = kw'$.

PROOF. We aim to apply [14, Chapter 4, Theorem 5.1]. For this, note that (f2) and (f3) imply that the matrix is irreducible in the functional sense (defined in [14]); this follows from (f2) alone when $N \geq 2$. Now let $\lambda \in \mathbb{C}$, $\text{Real } \lambda \geq 0$, and suppose that $g \in \tilde{\mathfrak{C}}^2$ satisfies $\mathcal{L}g = \lambda g$. That $\|g(x)\| \rightarrow 0$ as $|x| \rightarrow \infty$ follows from Lemma 2.5 with $\psi_0 \equiv 0$. The remark preceding this theorem together with [14, Chapter 4, Theorem 5.1, parts (1) and (2)] then yield (i) and (ii). \square

PROPOSITION 2.7. *There exists $\gamma > 0$ such that if $\lambda \in \mathbb{C}$ belongs to $\sigma(\mathcal{L}) \setminus \{0\}$, then $\text{Real } \lambda < -\gamma$.*

PROOF. Lemma 2.4 and Lemma 2.6(i) show that any non-zero point of $\sigma(\mathcal{L})$ lies in the open left-half complex plane. If there is a sequence $\{\lambda_n\} \subset \sigma(\mathcal{L}) \setminus \{0\}$ such that $\text{Real } \lambda_n \uparrow 0$ as $n \rightarrow \infty$, then by Lemma 2.1, $\{\text{Imag } \lambda_n\}$ is bounded. Whence there is a subsequence $\{\lambda_k\}$ and $\mu \in \sigma(\mathcal{L})$ (a closed set), $\text{Real } \mu = 0$, such that $\lambda_k \rightarrow \mu$ as $k \rightarrow \infty$. But this contradicts Lemma 2.4. \square

Lemma 2.6(ii) shows that the nullspace of \mathcal{L} is one-dimensional. We need additional information to exploit this. Recall that zero is an isolated eigenvalue of \mathcal{L} . Let Ω denote a ball in \mathbb{C} with centre zero such that $\sigma(\mathcal{L}) \cap \bar{\Omega} = \{0\}$. Then for $\lambda \in \partial\Omega$, $(\lambda I - \mathcal{L})^{-1} : \mathfrak{C} \rightarrow \mathfrak{C}$ is a bounded linear operator; a bounded linear operator \mathcal{P} is defined by

$$(14) \quad \mathcal{P} = \frac{1}{2\pi i} \int_{\partial\Omega} (\xi I - \mathcal{L})^{-1} d\xi,$$

(see [9] or [11]). Let $X = \mathfrak{C}$, $X_1 = \mathcal{P}X$ and $X_2 = (I - \mathcal{P})X$. [8, Chapter 1, Theorem 1.5.2] and [11, Proposition A.1.2] show that \mathcal{P} is a projection, $X = X_1 \oplus X_2$ and $\mathcal{P}X$ is a subset of the domain of \mathcal{L}^n for each n . Moreover, if \mathcal{L}_j is the restriction of \mathcal{L} to $X_j \cap \mathfrak{C}^2$, then

$$\begin{aligned} \mathcal{L}_1 : X_1 \rightarrow X_1 & \text{ is bounded, } \sigma(\mathcal{L}_1) = \{0\} \quad \text{and} \\ \mathcal{L}_2 : X_2 \cap \mathfrak{C}^2 \subset X_2 \rightarrow X_2, & \quad \sigma(\mathcal{L}_2) = \sigma(\mathcal{L}) \setminus \{0\} \quad (\neq \emptyset, \text{ by Lemma 2.4}). \end{aligned}$$

Note that since $\mathcal{P}, I - \mathcal{P}$ are bounded operators by definition, X_1 and X_2 are closed subspaces of X .

LEMMA 2.8. $X_1 = \text{span}\{w'\}$ and there exists $w^* \in X^*$ such that

$$(15) \quad \mathcal{P}g = w^*(g)w' \quad \text{for each } g \in X, \quad w^*(w') = 1.$$

PROOF. [11, Proposition A.2.2] shows that $\ker \mathcal{L} \subset X_1$. Since $0 \notin \sigma_{\text{ess}}(\mathcal{L})$, X_1 is finite-dimensional (see the footnote following the definition of $\sigma_{\text{ess}}(\mathcal{L})$). So $\sigma(\mathcal{L}_1)$ consists entirely of eigenvalues, the number of which, counted according to algebraic multiplicity, equals the dimension of X_1 . It is shown in [14, Chapter 4, Proof of Theorem 5.1(3)] that $\text{Range } \mathcal{L} \cap \text{span}\{w'\} = 0$. Thus zero is an eigenvalue of \mathcal{L}_1 of multiplicity one, whence $\ker \mathcal{L} = X_1$. Since \mathcal{P} is a bounded projection, the existence of w^* as in the statement of the lemma follows. \square

We will need two estimates on the behaviour of \mathcal{L}_2 . Define $\gamma_0 = -\sup\{\text{Real } z : z \in \sigma(\mathcal{L}_2)\}$. By Proposition 2.7, $\gamma_0 > 0$.

LEMMA 2.9. Given $\varepsilon \in (0, \gamma_0)$, there exists $M_\varepsilon \geq 1$ such that for $g \in X_2 \cap \mathfrak{C}^1$, $t > 0$,

$$(16) \quad \|e^{t\mathcal{L}_2}g\|_{\mathfrak{C}^1} \leq M_\varepsilon t^{-1/2} e^{-\gamma_\varepsilon t} \|g\|_{\mathfrak{C}}$$

and

$$(17) \quad \|e^{t\mathcal{L}_2}g\|_{\mathfrak{C}^1} \leq M_\varepsilon e^{-\gamma_\varepsilon t} \|g\|_{\mathfrak{C}^1},$$

where $\gamma_\varepsilon = \gamma_0 - \varepsilon$.

PROOF. Lemma 2.2 implies that the part of \mathcal{L} in \mathfrak{C}^1 generates an analytic semigroup in the Banach space \mathfrak{C}^1 . So there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for each $t > 0, g \in \mathfrak{C}^1$,

$$(18) \quad \|e^{t\mathcal{L}}g\|_{\mathfrak{C}^1} \leq M e^{\omega t} \|g\|_{\mathfrak{C}^1}.$$

Fix $\varepsilon \in (0, \gamma_0)$. We appeal to [11], in the notation of which, let $\alpha = 1/2$ and $n = 0$. The spaces $D_{\mathcal{L}}(1/2, p), 1 \leq p \leq \infty$ are defined in [11, p. 45]; note the last remark on that page. Now observe that

$$(19) \quad D_{\mathcal{L}}(1/2, 1) \hookrightarrow \mathfrak{C}^1.$$

This follows from Landau's inequality, [11, Propositions 2.2.2 and 1.2.13] with $\theta = 1/2$. This and [11, Proposition 2.3.3] with $\beta = 1/2$ and $p = 1$ together yield the existence of $\widehat{M} > 0$ such that for each $g \in X_2 \cap \mathfrak{C}^1$,

$$(20) \quad \|e^{t\mathcal{L}_2}g\|_{\mathfrak{C}^1} \leq \widehat{M} t^{-1/2} e^{-\gamma_\varepsilon t} \|g\|_{\mathfrak{C}} \quad \text{for each } t > 0.$$

In addition,

$$(21) \quad \mathfrak{C}^1 \hookrightarrow D_{\mathcal{L}}(1/2, \infty) \quad \text{and} \quad D_{\mathcal{L}}(\beta, \infty) \hookrightarrow \mathfrak{C}^1, \quad \beta \in (1/2, 1),$$

by [11, Theorem 3.1.12] with $\theta = 1/2$ and $\theta = \beta$, respectively. [11, Proposition 2.3.3] with $\beta \in (1/2, 1)$, $p = \infty$ and (21) give the existence of $\widehat{M} > 0$ such that for each $g \in X_2 \cap \mathfrak{C}^1$,

$$(22) \quad \begin{aligned} \|e^{t\mathcal{L}_2}g\|_{\mathfrak{C}^1} &\leq \widehat{M}t^{1/2-\beta}e^{-\gamma_\varepsilon t}\|g\|_{\mathfrak{C}^1} && \text{for each } t > 0, \\ &\leq \widehat{M}e^{-\gamma_\varepsilon t}\|g\|_{\mathfrak{C}^1} && \text{when } t \geq 1. \end{aligned}$$

It follows from (18) and (22) that there exists $\widetilde{M} > 0$ such that

$$(23) \quad \|e^{t\mathcal{L}_2}g\|_{\mathfrak{C}^1} \leq \widetilde{M}e^{-\gamma_\varepsilon t}\|g\|_{\mathfrak{C}^1} \quad \text{for all } t > 0.$$

(16) and (17) follow from (20) and (23). \square

3. Local stability

It is useful to formulate (6) as an abstract ordinary differential equation. Let $T > 0$ and let $v \in C(\mathbb{R} \times [0, T], \mathbb{R}^N)$ be such that v, v_t, v_x and v_{xx} are bounded and uniformly continuous on $\mathbb{R} \times (0, T)$. Define

$$y(t)(x) = v(x, t) - w(x), \quad (x, t) \in \mathbb{R} \times [0, T],$$

where w is the travelling wave introduced in (TW). Then v satisfies (6) if and only if $y \in C^1((0, T), \mathfrak{C}) \cap C((0, T), \mathfrak{C}^2)$ satisfies

$$(24) \quad y'(t) = \mathcal{L}(y(t)) + \mathcal{R}(y(t)), \quad t \in (0, T)$$

where $\mathcal{R} : \mathfrak{C}^1 \rightarrow \mathfrak{C}$ is given by

$$\mathcal{R}(y) = f(w + y, w' + y') - f(w, w') - d_p f[w, w']y' - d_q f[w, w']y, \quad y \in \mathfrak{C}^1.$$

Note that \mathcal{R} is continuously differentiable, and that $\|\mathcal{R}(y)\|_{\mathfrak{C}}/\|y\|_{\mathfrak{C}^1} \rightarrow 0$ as $\|y\|_{\mathfrak{C}^1} \rightarrow 0$.

Following [8], we adopt an elementary approach to proving local stability, based on the variation of constants formula and the estimates of Lemma 2.9. An alternative is to use centre-manifold theory and the existence of foliations (see [1]–[3]).

THEOREM 3.1. *Let $\varepsilon \in (0, \gamma_0)$. Then there exist $\nu_\varepsilon > 0$, $K_\varepsilon > 0$ and $\delta_\varepsilon > 0$ such that if $\varphi \in \mathfrak{C}^1$ satisfies*

$$(25) \quad \|\varphi - w(\cdot + \chi_0)\|_{\mathfrak{C}^1} < \nu_\varepsilon$$

for some $\chi_0 \in \mathbb{R}$, then there exists $\chi_\infty \in [\chi_0 - \delta_\varepsilon, \chi_0 + \delta_\varepsilon]$ such that

$$(25) \quad \|v^\varphi(\cdot, t) - w(\cdot + \chi_\infty)\|_{\mathfrak{C}^1} \leq K_\varepsilon e^{-\gamma_\varepsilon t}, \quad t > 0.$$

Note that K_ε and $\delta_\varepsilon > 0$ are independent of the exact choice of φ, χ_0 satisfying (25).

PROOF. We first prove a convergence result for (24), and then deduce Theorem 3.1 by interpreting this in terms of (6) and the travelling wave w . The idea for the proof comes from [8, Chapter 5, §1, Exercise 6]. For $\chi \in \mathbb{R}$, define $\widehat{w} : \mathbb{R} \rightarrow \mathfrak{C}^1$ by $\widehat{w}(\chi)(x) = w(x + \chi) - w(x)$, $x \in \mathbb{R}$. Then $\widehat{w}(0) = 0$, and for each $\chi \in \mathbb{R}$, $\mathcal{L}\widehat{w}(\chi) + \mathcal{R}(\widehat{w}(\chi)) = 0$, since $w(\cdot + \chi)$ is a stationary solution of (6). Since w satisfies (TW) and $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, $w \in \mathfrak{C}^3$. So $\widehat{w} : \mathbb{R} \rightarrow \mathfrak{C}^1$ is twice continuously differentiable, and

$$(27) \quad d\widehat{w}[\chi_0]\chi = \chi w'(\cdot + \chi_0) \quad \text{for each } \chi_0, \chi \in \mathbb{R}.$$

Let $\mathcal{H}(y, \chi) = w^*(y - \widehat{w}(\chi)) \in \mathbb{R}$, $(y, \chi) \in \mathfrak{C}^1 \times \mathbb{R}$, where w^* is as in Lemma 2.8. Then \mathcal{H} is continuously differentiable, $\mathcal{H}(0, 0) = 0$ and $d_\chi \mathcal{H}[0, 0]\chi = -\chi$ for each $\chi \in \mathbb{R}$. So it follows from the implicit function theorem that there is an open ball $B_{\mathfrak{C}^1}(\rho_0)$ in \mathfrak{C}^1 (centre 0, radius ρ_0), an open neighbourhood $(-\delta_0, \delta_0)$ of 0 in \mathbb{R} and a continuously differentiable function $\zeta : B_{\mathfrak{C}^1}(\rho_0) \rightarrow (-\delta_0, \delta_0)$ such that $\zeta(0) = 0$, $\mathcal{H}(y, \zeta(y)) = 0$ for $y \in B_{\mathfrak{C}^1}(\rho_0)$, and if $\mathcal{H}(y, \chi) = 0$ for some $y \in B_{\mathfrak{C}^1}(\rho_0), \chi \in (-\delta_0, \delta_0)$, then $\chi = \zeta(y)$. By (15), we can choose $\rho_0 > 0$ smaller if necessary so that $w^*(w'(\cdot + \chi)) > 1/2$ whenever $\chi = \zeta(y)$ for some $y \in B_{\mathfrak{C}^1}(\rho_0)$.

Proposition A.3 ensures that given initial data $y_0 \in \mathfrak{C}^1$, there is a unique local classical solution $y : (0, \tau(y_0)) \rightarrow \mathfrak{C}^2$ of (24) such that $\|y(t) - y_0\|_{\mathfrak{C}^1} \rightarrow 0$ as $t \rightarrow 0$. For $y_0 \in B_{\mathfrak{C}^1}(\rho_0)$, let $\widehat{t} \in (0, \tau(y_0))$ be such that $y(t) \in B_{\mathfrak{C}^1}(\rho_0)$ for each $t \in [0, \widehat{t}]$. For such t , define $\chi(t) = \zeta(y(t))$, where ζ is as given by the implicit function theorem above. Then $\chi(t) \in (-\delta_0, \delta_0)$ and $w^*(y(t)) = w^*(\widehat{w}(\chi(t)))$. Define $\widehat{y}(t) = y(t) - \widehat{w}(\chi(t))$. Since $w^*(\widehat{y}(t)) = 0$, $\widehat{y}(t) \in X_2$ (where X_2 is as defined before Lemma 2.8). Note that $\widehat{w}(\chi(\cdot)) = \widehat{w}(\zeta(y(\cdot)))$ and $\widehat{y}(\cdot)$ are both continuously differentiable on $(0, \widehat{t})$, and since $y \in C^1((0, \widehat{t}), \mathfrak{C})$ and X_2 is a closed subspace of \mathfrak{C} , $\widehat{y}'(t) \in X_2$ for $0 < t < \widehat{t}$.

Acting on (24) with w^* and using (27), the fact that $\widehat{w}(\chi)$ is a stationary solution of (24) for each χ and the properties of w^* together yield that for $0 < t < \widehat{t}$,

$$(28) \quad \chi'(t)w^*(w'(\cdot + \chi(t))) = w^*(\mathcal{R}(\widehat{w}(\chi(t)) + \widehat{y}(t)) - \mathcal{R}(\widehat{w}(\chi(t)))).$$

So

$$(29) \quad \chi'(t) = \varphi(\chi(t), \widehat{y}(t)), \quad t \in (0, \widehat{t}),$$

where we define

$$(30) \quad \varphi(\chi, \widehat{y}) = \frac{w^*(\mathcal{R}(\widehat{w}(\chi) + \widehat{y}) - \mathcal{R}(\widehat{w}(\chi)))}{w^*(w'(\cdot + \chi))}, \quad (\chi, \widehat{y}) \in \mathbb{R} \times \mathfrak{C}^1.$$

Similarly, acting on (24) with $I - \mathcal{P}$ (see (14)) gives that

$$(31) \quad \widehat{y}'(t) = \mathcal{L}_2 \widehat{y}(t) + \Psi(\chi(t), \widehat{y}(t)), \quad t \in (0, \widehat{t}),$$

where

$$(32) \quad \Psi(\chi, \widehat{y}) = (I - \mathcal{P})\{\mathcal{R}(\widehat{w}(\chi) + \widehat{y}) - \mathcal{R}(\widehat{w}(\chi))\} - (I - \mathcal{P})d\widehat{w}[\chi]\varphi(\chi, \widehat{y}).$$

Now, for $\widehat{y} \in \mathfrak{C}^1$ and $\chi \in \mathbb{R}$ with $|\chi| \leq 1$ and small enough, we have that

$$w^*(w'(\cdot + \chi)) > \frac{1}{2}, \quad |\varphi(\chi, \widehat{y})| \leq 2\|w^*\|_{\mathfrak{C}^*} K(\chi, \widehat{y}) \|\widehat{y}\|_{\mathfrak{C}^1}$$

where $K(\chi, \widehat{y}) = \sup_{0 \leq \theta \leq 1} \{\|d\mathcal{R}[\widehat{w}(\chi) + \theta\widehat{y}]\|_{L(\mathfrak{C}^1, \mathfrak{C})}\}$. Since $d\mathcal{R}[0] = 0$, $(\chi, \widehat{y}) \rightarrow 0$ as $|\chi| + \|\widehat{y}\|_{\mathfrak{C}^1} \rightarrow 0$. Also,

$$(33) \quad \begin{aligned} \|\Psi(\chi, \widehat{y})\|_{\mathfrak{C}} &\leq \|I - \mathcal{P}\|_{L(\mathfrak{C}, \mathfrak{C})} K(\chi, \widehat{y}) \|\widehat{y}\|_{\mathfrak{C}^1} \\ &\quad + \|I - \mathcal{P}\|_{L(\mathfrak{C}, \mathfrak{C})} \|d\widehat{w}[\chi]\|_{L(\mathbb{R}, \mathfrak{C}^1)} |\varphi(\chi, \widehat{y})|. \end{aligned}$$

So, since $\|d\widehat{w}[\chi]\|_{L(\mathbb{R}, \mathfrak{C}^1)}$ is bounded independently of $|\chi| \leq 1$, there exists a constant $\widehat{K} > 0$ such that

$$(34) \quad |\varphi(\chi, \widehat{y})| + \|\Psi(\chi, \widehat{y})\|_{\mathfrak{C}} \leq \widehat{K} K(\chi, \widehat{y}) \|\widehat{y}\|_{\mathfrak{C}^1},$$

where $K(\chi, \widehat{y}) \rightarrow 0$ as $|\chi| + \|\widehat{y}\|_{\mathfrak{C}^1} \rightarrow 0$. Henceforth fix $\varepsilon \in (0, \gamma_0)$. Choose $\sigma_\varepsilon > 0$ so that

$$(35) \quad M_{\varepsilon/2} \sigma_\varepsilon \int_0^\infty s^{-1/2} e^{-(\gamma_{\varepsilon/2} - \gamma_\varepsilon)s} ds = M_{\varepsilon/2} \sigma_\varepsilon \int_0^\infty s^{-1/2} e^{-\varepsilon/2s} ds < \frac{1}{2},$$

where $M_{\varepsilon/2} \geq 1$ is as in Lemma 2.9. Let $\widetilde{K} > 0$ be such that $K(\chi, \widehat{y}) < \widetilde{K}$ whenever $|\chi| < \delta_0$ and $\|\widehat{y}\|_{\mathfrak{C}^1} < \rho_0$. Now using (34), we can choose $\rho_\varepsilon \in (0, \rho_0)$, $\delta_\varepsilon \in (0, \delta_0)$ such that $\rho_\varepsilon < \gamma_\varepsilon^2 / (2\widetilde{K}\widehat{K})$ and

$$(36) \quad \begin{aligned} \|\Psi(\chi, \widehat{y})\|_{\mathfrak{C}} &\leq \sigma_\varepsilon \|\widehat{y}\|_{\mathfrak{C}^1}, \quad \|\widehat{w}(\chi) + \widehat{y}\|_{\mathfrak{C}^1} \leq \rho_0/2 \\ &\quad \text{for all } (\chi, \widehat{y}) \text{ with } |\chi| \leq \delta_\varepsilon \text{ and } \|\widehat{y}\|_{\mathfrak{C}^1} \leq \rho_\varepsilon. \end{aligned}$$

Let $\nu_\varepsilon \in (0, \rho_0)$ be such that

$$(37) \quad \|y_0\|_{\mathfrak{C}^1} < \nu_\varepsilon \Rightarrow |\zeta(y_0)| < \delta_\varepsilon/2 \text{ and } \|y_0\|_{\mathfrak{C}^1} + \|\widehat{w}(\zeta(y_0))\|_{\mathfrak{C}^1} < \rho_\varepsilon / (2M_{\varepsilon/2}).$$

Fix initial data $y_0 \in \mathfrak{C}^1$ with $\|y_0\|_{\mathfrak{C}^1} < \nu_\varepsilon$. Define $t_0 = \sup_{0 \leq t < \tau(y_0)} \{t : y(s) \in B_{\mathfrak{C}^1}(\rho_0) \text{ for all } s \in [0, t]\}$. For $t \in [0, t_0)$, $\chi(t) = \zeta(y(t))$ and $\widehat{y}(t) = y(t) - \widehat{w}(\chi(t))$ are well-defined and have the properties described above. By the choice of ν_ε , $|\chi(0)| < \delta_\varepsilon/2$ and $\|\widehat{y}(0)\|_{\mathfrak{C}^1} < \rho_\varepsilon / (2M_{\varepsilon/2})$. Define $m(t) = \sup_{0 \leq s \leq t} \{e^{\gamma_\varepsilon s} \|\widehat{y}(s)\|_{\mathfrak{C}^1}\}$, $t \in [0, t_0)$. Then since \widehat{y} satisfies (31) and $\gamma_\varepsilon = \gamma_0 - \varepsilon$,

it follows from the variation of constants formula, Lemma 2.9, (35) and (36) that for $0 \leq s \leq t < t_0$,

$$\begin{aligned} e^{\gamma_\varepsilon s} \|\widehat{y}(s)\|_{\mathfrak{C}^1} &= e^{\gamma_\varepsilon s} \left\| e^{s\mathcal{L}_2} \widehat{y}(0) + \int_0^s e^{(s-\tilde{s})\mathcal{L}_2} \Psi(\chi(\tilde{s}), \widehat{y}(\tilde{s})) d\tilde{s} \right\|_{\mathfrak{C}^1} \\ &\leq M_{\varepsilon/2} e^{(\gamma_\varepsilon - \gamma_{\varepsilon/2})s} \|\widehat{y}(0)\|_{\mathfrak{C}^1} \\ &\quad + e^{\gamma_\varepsilon s} \sigma_\varepsilon M_{\varepsilon/2} \int_0^s (s-\tilde{s})^{-1/2} e^{-\gamma_{\varepsilon/2}(s-\tilde{s})} \|\widehat{y}(\tilde{s})\|_{\mathfrak{C}^1} d\tilde{s} \\ &\leq M_{\varepsilon/2} \|\widehat{y}(0)\|_{\mathfrak{C}^1} + m(t)/2. \end{aligned}$$

Whence $m(t) \leq 2M_{\varepsilon/2} \|\widehat{y}(0)\|_{\mathfrak{C}^1}$ for each $t \in [0, t_0)$. It follows, using (29), (34), that

$$(38) \quad \|\widehat{y}(t)\|_{\mathfrak{C}^1} \leq \rho_\varepsilon e^{-\gamma_\varepsilon t} \text{ and } |\chi'(t)| = |\varphi(\chi(t), \widehat{y}(t))| \leq \widehat{K} \widetilde{K} \rho_\varepsilon e^{-\gamma_\varepsilon t}, \quad t \in (0, t_0).$$

This, together with the facts that $|\chi(0)| < \delta_\varepsilon/2$ and $\rho_\varepsilon < \gamma_\varepsilon^2/(2\widehat{K}\widetilde{K})$, yields that for each $t \in [0, t_0)$,

$$(39) \quad |\chi(t)| \leq \delta_\varepsilon/2 + \widehat{K} \widetilde{K} \rho_\varepsilon \gamma_\varepsilon^{-1} [1 - e^{-\gamma_\varepsilon t}] < \delta_\varepsilon.$$

Now it follows from the definition of t_0 , (36), (38) and (39) that $t_0 = \tau(y_0)$. And Proposition A.4 shows that if $\tau(y_0) < \infty$, then $\sup_{0 \leq s \leq t} \|y(s)\|_{\mathfrak{C}^1} \rightarrow \infty$ as $t \uparrow \tau(y_0)$. So $t_0 = \tau(y_0) = \infty$, and (38) and (39) hold for all $t \geq 0$. Since $|\chi'(\cdot)| \in L^1((0, \infty), \mathbb{R})$ and $|\chi(t)| \leq \delta_\varepsilon$ for all $t \geq 0$, there exists $\widehat{\chi} \in [-\delta_\varepsilon, \delta_\varepsilon]$ such that

$$(40) \quad |\widehat{\chi} - \chi(t)| \leq \widehat{K} \widetilde{K} \rho_\varepsilon \gamma_\varepsilon^{-1} e^{-\gamma_\varepsilon t}, \quad t > 0.$$

We now rewrite (38) and (40) in terms of the travelling wave w . Recall that y is a solution of (24) with initial data y_0 if and only if $v^\varphi(\cdot, t) = y(t) + w$ is a solution of (6) with initial data $\varphi = y_0 + w$, and that $\widehat{w}(\chi)(x) = w(x + \chi) - w(x)$ for $x, \chi \in \mathbb{R}$. So $\|y_0\|_{\mathfrak{C}^1} = \|\varphi - w\|_{\mathfrak{C}^1}$, and

$$(41) \quad \|\widehat{y}(t)\|_{\mathfrak{C}^1} = \|y(t) - \widehat{w}(\chi(t))\|_{\mathfrak{C}^1} = \|v^\varphi(\cdot, t) - w(\cdot + \chi(t))\|_{\mathfrak{C}^1}.$$

Hence if $\|\varphi - w\|_{\mathfrak{C}^1} \leq \nu_\varepsilon$, (38) and (40) give that

$$\begin{aligned} \|v^\varphi(\cdot, t) - w(\cdot + \widehat{\chi})\|_{\mathfrak{C}^1} &\leq \|v^\varphi(\cdot, t) - w(\cdot + \chi(t))\|_{\mathfrak{C}^1} \\ &\quad + \|w(\cdot + \chi(t)) - w(\cdot + \widehat{\chi})\|_{\mathfrak{C}^1} \\ &\leq \rho_\varepsilon e^{-\gamma_\varepsilon t} + |\chi(t) - \widehat{\chi}| \|w'\|_{\mathfrak{C}^1} \leq K_\varepsilon e^{-\gamma_\varepsilon t}, \end{aligned}$$

where $K_\varepsilon = \rho_\varepsilon \{1 + \widehat{K} \widetilde{K} \gamma_\varepsilon^{-1} \|w'\|_{\mathfrak{C}^1}\}$. To complete the proof, note that if $\varphi \in \mathfrak{C}^1$ satisfies $\|\varphi - w(\cdot + \chi_0)\|_{\mathfrak{C}^1} < \nu_\varepsilon$ for some $\chi_0 \in \mathbb{R}$, then $\|\varphi(\cdot - \chi_0) - w(\cdot)\|_{\mathfrak{C}^1} < \nu_\varepsilon$. The above analysis immediately implies that $\|v^\varphi(\cdot, t) - w(\cdot + \widehat{\chi} + \chi_0)\|_{\mathfrak{C}^1} \leq K_\varepsilon e^{-\gamma_\varepsilon t}$ for all $t \geq 0$. The result follows. \square

4. Global stability for monotone initial data

We turn now to the global stability of the wave w . Note first that if $\varphi \in \mathcal{C}^1$ satisfies $E^- \leq \varphi(x) \leq E^+$ for all $x \in \mathbb{R}$, then it follows from Theorem A.7 that the initial value problem (6) has a unique classical solution v^φ that exists for all time, and that $E^- \leq v^\varphi(x, t) \leq E^+$ for all $x \in \mathbb{R}$, $t \geq 0$.

In this section, we consider the initial-value problem (6) with initial data $\varphi \in \mathcal{C}^1$ satisfying the following conditions :

- ($\varphi 1$) $\varphi(x) \rightarrow E^\pm$ as $x \rightarrow \pm\infty$, and $\varphi'(x) \rightarrow 0$ as $|x| \rightarrow \infty$,
- ($\varphi 2$) $\varphi'(x) \geq 0$ for each $x \in \mathbb{R}$.

Our approach is similar to that of [14, Chapter 5, Theorem 6.1].

THEOREM 4.1. *Let $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfy (f1)–(f4) and $\varphi \in \mathcal{C}^1$ satisfy ($\varphi 1$)–($\varphi 2$). Then there exists $\chi_\infty \in \mathbb{R}$ such that for each $\varepsilon \in (0, \gamma_0)$, there exists $N_\varepsilon > 0$ such that the solution v^φ of (6) with initial data φ satisfies*

$$(42) \quad \|v^\varphi(\cdot, t) - w(\cdot + \chi_\infty)\|_{\mathcal{C}^1} \leq N_\varepsilon e^{-\gamma_\varepsilon t} \quad \text{for all } t > 0.$$

PROOF. The idea is to construct a function φ^* , from φ and the wave w , such that the solution v^{φ^*} of (6) satisfies (42), and then to use a homotopy argument to deduce the corresponding result for φ .

Fix $\varepsilon \in (0, \gamma_0)$. We begin with the construction of φ^* . Let ν_ε be as in (37). Choose $\eta_1 > 0$ sufficiently large then that

$$(43) \quad \pm x \geq +\eta_1 \Rightarrow \|\varphi(x) - E^\pm\|, \quad \|w(x) - E^\pm\|, \quad \|w'(x)\|, \quad \|\varphi'(x)\| < \nu_\varepsilon/4.$$

Choose $\eta_2 > \eta_1 + 1$ so that $\varphi(\eta_2) > w(\eta_1)$ and $\varphi(-\eta_2) < w(-\eta_1)$. Define $\varphi^* : \mathbb{R} \rightarrow \mathbb{R}^N$ by $\varphi^*(x) = w(x)$ for $|x| \leq \eta_1$ and $\varphi^*(x) = \varphi(x)$ for $|x| \geq \eta_2$; for $|x| \in [\eta_1, \eta_2]$, define $\varphi^*(x)$ so that $\varphi^* \in \mathcal{C}^1$ is increasing and $\|(\varphi^*)'(x)\| < \nu_\varepsilon/4$ for each x , $|x| \geq \eta_1$. By construction,

$$(44) \quad \|\varphi^* - w\|_{\mathcal{C}^1} < \nu_\varepsilon/2.$$

Here is the construction that underlies the homotopy argument. As in [14], define

$$(45) \quad \varphi_\tau(x) = \min\{\varphi(x), \varphi^*(x - \tau)\}, \quad \tau \in \mathbb{R}, \quad x \in \mathbb{R}.$$

The minimum is calculated componentwise. For each τ , φ_τ is clearly continuous and increasing. It also follows directly from (45) that for each fixed $x \in \mathbb{R}$, $\varphi_\tau(x)$ is a decreasing function of τ . The following crucial property of φ_τ is proved in [14];

$$(46) \quad \varphi_{-2\eta_2}(x) = \varphi(x) \quad \text{and} \quad \varphi_{2\eta_2}(x) = \varphi^*(x - 2\eta_2) \quad \text{for all } x \in \mathbb{R}.$$

The existence theory for the initial-value problem for (6) in the Appendix requires the initial data in \mathcal{C}^1 . We introduce mollifications of φ_τ in order to

consider τ -dependent initial-value problems. For $b \in (0, 1)$, let $\kappa_b : \mathbb{R} \rightarrow [0, \infty)$ be a standard normalised mollifier, supported in $[-b, b]$ (see, for example, [6]). For $\tau \in \mathbb{R}$, $b \in (0, 1)$, $x \in \mathbb{R}$, let

$$(47) \quad \psi_{\tau,b}(x) = (\varphi_\tau * \kappa_b)(x) = \int_{-\infty}^{\infty} \varphi_\tau(x-s)\kappa_b(s) ds.$$

By construction, $E^- \leq \psi_{\tau,b}(x) \leq E^+$ for all x . It follows from Theorem A.7 that the initial-value problem (6) with initial data $\psi_{\tau,b}$ has a unique classical solution $v^{\psi_{\tau,b}}$ that exists for all time, and that

$$(48) \quad E^- \leq v^{\psi_{\tau,b}}(x, t) \leq E^+ \quad \text{for all } x \in \mathbb{R}, t \geq 0.$$

The approach is to advance the parameter τ with step $-h < 0$ (to be determined) from $\tau = 2\eta_2$ to $\tau = -2\eta_2$, at each stage proving that the solution $v^{\psi_{\tau,b}}$ with initial data $\psi_{\tau,b}$ converges in \mathfrak{C}^1 to a translate of w . At $\tau = -2\eta_2$, the initial data is $\varphi * \kappa_b$, by (46); letting $b \rightarrow 0$ will then yield the required result.

We seek $h_\varepsilon > 0$, independent of $b \in (0, 1)$, $\tau \in \mathbb{R}$, $T \geq 1$, such that

$$(49) \quad \|v^{\psi_{\tau-h_\varepsilon,b}}(\cdot, T) - v^{\psi_{\tau,b}}(\cdot, T)\|_{\mathfrak{C}^1} \leq \nu_\varepsilon/4.$$

By Landau's inequality,

$$(50) \quad \|(v^{\psi_{\tau-h,b}} - v^{\psi_{\tau,b}})_x(\cdot, T)\|_{\mathfrak{C}} \leq 2\|(v^{\psi_{\tau-h,b}} - v^{\psi_{\tau,b}})(\cdot, T)\|_{\mathfrak{C}}^{1/2} \|(v^{\psi_{\tau-h,b}} - v^{\psi_{\tau,b}})_{xx}(\cdot, T)\|_{\mathfrak{C}}^{1/2}$$

for each $b \in (0, 1)$, $\tau \in \mathbb{R}$, $T \geq 1$ and $h > 0$. We now show that the first factor on the right of (50) is small when h is small. Note first that for $x \in \mathbb{R}$, $\tau \in \mathbb{R}$, $h > 0$,

$$(51) \quad \varphi_\tau(x) \leq \varphi_{\tau-h}(x) \leq \varphi_\tau(x+h).$$

Since mollification preserves ordering and commutes with translation, it follows that for $b \in (0, 1)$,

$$(52) \quad \psi_{\tau,b}(x) \leq \psi_{\tau-h,b}(x) \leq \psi_{\tau,b}(x+h).$$

Now, since f satisfies (f1)–(f2), the comparison principle Theorem A.2 yields that

$$(53) \quad v^{\psi_{\tau,b}}(x, t) \leq v^{\psi_{\tau-h,b}}(x, t) \leq v^{\psi_{\tau,b}}(x+h, t), \quad x \in \mathbb{R}, t > 0.$$

So by the Mean Value Inequality, for $t > 0$, $x \in \mathbb{R}$,

$$(54) \quad \|v^{\psi_{\tau-h,b}}(x, t) - v^{\psi_{\tau,b}}(x, t)\| \leq \|v^{\psi_{\tau,b}}(x+h, t) - v^{\psi_{\tau,b}}(x, t)\| \leq h\|(v^{\psi_{\tau,b}})_x(\cdot, t)\|_{\mathfrak{C}}.$$

By Theorem A.8 there exists $K_1 > 0$, independent of $t \geq 1, \tau \in \mathbb{R}, b \in (0, 1)$, such $\|(v^{\psi_{\tau,b}})_x(\cdot, t)\|_{\mathfrak{C}} \leq K_1$. Hence for each $h > 0, t \geq 1, \tau \in \mathbb{R}, b \in (0, 1)$,

$$(55) \quad \|v^{\psi_{\tau-h,b}}(\cdot, t) - v^{\psi_{\tau,b}}(\cdot, t)\|_{\mathfrak{C}} \leq K_1 h.$$

It follows from (48) and Theorem A.8 that the second factor on the right of (50) is bounded independently of $\tau \in \mathbb{R}, h > 0, b \in (0, 1), T \geq 1$. The existence of $h_\varepsilon > 0$ satisfying (49), independent of $b \in (0, 1), \tau \in \mathbb{R}$ and $T \geq 1$, thus follows from (50) and (55). We choose $h_\varepsilon > 0$ smaller if necessary so that there exists $n \in \mathbb{N}$ such that

$$(56) \quad 4\eta_2 = nh_\varepsilon.$$

Now $\|\varphi^* * \kappa_b - w\|_{\mathfrak{C}^1} \rightarrow 0$ as $b \rightarrow 0$, so it follows from (44) that for $b \in (0, b_0)$ say, $\|\varphi^* * \kappa_b - w\|_{\mathfrak{C}^1} < \nu_\varepsilon$. Hence $\|\psi_{2\eta_2,b} - w(\cdot - 2\eta_2)\|_{\mathfrak{C}^1} < \nu_\varepsilon$. With $\gamma_\varepsilon, K_\varepsilon, \delta_\varepsilon > 0$ (independent of b) as in Theorem 3.1, there exists $\chi_{2\eta_2,b} \in [-2\eta_2 - \delta_\varepsilon, -2\eta_2 + \delta_\varepsilon]$ such that

$$(57) \quad \|v^{\psi_{2\eta_2,b}}(\cdot, t) - w(\cdot + \chi_{2\eta_2,b})\|_{\mathfrak{C}^1} \leq K_\varepsilon e^{-\gamma_\varepsilon t} \quad \text{for all } t > 0.$$

Next define

$$(58) \quad T_\varepsilon = \max \left\{ 1, \frac{1}{\gamma_\varepsilon} \log \frac{4K_\varepsilon}{\nu_\varepsilon} \right\}.$$

(Clearly T_ε is independent of $b \in (0, b_0)$.) So, by (57) and (58),

$$(59) \quad \|v^{\psi_{2\eta_2,b}}(\cdot, T_\varepsilon) - w(\cdot + \chi_{2\eta_2,b})\|_{\mathfrak{C}^1} \leq \nu_\varepsilon/4.$$

Together with (49), this yields that

$$(60) \quad \|v^{\psi_{2\eta_2-h_\varepsilon,b}}(\cdot, T_\varepsilon) - w(\cdot + \chi_{2\eta_2,b})\|_{\mathfrak{C}^1} \leq \nu_\varepsilon/2.$$

So by Theorem 3.1, there exists $\chi_{2\eta_2-h_\varepsilon,b} \in [\chi_{2\eta_2,b} - \delta_\varepsilon, \chi_{2\eta_2,b} + \delta_\varepsilon] \subset [-2\eta_2 - 2\delta_\varepsilon, -2\eta_2 + 2\delta_\varepsilon]$ such that for $t > T_\varepsilon$,

$$(61) \quad \|v^{\psi_{2\eta_2-h_\varepsilon,b}}(\cdot, t) - w(\cdot + \chi_{2\eta_2-h_\varepsilon,b})\|_{\mathfrak{C}^1} \leq K_\varepsilon e^{-\gamma_\varepsilon(t-T_\varepsilon)}.$$

Arguing by induction, it follows that given $m \in \mathbb{N}$, there exist $\chi_{2\eta_2-kh_\varepsilon,b} \in [-2\eta_2 - k\delta_\varepsilon, -2\eta_2 + k\delta_\varepsilon]$ for each $0 \leq k \leq m$ such that

$$(62) \quad \|v^{\psi_{2\eta_2-mh_\varepsilon,b}}(\cdot, mT_\varepsilon) - w(\cdot + \chi_{2\eta_2-(m-1)h_\varepsilon,b})\|_{\mathfrak{C}^1} \leq \nu_\varepsilon/2,$$

and for $t > mT_\varepsilon$,

$$(63) \quad \|v^{\psi_{2\eta_2-mh_\varepsilon,b}}(\cdot, t) - w(\cdot + \chi_{2\eta_2-mh_\varepsilon,b})\|_{\mathfrak{C}^1} \leq K_\varepsilon e^{-\gamma_\varepsilon(t-mT_\varepsilon)}.$$

In particular, (62) and (63) hold for n satisfying (56). Since $\psi_{-2\eta_2,b} = \varphi^* * \kappa_b$, this yields that for each $b \in (0, b_0)$, there exists $\chi_{-2\eta_2+h_\varepsilon,b} \in [-2\eta_2 - (n-1)\delta_\varepsilon, -2\eta_2 + (n-1)\delta_\varepsilon]$ such that

$$(64) \quad \|v^{\varphi^* * \kappa_b}(\cdot, nT_\varepsilon) - w(\cdot + \chi_{-2\eta_2+h_\varepsilon,b})\|_{\mathfrak{C}^1} \leq \nu_\varepsilon/2.$$

We now let $b \rightarrow 0$. The interval $[-2\eta_2 - (n-1)\delta_\varepsilon, -2\eta_2 + (n-1)\delta_\varepsilon]$ is independent of $b \in (0, b_0)$. So there is a sequence $\{b_k\} \subset (0, b_0)$, $b_k \downarrow 0$ and $\chi_\varepsilon \in [-2\eta_2 - (n-1)\delta_\varepsilon, -2\eta_2 + (n-1)\delta_\varepsilon]$ such that

$$(65) \quad \chi_{-2\eta_2+h_\varepsilon, b_k} \rightarrow \chi_\varepsilon \quad \text{as } k \rightarrow \infty.$$

Thus there exists $k_0 \in \mathbb{N}$ such that

$$(66) \quad k \geq k_0 \Rightarrow \|w(\cdot + \chi_{-2\eta_2+h_\varepsilon, b_k}) - w(\cdot + \chi_\varepsilon)\|_{\mathfrak{C}^1} \leq \nu_\varepsilon/4.$$

Proposition A.3 yields the existence of $r, K > 0$ such that for n as in (56) and $\hat{\varphi}, \tilde{\varphi} \in \mathfrak{C}^1$,

$$(67) \quad \|\hat{\varphi} - \tilde{\varphi}\|_{\mathfrak{C}^1} \leq r \Rightarrow \|v^{\hat{\varphi}}(\cdot, nT_\varepsilon) - v^{\tilde{\varphi}}(\cdot, nT_\varepsilon)\|_{\mathfrak{C}^1} \leq K\|\hat{\varphi} - \tilde{\varphi}\|_{\mathfrak{C}^1}.$$

Hence since $\|\varphi - \varphi * \kappa_b\|_{\mathfrak{C}^1} \rightarrow 0$ as $b \rightarrow 0$, there exists $k_1 \in \mathbb{N}$ such that

$$(68) \quad k \geq k_1 \Rightarrow \|v^\varphi(\cdot, nT_\varepsilon) - v^{\varphi * \kappa_{b_k}}(\cdot, nT_\varepsilon)\|_{\mathfrak{C}^1} \leq \nu_\varepsilon/4.$$

So, by (64), (66) and (68),

$$(69) \quad \|v^\varphi(\cdot, nT_\varepsilon) - w(\cdot + \chi_\varepsilon)\|_{\mathfrak{C}^1} \leq \nu_\varepsilon.$$

Theorem 3.1 yields that

$$(70) \quad \|v^\varphi(\cdot, t) - w(\cdot + \chi_\varepsilon)\|_{\mathfrak{C}^1} \leq K_\varepsilon e^{-\gamma_\varepsilon(t-nT_\varepsilon)} \quad \text{for } t > nT_\varepsilon.$$

Since v^φ is independent of ε , and w is not periodic, it is immediate that $\chi_{\varepsilon_1} = \chi_{\varepsilon_2}$ for any $\varepsilon_1, \varepsilon_2 \in (0, \gamma_0)$. The result follows. \square

5. Global stability for general initial data

We will invoke an idea from [13]. First a preliminary lemma, which is a modification of [13, Lemma 3.3]. This result will be used later, in the proof of Theorem 5.3, as part of an argument by contradiction.

LEMMA 5.1. *Let $D, G : \mathbb{R} \times [0, \infty) \rightarrow M^{N \times N}$ be continuous $N \times N$ -matrix-valued-functions, uniformly bounded on $\mathbb{R} \times [0, \infty)$, such that $D(x, t)$ is diagonal and the off-diagonal elements of $G(x, t)$ are non-negative for each $(x, t) \in \mathbb{R} \times [0, \infty)$. Let \mathfrak{h} be a non-negative, uniformly bounded solution of*

$$(71) \quad \mathfrak{h}_t(x, t) = A\mathfrak{h}_{xx}(x, t) + D(x, t)\mathfrak{h}_x(x, t) + G(x, t)\mathfrak{h}(x, t), \quad (x, t) \in \mathbb{R} \times [0, \infty),$$

such that \mathfrak{h}_t is uniformly bounded for $t \geq 1/2$ and there exist $\mu_0, M_0 > 0$ such that for each $t \geq 0$,

$$(72) \quad \sup_{x \in \mathbb{R}} (\min_{1 \leq i \leq N} \mathfrak{h}_i(x, t)) = \max_{|x| \leq M_0} (\min_{1 \leq i \leq N} \mathfrak{h}_i(x, t)) \geq \mu_0.$$

Then for each $M \geq M_0$, there exists $\alpha(M) > 0$ such that for all $t \geq 1$,

$$(73) \quad \min_{|x| \leq M} \min_{1 \leq i \leq N} \mathfrak{h}_i(x, t) \geq \alpha(M).$$

PROOF. Let $M \geq M_0$ and recall that $\mathbf{e} = (1, \dots, 1)$. It follows from (72) that for each $T \geq 0$, there exists $x^T \in [-M_0, M_0]$ such that $\mathfrak{h}(x^T, T) \geq \mu_0 \mathbf{e}$. Furthermore, $\mathfrak{h}_t(x, t)$ is bounded independently of $x \in \mathbb{R}$, $t \geq 1/2$, so there exists $T_0 \in (0, 1/2)$, independent of $T \geq 1$, such that

$$(74) \quad T \geq 1, |\hat{t}| \leq T_0 \Rightarrow \mathfrak{h}(x^T, T + \hat{t}) \geq \epsilon \mu_0 / 2.$$

We will construct a strictly positive function which lies beneath $\mathfrak{h}(x, t)$ for all $t \geq 1$. By the hypotheses on D and G , there are constant diagonal matrices D^-, D^+ and a constant negative-definite diagonal matrix G^- such that

$$(75) \quad G_{ij}^- \leq G_{ij}(x, t) \quad \text{and} \quad D_{ij}^- \leq D_{ij}(x, t) \leq D_{ij}^+$$

for all $x \in \mathbb{R}$, $t \geq 0$, $i, j \in \{1, \dots, N\}$. Consider the two initial-boundary-value problems for $u^+ : [0, \infty) \times [0, 2T_0] \rightarrow \mathbb{R}^N$ and $u^- : (-\infty, 0] \times [0, 2T_0] \rightarrow \mathbb{R}^N$;

$$\begin{aligned} u_t^\pm &= Au_{xx}^\pm + D^\pm u_x^\pm + G^- u^\pm, & (\pm x, t) \in (0, \infty) \times (0, 2T_0), \\ u^\pm(0, t) &= \frac{\mu_0}{2} \mathbf{e} & \text{for } t \in [0, 2T_0], \\ u^\pm(x, 0) &= 0 & \text{for } \pm x \in (0, \infty), \\ u^\pm(x, t) &\rightarrow 0 & \text{as } \pm x \rightarrow \infty. \end{aligned}$$

Since A, D^\pm, G^- are diagonal, we can solve these explicitly using Laplace transforms to find that for each $i \in \{1, \dots, N\}$ and $(\pm x, t) \in (0, \infty) \times [0, 2T_0]$,

$$(76) \quad u_i^\pm(x, t) = \pm \left(\frac{A_i}{4\pi} \right)^{1/2} x e^{-D_i^\pm x/2} \cdot \int_0^t s^{-3/2} \exp \left[- \left(\frac{(D_i^\pm)^2}{A_i} - G_i^- \right) s - \frac{A_i x^2}{4s} \right] ds.$$

We will show that $u_x^+(x, t) < 0$ for all $x > 0, t > 0$. (76) yields that for each $i \in \{1, \dots, N\}$,

$$\begin{aligned} (u_i^+)_x(x, t) &= \left(\frac{A_i}{4\pi} \right)^{1/2} e^{-D_i^+ x/2} \\ &\cdot \int_0^t \left\{ 1 - \frac{D_i^+ x}{2} - \frac{A_i x^2}{2s} \right\} s^{-3/2} \exp \left[- \left(\frac{(D_i^+)^2}{A_i} - G_i^- \right) s - \frac{A_i x^2}{4s} \right] ds. \end{aligned}$$

Fix $t \in (0, 2T_0]$ and let

$$x_+^t = \inf \{ x > 0 : u_x^+(s, t) < 0 \text{ for each } s \in [x, \infty) \}.$$

The formula for u_x^+ shows that $u_x^+(x, t) < 0$ for x sufficiently large. So $x_+^t \in [0, \infty)$. Suppose that $x_+^t > 0$. Then for some $i \in \{1, \dots, N\}$, $(u_i^+)_x(x_+^t, t) = 0$. So

$$A_i (u_i^+)_x(x_+^t, t) = (u_i^+)_t(x_+^t, t) - G_{ii}^- u_i^+(x_+^t, t).$$

Now $u_i^+ > 0$, $G_{ii}^- < 0$ and it is clear from (76) that $(u_i^+)_t > 0$. So since $A_i > 0$, $(u_i^+)_{xx}(x_+^t, t) > 0$. But this implies that u_i^+ has a strict local minimum at x_+^t , which contradicts the fact that $(u_i^+)_x(x, t) < 0$ for all $x > x_+^t$. Whence $x_+^t = 0$. A similar argument shows that $u_x^-(x, t) > 0$ for each $t > 0, x < 0$.

Fix $T \geq 1$. Let $u^{T,+} : [x^T, \infty) \times [T - T_0, T + T_0] \rightarrow \mathbb{R}^N$, $u^{T,-} : (-\infty, x^T] \times [T - T_0, T + T_0] \rightarrow \mathbb{R}^N$ denote the unique solutions of the two initial-boundary-value problems

$$\begin{aligned} u_t^{T,\pm} &= Au_{xx}^{T,\pm} + D^\pm u_x^{T,\pm} + G^- u^{T,\pm}, & (\pm\{x - x^T\}, t) &\in (0, \infty) \times (T - T_0, T + T_0), \\ u^{T,\pm}(x^T, t) &= \epsilon\mu_0/2 & \text{for } t &\in [T - T_0, T + T_0], \\ u^{T,\pm}(x, T - T_0) &= 0 & \text{for } \pm\{x - x^T\} &\in (0, \infty), \\ u^{T,\pm}(x, t) &\rightarrow 0 & \text{as } \pm x &\rightarrow \infty. \end{aligned}$$

Clearly,

$$u^{T,\pm}(x, t) = u^\pm(x - x^T, t - T + T_0), \quad (\pm\{x - x^T\}, t) \in [0, \infty) \times [T - T_0, T + T_0].$$

So, since $\pm u_x^\pm < 0$ for $t, \pm x > 0$,

$$(78) \quad \min_{x \in [x^T, M]} u^{T,+}(x, T) \geq \min_{x \in [0, M_0 + M]} u^+(x, T_0) = u^+(M_0 + M, T_0),$$

$$(77) \quad \min_{x \in [-M, x^T]} u^{T,-}(x, T) \geq \min_{x \in [-M_0 - M, 0]} u^-(x, T_0) = u^-(-M_0 - M, T_0).$$

Now $u^{T,\pm}(x, t) > 0, \pm u_x^{T,\pm}(x, t) < 0$ for $(\pm\{x - x^T\}, t) \in (0, \infty) \times (T - T_0, T + T_0)$, so it follows from (75) and (77) that for such (x, t) ,

$$(80) \quad u_t^{T,\pm}(x, t) - Au_{xx}^{T,\pm}(x, t) - D(x, t)u_x^{T,\pm}(x, t) - G(x, t)u^{T,\pm}(x, t) \leq 0.$$

So, since (74) holds and \mathfrak{h} is non-negative, it follows from the positivity Theorem A.1(i) that

$$(81) \quad \mathfrak{h}(x, t) \geq u^{T,\pm}(x, t), \quad (\pm\{x - x^T\}, t) \in [0, \infty) \times [T - T_0, T + T_0].$$

Hence, by (78), (79), (81),

$$(82) \quad \min_{x \in [-M, M]} \mathfrak{h}(x, T) \geq \min\{u^0(M_0 + M), T_0, v^0(-M_0 - M, T_0)\}.$$

The right-hand side of (82) is a strictly positive number independent of $x \in [-M, M]$, $T \geq 1$. The result follows. \square

For $\varphi \in \mathfrak{C}^1$, define its omega limit set

$$(83) \quad W(\varphi) = \{\psi \in \mathfrak{C}^1 : \text{there is a sequence } t_n \rightarrow \infty \\ \text{such that } \|v^\varphi(\cdot, t_n) - \psi\|_{\mathfrak{C}^1} \rightarrow 0\}.$$

Theorem A.6 gives conditions on the initial data φ under which wave-dependent sub- and super-solutions for (6) can be constructed. This yields important information about $W(\varphi)$.

LEMMA 5.2. *Let $\hat{\eta} > 0$ be as in Theorem A.6, and let $\varphi \in \mathcal{C}^1$ satisfy (110), (111) for some $\eta \in (0, \hat{\eta})$. Then*

- (i) $W(\varphi)$ is nonempty and compact in \mathcal{C}^1 ,
- (ii) there exists $\hat{x}(\varphi) \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $\psi \in W(\varphi)$,

$$w(x - \hat{x}(\varphi)) \leq \psi(x) \leq w(x + \hat{x}(\varphi)),$$

- (iii) $(\psi)'(x) \rightarrow 0$ as $|x| \rightarrow \infty$ for each $\psi \in W(\varphi)$,
- (iv) if $\psi \in W(\varphi)$, then $v^\psi(\cdot, t) \in W(\varphi)$ for all $t \geq 0$ and $W(\psi) \subset W(\varphi)$.

PROOF. The *a priori* estimates of Theorem A.8, the Arzela–Ascoli theorem and estimate (112) of Theorem A.6 together show (i). Estimate (112) also yields (ii). (iii) follows from (ii), Theorem A.8 and Landau’s inequality on a half-line. (iv) is a consequence of definition (83), the last part of Proposition A.3 and the semigroup property of solutions of (6). \square

The next theorem is the key. We include a proof for completeness; the approach is a minor modification of [13, Lemma 3.4].

THEOREM 5.3. *Let $\varphi \in \mathcal{C}^1$ be as in Lemma 5.2. Then there exists $\psi_0 \in W(\varphi)$, $\psi_0(x) \rightarrow E^\pm$ as $x \rightarrow \pm\infty$, $(\psi_0)'(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and $(\psi_0)'(x) \geq 0$ for each $x \in \mathbb{R}$.*

PROOF. Define $\mathcal{F} : W(\varphi) \rightarrow [0, \infty]$ by

$$(84) \quad \mathcal{F}(\psi) = \inf\{\chi_0 > 0 : \psi(x + \chi) \geq \psi(x) \text{ for all } \chi \geq \chi_0, x \in \mathbb{R}\}.$$

Note that since $W(\varphi) \subset \mathcal{C}$, $\psi(x + \mathcal{F}(\psi)) \geq \psi(x)$ for each $x \in \mathbb{R}$, $\psi \in W(\varphi)$. Lemma 5.2(ii) shows that $\mathcal{F}(\psi) < \infty$ for each $\psi \in W(\varphi)$. It follows from Lemma 5.2(i) that \mathcal{F} attains its minimum \mathcal{F}_0 at a point $\psi_0 \in W(\varphi)$. Lemma 5.2(ii), (iii) ensure that $\psi_0(x) \rightarrow E^\pm$ as $x \rightarrow \pm\infty$ and $(\psi_0)'(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

If $\mathcal{F}_0 = 0$, then $(\psi_0)'(x) \geq 0$ for each $x \in \mathbb{R}$. So suppose, for contradiction, that $\mathcal{F}_0 > 0$. We consider the solution v^{ψ_0} of (6) with initial data ψ_0 . Note first that Lemma 5.2(iv) states that $v^{\psi_0}(\cdot, t) \in W(\varphi)$ for all $t \geq 0$. By the choice of ψ_0 as the minimiser of \mathcal{F} and Theorem A.2, $\mathcal{F}(v^{\psi_0}(\cdot, t)) = \mathcal{F}_0$ for all $t \geq 0$, so

$$v^{\psi_0}(x + \mathcal{F}_0, t) \geq v^{\psi_0}(x, t) \quad \text{for all } x \in \mathbb{R}, t \geq 0.$$

In fact, since $\mathcal{F}_0 > 0$ and Lemma 5.2(ii) holds, there exist $\mu_0 > 0, M_0 > 0$ such that for all $t \geq 0$,

$$(85) \quad \sup_{x \in \mathbb{R}} \left(\min_{1 \leq i \leq N} v_i^{\psi_0}(x + \mathcal{F}_0, t) - v_i^{\psi_0}(x, t) \right) \\ = \max_{|x| \leq M_0} \left(\min_{1 \leq i \leq N} v_i^{\psi_0}(x + \mathcal{F}_0, t) - v_i^{\psi_0}(x, t) \right) \geq \mu_0.$$

Let q_0, ϵ^\pm, ν be as in the preamble to Theorem A.5. By Theorem A.8, $\|v_{xx}^{\psi_0}(\cdot, t)\|_{\mathfrak{C}}$ is bounded independently of $t \geq 1$. So it follows from Lemma 5.2(ii) and Landau's inequality on a half line that $v_x^{\psi_0}(x, t) \rightarrow 0$ as $|x| \rightarrow \infty$ at a rate independent of $t \geq 1$. Thus Lemma 5.2(ii) and (101) give that there exists $M \geq M_0$ such that for all $t \geq 0, \sigma \in [0, 1]$ and each $\mathcal{F} \in [0, \mathcal{F}_0]$,

$$(86) \quad \pm x \geq M \Rightarrow d_q f[\sigma v^{\psi_0}(x + \mathcal{F}, t) + (1 - \sigma)v^{\psi_0}(x, t), \sigma v_x^{\psi_0}(x + \mathcal{F}, t) \\ + (1 - \sigma)v_x^{\psi_0}(x, t)] \mathfrak{e}^\pm \leq -\epsilon^\pm \nu / 2.$$

For $\delta \geq 0$, define

$$(87) \quad \mathfrak{h}^\delta(x, t) = v^{\psi_0}(x + \mathcal{F}_0 - \delta, t) - v^{\psi_0}(x, t), \quad x \in \mathbb{R}, t \geq 0.$$

Then $\mathfrak{h}^0 \geq 0$, and for each $\delta \geq 0, \mathfrak{h}^\delta$ is a solution of

$$(88) \quad \mathfrak{h}_t^\delta(x, t) = A \mathfrak{h}_{xx}^\delta(x, t) + c \mathfrak{h}_x^\delta(x, t) + D^\delta(x, t) \mathfrak{h}_x^\delta(x, t) + G^\delta(x, t) \mathfrak{h}^\delta(x, t),$$

where

$$D^\delta(x, t) = \int_0^1 d_p f[\sigma v^{\psi_0}(x + \mathcal{F}_0 - \delta, t) \\ + (1 - \sigma)v^{\psi_0}(x, t), \sigma v_x^{\psi_0}(x + \mathcal{F}_0 - \delta, t) + (1 - \sigma)v_x^{\psi_0}(x, t)] d\sigma, \\ G^\delta(x, t) = \int_0^1 d_q f[\sigma v^{\psi_0}(x + \mathcal{F}_0 - \delta, t) \\ + (1 - \sigma)v^{\psi_0}(x, t), \sigma v_x^{\psi_0}(x + \mathcal{F}_0 - \delta, t) + (1 - \sigma)v_x^{\psi_0}(x, t)] d\sigma.$$

Since f satisfies (f1) and (f2), the matrices $cI + D^0, G^0$ satisfy the hypotheses on D, G respectively in Lemma 5.1. Also, Theorem A.8 shows that $\mathfrak{h}_t^0(x, t)$ is bounded independently of $x \in \mathbb{R}, t \geq 1/2$. So, with M as in (86), Lemma 5.1 (applied to the function \mathfrak{h}^0) together with (85) imply the existence of $\alpha(M) > 0$ such that for each $t \geq 1$,

$$(89) \quad |x| \leq M \Rightarrow \mathfrak{h}^0(x, t) \geq \alpha(M).$$

Theorem A.8 shows that $(v^{\psi_0})_x(x, t)$ is bounded independently of $x \in \mathbb{R}, t \geq 1$. So there exists $\delta(M) \in (0, \mathcal{F}_0)$ such that for each $t \geq 1$,

$$(90) \quad |x| \leq M, \delta \in [0, \delta(M)] \Rightarrow \mathfrak{h}^\delta(x, t) \geq \alpha(M)/2.$$

Now, by Lemma 5.2(i) and (iv), there is a sequence $t_n \rightarrow \infty$ and $\psi_1 \in W(\psi_0) \subset W(\varphi)$ such that

$$(91) \quad \|v^{\psi_0}(\cdot, t_n) - \psi_1\|_{\mathfrak{C}^1} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Fix $\delta \in [0, \delta(M)]$. (90) and (91) show that $\psi_1(x + \mathcal{F}_0 - \delta) \geq \psi_1(x)$ for all $x \in [-M, M]$. Consider $x \geq M$. Now \mathfrak{h}^δ is uniformly bounded, $\mathfrak{h}^\delta(M, t) \geq \alpha(M)/2$ for $t \geq 1$ and (86) holds. So Theorem A.1(i) (applied to (88)) shows that there is a constant $K^\delta > 0$ such that for all $x \geq M$, $t \geq 1$,

$$(92) \quad \mathfrak{h}^\delta(x, t) \geq -K^\delta e^{-\nu/2t} \mathfrak{e}^+.$$

Whence $\psi_1(x + \mathcal{F}_0 - \delta) \geq \psi_1(x)$ for each $x \geq M$. Similarly, $\psi_1(x + \mathcal{F}_0 - \delta) \geq \psi_1(x)$ for $x \leq -M$. So

$$(93) \quad \psi_1(x + \mathcal{F}_0 - \delta) \geq \psi_1(x) \quad \text{for all } x \in \mathbb{R}, \delta \in [0, \delta(M)].$$

But it follows from (91) and the fact that $\mathcal{F}(v^{\psi_0}(\cdot, t)) = \mathcal{F}_0$ for all $t \geq 0$ that $\mathcal{F}(\psi_1) = \mathcal{F}_0$. This contradicts (93). Thus $\mathcal{F}_0 = 0$ and the result follows. \square

The main result of this paper is the following.

THEOREM 5.4. *Let $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfy (f1)–(f4). Let $\hat{\eta} > 0$ be as in Theorem A.6, and let φ satisfy (110), (111) for some $\eta \in (0, \hat{\eta})$. Then there exists $\chi_\infty \in \mathbb{R}$ such that for each $\varepsilon \in (0, \gamma_0)$, there exists $N_\varepsilon > 0$ such that the solution v^φ of (6) with initial data φ satisfies*

$$(94) \quad \|v^\varphi(\cdot, t) - w(\cdot + \chi_\infty)\|_{\mathfrak{C}^1} \leq N_\varepsilon e^{-\gamma_\varepsilon t} \quad \text{for all } t > 0.$$

PROOF. Theorem 5.3, Theorem 4.1 and Lemma 5.2(iv) show that there exists $\chi_\infty \in \mathbb{R}$ such that $w(\cdot + \chi_\infty) \in W(\varphi)$. The result then follows from Theorem 3.1. \square

This implies a uniqueness result for travelling-wave solutions of (3).

COROLLARY 5.5. *Let $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfy (f1)–(f4). Let $\hat{\eta} > 0$ be as in Theorem A.6, and let $\varphi \in \mathfrak{C}^1$ satisfy (110), (111) for some $\eta \in (0, \hat{\eta})$. Suppose that there exists $\hat{c} \in \mathbb{R}$ such that $u(x, t) := \varphi(x - \hat{c}t)$ is a travelling-wave solution of (3). Then $\hat{c} = c$ and there exists $\chi_\infty \in \mathbb{R}$ such that $\varphi(\cdot) = w(\cdot + \chi_\infty)$. (Here w, c are as in (TW).)*

PROOF. Theorem 5.4 shows that there exists $\chi_\infty \in \mathbb{R}$ such that

$$(95) \quad \|u(\cdot + ct, t) - w(\cdot + \chi_\infty)\|_{\mathfrak{C}^1} = \|\varphi(\cdot + \{c - \hat{c}\}t) - w(\cdot + \chi_\infty)\|_{\mathfrak{C}^1} \rightarrow 0$$

as $t \rightarrow \infty$. Suppose that $c > \hat{c}$. Since $w(x) \rightarrow E^-$ as $x \rightarrow -\infty$, we can choose $\hat{x} \in \mathbb{R}$ such that $w(\hat{x} + \chi_\infty) < E^+ - \hat{\eta}\mathfrak{e}^+$. But since φ satisfies (111) and $c - \hat{c} > 0$, $\varphi(\hat{x} + \{c - \hat{c}\}t) > E^+ - \hat{\eta}\mathfrak{e}^+$ for t sufficiently large. This contradicts (95), so

$c \leq \widehat{c}$. A similar argument shows that $c \geq \widehat{c}$. Whence $c = \widehat{c}$. The result now follows from (95). \square

Appendix

Comparison theorem. For $T > 0$, define

$$\Gamma_T = \{v \in C(\mathbb{R} \times [0, T], \mathbb{R}^N) : v_t, v_x, v_{xx} \text{ are continuous on } \mathbb{R} \times (0, T)\},$$

$$\Gamma_T^+ = \{v \in C([0, \infty) \times [0, T], \mathbb{R}^N) : v_t, v_x, v_{xx} \text{ are continuous on } (0, \infty) \times (0, T)\}.$$

For $v \in \Gamma_T$, $(x, t) \in \mathbb{R} \times (0, T]$ (or $v \in \Gamma_T^+$, $(x, t) \in (0, \infty) \times (0, T)$), define

$$(96) \quad \mathcal{M}(v)(x, t) = -v_t(x, t) + Av_{xx}(x, t) + D(x, t)v_x(x, t) + G(x, t)v(x, t),$$

and

$$(97) \quad \mathcal{N}(v)(x, t) = -v_t(x, t) + Av_{xx}(x, t) + cv_x(x, t) + f(v(x, t), v_x(x, t)),$$

where A satisfies (a), $c \in \mathbb{R}$, $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies (f1)–(f2) and $D, G : \mathbb{R} \times [0, T] \rightarrow M^{N \times N}$ are continuous $N \times N$ matrix-valued functions, bounded on $\mathbb{R} \times [0, T]$, such that D is diagonal and the off-diagonal elements of G are non-negative. [14, Chapter 5, Lemma 5.2 and Theorem 5.3] yield the following positivity results.

THEOREM A.1.

- (i) Let $v \in \Gamma_T^+$ be such that v is bounded on $[0, \infty) \times [0, T]$ and $\mathcal{M}(v)(x, t) \leq 0$ for $(x, t) \in (0, \infty) \times (0, T]$. If $v(x, 0) \geq 0$ for all $x \in \mathbb{R}$ and $v(0, t) \geq 0$ for each $t \in [0, T]$, then $v(x, t) \geq 0$ for all $(x, t) \in [0, \infty) \times [0, T]$.
- (ii) Let $v \in \Gamma_T$ be such that v is bounded on $\mathbb{R} \times [0, T]$ and $\mathcal{M}(v)(x, t) \leq 0$ for $(x, t) \in \mathbb{R} \times (0, T]$. If $v(x, 0) \geq 0$ for all $x \in \mathbb{R}$, then $v(x, t) \geq 0$ for all $(x, t) \in \mathbb{R} \times [0, T]$.

The following comparison principle for (6) is a straightforward consequence of Theorem A.1(ii).

THEOREM A.2. Let $v, \tilde{v} \in \Gamma_T$ be such that $v, \tilde{v}, v_x, \tilde{v}_x$ are bounded on $\mathbb{R} \times (0, T]$, $\mathcal{N}(\tilde{v})(x, t) \leq 0$ and $\mathcal{N}(v)(x, t) \geq 0$ for $(x, t) \in \mathbb{R} \times (0, T]$. Suppose that $\tilde{v}(x, 0) - v(x, 0) \geq 0$ for all $x \in \mathbb{R}$. Then $\tilde{v}(x, t) \geq v(x, t)$ for all $(x, t) \in \mathbb{R} \times [0, T]$.

Global existence and a priori bounds. The abstract existence theory of [11] applies to the concrete problem

$$(98) \quad v_t = Av_{xx} + cv_x + f(v, v_x), \quad x \in \mathbb{R}, \quad t > 0, \quad v(x, t) \in \mathbb{R}^N,$$

$$(99) \quad v(\cdot, 0) = \varphi,$$

where A satisfies (a), $c \in \mathbb{R}$ and $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$.

The local existence of a unique solution of (98), (99) and continuous dependence on the initial data (99) are a consequence of [11, Theorem 7.1.2, Propositions 7.1.9 and 7.1.10 and Remark 7.1.12]. \mathfrak{C}^1 is a suitable choice of space between \mathfrak{C}^2 and \mathfrak{C} for the initial data φ – see [11], the embeddings (19) and (21) and Lemma 2.2. The result is the following.

PROPOSITION A.3. *Let $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ and $\varphi \in \mathfrak{C}^1$. Then there exists a maximal $\tau(\varphi) \in (0, \infty]$ such that there exists a function $V^\varphi \in C^1((0, \tau(\varphi)), \mathfrak{C}) \cap C((0, \tau(\varphi)), \mathfrak{C}^2) \cap C([0, \tau(\varphi)), \mathfrak{C}^1)$ such that v^φ defined by $v^\varphi(x, t) = V^\varphi(t)(x)$ for each $x \in \mathbb{R}$, $t \in [0, \tau(\varphi))$ satisfies (98), (99). Moreover, there is a unique function $V^\varphi : [0, \tau(\varphi)) \rightarrow \mathfrak{C}^1$ with these properties. In addition, given $0 < T < \tau(\varphi)$, there exist $r, K > 0$, depending on φ and T , such that if $\tilde{\varphi} \in \mathfrak{C}^1$ is such that $\|\varphi - \tilde{\varphi}\|_{\mathfrak{C}^1} < r$, then $\tau(\tilde{\varphi}) \geq T$ and*

$$\|v^\varphi(\cdot, t) - v^{\tilde{\varphi}}(\cdot, t)\|_{\mathfrak{C}^1} \leq K \|\varphi - \tilde{\varphi}\|_{\mathfrak{C}^1} \quad \text{for each } 0 \leq t \leq T.$$

Under a growth hypothesis on f , the following global existence result, conditional on an *a priori* bound on $\|v(\cdot, t)\|_{\mathfrak{C}}$, is a consequence of [11, Propositions 7.1.8, 7.1.10 and 7.2.2].

PROPOSITION A.4. *Suppose that $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies the growth condition (f4). Let $\varphi \in \mathfrak{C}^1$ be such that*

$$(100) \quad \sup_{0 \leq \tilde{t} < \tau(\varphi)} \|v^\varphi(\cdot, \tilde{t})\|_{\mathfrak{C}} = K < \infty,$$

where v^φ and $\tau(\varphi)$ are as in Proposition A.3. Then $\tau(\varphi) = \infty$.

Sub- and supersolutions. Theorem A.2 enables verification of condition (100) under additional hypotheses on f and φ . Suppose that $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies (f1)–(f4). Let $e_0 = \min_{1 \leq i \leq N} \{E_i^+ - E_i^-\} > 0$. Conditions (f2)–(f3) and the Perron–Frobenius Theorem together imply the existence of $\nu^+, \nu^- > 0$ and vectors $\mathbf{e}^+, \mathbf{e}^- \in \mathbb{R}^N$, $\mathbf{e}^\pm > 0$, $\|\mathbf{e}^\pm\| = 1$ such that $d_q f[E^\pm, 0]\mathbf{e}^\pm = -\nu^\pm \mathbf{e}^\pm$. Since $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$, it follows that there exist $p_0, \nu > 0$, $q_0 \in (0, e_0/2)$, $\eta_0 \in (0, e_0/2)$ such that

$$(101) \quad \left. \begin{array}{l} q \in \mathbb{R}^N, \|q\| \leq q_0, \\ p \in \mathbb{R}^N, \|p\| \leq p_0, \\ \eta \in (0, \eta_0) \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} d_q f[E^\pm + q - \eta \mathbf{e}^\pm, p]\mathbf{e}^\pm < -\nu \mathbf{e}^\pm, \\ d_q f[E^\pm + q + \eta \mathbf{e}^\pm, p]\mathbf{e}^\pm < -\nu \mathbf{e}^\pm, \\ f(E^\pm + q - \eta \mathbf{e}^\pm, p) - f(E^\pm + q, p) \geq \nu \eta \mathbf{e}^\pm, \\ f(E^\pm + q + \eta \mathbf{e}^\pm, p) - f(E^\pm + q, p) \leq \nu \eta \mathbf{e}^\pm. \end{array} \right.$$

Suppose that (TW) holds. Since $w'(x) \rightarrow 0$ as $|x| \rightarrow \infty$, we can choose q_0, η_0 smaller if necessary to ensure that

$$(102) \quad \|w(x) - E^\pm\| < q_0 + \eta_0 \Rightarrow \|w'(x)\| < p_0.$$

Let $\mathbf{p} \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ be such that $\mathbf{p}_i(q) = \tilde{\mathbf{p}}_i(q_i)$ for each $i \in \{1, \dots, N\}$, $q \in \mathbb{R}^N$ (the i th component of \mathbf{p} depends only on the i th component of its argument), where $\tilde{\mathbf{p}}_i \in C^\infty(\mathbb{R}, \mathbb{R})$ is a smooth monotone function with

$$(103) \quad \tilde{\mathbf{p}}_i(\omega) = \mathbf{e}_i^+ \text{ when } |E_i^+ - \omega| \leq q_0 \quad \text{and} \quad \tilde{\mathbf{p}}_i(\omega) = \mathbf{e}_i^- \text{ when } |E_i^- - \omega| \leq q_0.$$

The following construction of sub- and super-solutions is an extension, to nonlinearities f depending on v_x , of constructions in [7] and [12].

THEOREM A.5. *There exist $\alpha_0 > 0$ and $\hat{\eta} \in (0, \eta_0]$ such that for any $x_0, x_1 \in \mathbb{R}$ and any $\eta \in [0, \hat{\eta}]$,*

$$\mathcal{N}(s_{\eta, x_0})(x, t) \geq 0 \quad \text{and} \quad \mathcal{N}(S_{\eta, x_1})(x, t) \leq 0 \quad \text{for all } x \in \mathbb{R}, t \geq 0,$$

where for each $i \in \{1, \dots, N\}$,

$$(104) \quad (s_{\eta, x_0})_i(x, t) = w_i(x - x_0 + \eta\alpha_0 e^{-\nu t}) - \eta e^{-\nu t} \tilde{\mathbf{p}}_i(w_i(x - x_0 + \eta\alpha_0 e^{-\nu t}))$$

and

$$(105) \quad (S_{\eta, x_1})_i(x, t) = w_i(x + x_1 - \eta\alpha_0 e^{-\nu t}) + \eta e^{-\nu t} \tilde{\mathbf{p}}_i(w_i(x + x_1 - \eta\alpha_0 e^{-\nu t})).$$

Here the c in (97) is the velocity of the wave w .

PROOF. Let $x_0, x_1 \in \mathbb{R}$ be arbitrary. Let $\alpha_0 > 0$ (to be fixed later), and let $\eta \in (0, \eta_0]$. Define s_{η, x_0} and S_{η, x_1} as in (104) and (105). We will prove the result for s_{η, x_0} ; the argument for S_{η, x_1} is similar.

Unless otherwise indicated, w, w' are to be evaluated at the point $(x - x_0 + \eta\alpha_0 e^{-\nu t})$. Fix $t \geq 0$. First let x be such that $\|E^+ - w(x - x_0 + \eta\alpha_0 e^{-\nu t})\| \leq q_0/2$. For such x , $\tilde{\mathbf{p}}'_i(w_i(x - x_0 + \eta\alpha_0 e^{-\nu t})) = 0$ and $\tilde{\mathbf{p}}_i(w_i(x - x_0 + \eta\alpha_0 e^{-\nu t})) = \mathbf{e}_i^+$ for each $i \in \{1, \dots, N\}$. Hence (101), (102) together with the facts that w is a stationary solution of (6) and that $w'(s) > 0$ for all s yield that

$$\begin{aligned} \mathcal{N}(s_{\eta, x_0})(x, t) &= \nu\eta\alpha_0 e^{-\nu t} w' - \nu\eta e^{-\nu t} \mathbf{e}^+ + f(w - \eta e^{-\nu t} \mathbf{e}^+, w') - f(w, w') \\ &\geq \nu\eta e^{-\nu t} \mathbf{e}^+ - \nu\eta e^{-\nu t} \mathbf{e}^+ = 0. \end{aligned}$$

Similarly, $\mathcal{N}(s_{\eta, x_0})(x, t) \geq 0$ when $\|E^- - w(x - x_0 + \eta\alpha_0 e^{-\nu t})\| \leq q_0/2$.

Now let $x \in \mathbb{R}$ be such that

$$\|E^- - w(x - x_0 + \eta\alpha_0 e^{-\nu t})\| \geq q_0/2 \quad \text{and} \quad \|E^+ - w(x - x_0 + \eta\alpha_0 e^{-\nu t})\| \geq q_0/2.$$

Since $w' > 0$, there exists $\beta > 0$, depending only on w and q_0 , such that for each $i \in \{1, \dots, N\}$,

$$(106) \quad \|E^- - w(s)\| \geq q_0/2 \quad \text{and} \quad \|E^+ - w(s)\| \geq q_0/2 \Rightarrow w'_i(s) \geq \beta.$$

Let $i \in \{1, \dots, N\}$. Since w is a stationary solution of (6),

$$\begin{aligned} \mathcal{N}_i(s_{\eta, x_0})(x, t) &= \nu\eta\alpha_0 e^{-\nu t} w'_i - \nu\eta e^{-\nu t} \tilde{\mathbf{p}}_i(w_i) - \nu\eta^2 \alpha_0 e^{-2\nu t} \tilde{\mathbf{p}}'_i(w_i) w'_i \\ &\quad - \eta e^{-\nu t} A_i[\tilde{\mathbf{p}}''_i(w_i)(w'_i)^2 + \tilde{\mathbf{p}}'_i(w_i)w''_i] - \eta e^{-\nu t} c\tilde{\mathbf{p}}'_i(w_i)w'_i \\ &\quad + f_i(w - \mathbf{p}\eta e^{-\nu t}, w' - \eta e^{-\nu t} d\mathbf{p}[w]w') - f_i(w, w'). \end{aligned}$$

By the Mean Value Theorem and the properties of \mathbf{p} and w ,

$$(107) \quad f_i(w - \mathbf{p}\eta e^{-\nu t}, w' - \eta e^{-\nu t} d\mathbf{p}[w]w') - f_i(w, w') = \mathbf{q}_i(x, t)\eta e^{-\nu t},$$

where $\mathbf{q}_i(x, t)$ is bounded independently of $x \in \mathbb{R}$ and $t \geq 0$. So

$$(108) \quad \begin{aligned} \mathcal{N}_i(s_{\eta, x_0})(x, t) &= \eta e^{-\nu t} \{\mathbf{q}_i(x, t) - \nu\tilde{\mathbf{p}}_i(w_i) \\ &\quad - A_i[\tilde{\mathbf{p}}''_i(w_i)(w'_i)^2 + \tilde{\mathbf{p}}'_i(w_i)w''_i] - c\tilde{\mathbf{p}}'_i(w_i)w'_i\} \\ &\quad + \nu\eta\alpha_0 e^{-\nu t} w'_i \{1 - \eta e^{-\nu t} \tilde{\mathbf{p}}'_i(w_i)\}. \end{aligned}$$

Since $d\mathbf{p}[\cdot]$ is uniformly bounded, there exists $\hat{\eta} \in (0, \eta_0]$ such that

$$(109) \quad \eta \in (0, \hat{\eta}] \Rightarrow 1 - \eta e^{-\nu t} |\tilde{\mathbf{p}}'_i(\omega)| \geq 1/2 \quad \text{for each } \omega \in \mathbb{R}.$$

(We need that $1 + \eta e^{-\nu t} \tilde{\mathbf{p}}'_i(\omega) \geq 1/2$ for the analysis of S_{η, x_1} .) So since w'_i satisfies (106),

$$\begin{aligned} \mathcal{N}_i(s_{\eta, x_0})(x, t) &\geq \eta e^{-\nu t} \{\mathbf{q}_i(x, t) - \nu\tilde{\mathbf{p}}_i(w_i) \\ &\quad - A_i[\tilde{\mathbf{p}}''_i(w_i)(w'_i)^2 + \tilde{\mathbf{p}}'_i(w_i)w''_i] - c\tilde{\mathbf{p}}'_i(w_i)w'_i + \nu\beta\alpha_0/2\}. \end{aligned}$$

Whence we can choose $\alpha_0 > 0$, dependent on \mathbf{p} and w but independent of x and t , such that $\mathcal{N}_i(s_{\eta, x_0})(x, t) \geq 0$. The result follows. \square

THEOREM A.6. *Suppose that $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies (f1)–(f4). Then there exists $\hat{\eta} > 0$ such that if $\varphi \in \mathcal{C}^1$ is such that there exists $\eta \in (0, \hat{\eta})$ with*

$$(110) \quad E^- - \eta\mathbf{e}^- \leq \varphi(x) \leq E^+ + \eta\mathbf{e}^+ \quad \text{for all } x \in \mathbb{R},$$

and

$$(111) \quad \limsup_{x \rightarrow \infty} |\varphi_i(x) - E_i^+| \leq \eta\mathbf{e}_i^+, \quad \limsup_{x \rightarrow -\infty} |\varphi_i(x) - E_i^-| \leq \eta\mathbf{e}_i^-$$

for each $i \in \{1, \dots, N\}$, then $\tau(\varphi) = \infty$, and there exist $x_0(\varphi), x_1(\varphi) \in \mathbb{R}$ such that

$$(112) \quad s_{\hat{\eta}, x_0(\varphi)}(x, t) \leq v^\varphi(x, t) \leq S_{\hat{\eta}, x_1(\varphi)}(x, t) \quad \text{for all } x \in \mathbb{R}, t \geq 0.$$

PROOF. Let $\hat{\eta}$ be as in Theorem A.5, and let $\varphi \in \mathcal{C}^1$ satisfy (110), (111) for some $\eta \in (0, \hat{\eta})$. Now given $x_0, x_1 \in \mathbb{R}$,

$$(113) \quad s_{\hat{\eta}, x_0}(x, 0) = w(x - x_0 + \hat{\eta}\alpha_0) - \hat{\eta}\mathbf{p}(w(x - x_0 + \hat{\eta}\alpha_0)),$$

$$(114) \quad S_{\hat{\eta}, x_1}(x, 0) = w(x + x_1 - \hat{\eta}\alpha_0) + \hat{\eta}\mathbf{p}(w(x + x_1 - \hat{\eta}\alpha_0)),$$

for each $x \in \mathbb{R}$. Recall (103). From (110), (111), (113), (114) and the fact that $\eta < \hat{\eta}$, it follows that there exist $x_0(\varphi), x_1(\varphi) \in \mathbb{R}$ such that

$$(115) \quad s_{\hat{\eta}, x_0(\varphi)}(x, 0) \leq \varphi(x) \leq S_{\hat{\eta}, x_1(\varphi)}(x, 0) \quad \text{for all } x \in \mathbb{R}.$$

This, together with Theorem A.5, allow application of Theorem A.2 to get that

$$(116) \quad s_{\hat{\eta}, x_0(\varphi)}(x, t) \leq v^\varphi(x, t) \leq S_{\hat{\eta}, x_1(\varphi)}(x, t) \quad \text{for all } x \in \mathbb{R}, 0 \leq t < \tau(\varphi).$$

Hence condition (100) is satisfied. The result follows from Proposition A.4. \square

The wave-dependent sub- and super-solutions constructed above are valuable in analysing the stability of the wave w . The following is another, simple but useful, route to verification of condition (100).

THEOREM A.7. *Suppose that $f \in C^1(\mathbb{R}^N \times \mathbb{R}^N, \mathbb{R}^N)$ satisfies (f1)–(f4). Then there exists $\hat{\eta} > 0$ such that if $\varphi \in \mathfrak{C}^1$ is such that there exists $\eta \in [0, \hat{\eta}]$ such that*

$$(117) \quad E^- - \eta \mathbf{e}^- \leq \varphi(x) \leq E^+ + \eta \mathbf{e}^+ \quad \text{for all } x \in \mathbb{R},$$

then $\tau(\varphi) = \infty$, and

$$(118) \quad E^- - \eta \mathbf{e}^- \leq v^\varphi(x, t) \leq E^+ + \eta \mathbf{e}^+ \quad \text{for all } x \in \mathbb{R}, t \geq 0.$$

PROOF. Let $\hat{\eta}$ be as in Theorem A.5 and let $\varphi \in \mathfrak{C}^1$ satisfy (117) for some $\eta \in [0, \hat{\eta}]$. Then since $f(E^+, 0) = f(E^-, 0) = 0$ (by (f3)), it follows from (101) that $f(E^- - \eta \mathbf{e}^-, 0) > 0, f(E^+ + \eta \mathbf{e}^+, 0) < 0$. So with \mathcal{N} as defined in (97), $\mathcal{N}(E^- - \eta \mathbf{e}^-, 0) > 0$ and $\mathcal{N}(E^+ + \eta \mathbf{e}^+, 0) < 0$. It then follows from Theorem A.2 that

$$(119) \quad E^- - \eta \mathbf{e}^- \leq v^\varphi(x, t) \leq E^+ + \eta \mathbf{e}^+ \quad \text{for } 0 \leq t \leq \tau(\varphi).$$

Whence condition (100) is satisfied. The result follows from Proposition A.4. \square

A priori bounds. The derivatives of v^φ can be estimated independently of the exact choice of φ satisfying (117), as follows.

THEOREM A.8. *Let $f, \hat{\eta}$ be as in Theorem A.7 and let $t_0 > 0$. Then there exists $K(t_0) > 0$ such that if $\varphi \in \mathfrak{C}^1$ satisfies (117) for some $\eta \in [0, \hat{\eta}]$, then for all $t \geq t_0$,*

$$(120) \quad \|v^\varphi(\cdot, t)\|_{\mathfrak{C}^2} \leq K(t_0).$$

PROOF. Since f satisfies (f1), (f4) and (118) holds, the single-equation analysis of [10, Chapter V, §3, Theorem 3.1] implies the existence of $K_1(t_0) > 0$ such that $\|v^\varphi(\cdot, t)\|_{\mathfrak{C}^1} \leq K_1(t_0)$ for all $t \geq t_0$. This enables application of [10, Chapter VII, §5, Theorem 5.1] to obtain (120). \square

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ELAINE C. M. CROOKS
 Department of Mathematical Sciences
 University of Bath
 Bath BA2 7AY, UNITED KINGDOM

Current address: Balliol College
 Oxford, OX1 3BJ, UNITED KINGDOM

E-mail address: crooks@maths.ox.ac.uk