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BIFURCATION PROBLEMS FOR SUPERLINEAR ELLIPTIC INDEFINITE EQUATIONS

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Abstract. In this paper, we are dealing with the following superlinear elliptic problem:

$$\text{(P)} \qquad \begin{cases} -\Delta u = \lambda u + h(x)u^p & \text{in } \mathbb{R}^N, \\ u \geq 0, \end{cases}$$

where h is a C^2 function from $\mathbb{R}^N$ to $\mathbb{R}$ changing sign such that $\Omega^+ := \{x \in \mathbb{R}^N \mid h(x) > 0\}$, $\Gamma := \{x \in \mathbb{R}^N \mid h(x) = 0\}$ are bounded.

For $1 < p < (n+2)/(n-2)$ we prove the existence of global and connected branches of solutions of (P) in $\mathbb{R}^- \times H^1(\mathbb{R}^N)$ and in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. The proof is based upon a local approach.

1. Introduction

In this paper, we consider the following superlinear elliptic problem:

$$\text{(P)} \qquad \begin{cases} -\Delta u = \lambda u + h(x)u^p & \text{in } \mathbb{R}^N, \\ u \geq 0. \end{cases}$$

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We suppose that h satisfies the following assumptions:

(H1) $h \in C^2(\mathbb{R}^N, \mathbb{R})$, $\Omega^+ := \{x \in \mathbb{R}^N \mid h(x) > 0\}$ is bounded.

(H2) For all $x \in \Gamma := \{x \in \mathbb{R}^N \mid h(x) = 0\}$, $\nabla h(x) \neq 0$.

Clearly (H1) and (H2) imply that

$$\Gamma = \overline{\Omega^+} \cap \overline{\Omega^-} \quad \text{and it is bounded.}$$

Our purpose is to prove the existence of solutions and to give the structure of solutions set with respect to the bifurcation parameter λ. The method we use involves studying a "local problem", (P_{Ω_R}), in a bounded domain $\Omega_R \supset B_R$ where B_R is the ball centered at 0 and with radius R

$$(\mathrm{P}_{\Omega_R}) \qquad \begin{cases} -\Delta u = \lambda u + h(x)u^p & \text{in } \Omega_R, \\ u \in H_0^1(\Omega_R) & u \geq 0, \end{cases}$$

and then we pass to the limit when R goes to ∞.

To our knowledge, this type of superlinear problems has mainly been investigated when $h(x)$ is strictly positive, see e.g. ([12]) and ([11]). In this direction, we can also cite [2] and [20]. In these two works, the authors consider the global bifurcation problem:

$$(\mathrm{P}_{\Omega}) \qquad \begin{cases} -\Delta u = \lambda u + h(x)u^p & \text{in } \Omega, \\ u(x) = 0 \text{ for all } x \in \partial\Omega & u \geq 0, \end{cases}$$

where Ω is unbounded (precisely $\Omega = \mathbb{R} \times [-\pi/2, \pi/2]$ in [2] and $\Omega = \mathbb{R}^N$ in [20]) and h is strictly positive and it satisfies symmetric assumptions. They prove the existence of a global connected branch which bifurcates from the essential spectrum in $\mathbb{R}^- \times L^\sigma(\mathbb{R}^N)$, with σ depending on N, p and the asymptotic behaviour of h; they use a local approach. The assumptions about symmetry of h yield symmetric properties of solutions of (P_Ω). For the local problem the uniform bounds had been proved by Gidas and Spruck in [14] while, using the symmetry, the compactness of the solutions are obtained studying an ODE.

On the other hand when $h(x)$ changes sign, the nature of the problem is completely different and requires new tools. Let us mention for example the papers of Alama and Tarantello [1] and of Ramos, Terracini and Troestler [19]. The nature of the problem studied in the present paper is closer to the work of Berestycki, Capuzzo Dolcetta and Nirenberg [3]. They use a blow up technique combined with some Liouville theorems in cones, to obtain uniform a priori bounds and some existence results for equation (P_Ω) with Ω a bounded domains for $1 < p < (n+2)/(n-1)$ and $\lambda = 0$. In that paper they ask whether the results were still true for all p subcritical. In [7] Chen and Li answer positively to that question i.e. they obtain some a priori bounds for positive solutions when

p is subcritical (i.e. $p < (N+2)/(N-2)$). Precisely they consider the following problem

$$\begin{cases} -\Delta u = h(x)u^p & \text{in } \Omega, \\ u \in H_0^1(\Omega) & u \geq 0, \end{cases}$$

where h satisfies (H1), (H2) and $\Gamma \subset \Omega$. They prove that every solution is uniformly bounded and that the a priori bound depends only on the geometry of Ω, p and h. The very elegant proof of this result is carried out dividing the domain in three regions and then solving the following steps:

(1) boundedness of solutions in the region where $h(x) \leq -\delta$, for a fixed $\delta > 0$,
(2) boundedness of solutions in the region where $|h(x)|$ is small,
(3) boundedness of solutions in the region where $h(x) \geq \delta$.

Each step involves different techniques:

(1) In the region where $h(x)$ is strictly negative, the uniform estimate is obtained by an Harnack inequality and an integral estimate.
(2) In the region where $|h(x)|$ is small, the a priori bound results from the moving plane technique and from the above estimate.
(3) In the last region, a classical blow up analysis is used in each peak of the solution.

Chen and Li had already used a similar technique to treat the critical case, see [6].

In the present work, we prove the existence of global connected branches of solutions of (P) in $\mathbb{R} \times H^1(\mathbb{R}^N)$ and in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$.

Before describing our results let us mention that to our knowledge global bifurcation in unbounded domains with indefinite non-linearity has only been treated by Cingolani and Gamez in [8] where they consider the following problem:

$$\begin{cases} -\Delta u = \lambda h_1(x)u + h_2(x)u^p & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) & u \geq 0, \end{cases}$$

where h_1, h_2 change sign and among other hypothesis they are in some L^q spaces which ensure the existence of two isolated eigenvalues for the above problem and hence they can use a local approach and prove the existence of a global branch bounded and connected in $\mathbb{R} \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ bifurcating from the two eigenvalues. Let us mention that since they use the result of Berestycki, Capuzzo Dolcetta and Nirenberg the range of p is bounded by $(n+2)/(n-1)$.

Here we consider the branches $\mathcal{C}_R$ of solutions of the problem (P_{Ω_R}) and we want to study their behaviour as R tends to ∞. The existence of $\mathcal{C}_R$ is obtained by the global bifurcation theory of Rabinowitz (see [18]). To give the behaviour of $\mathcal{C}_R$, we need some uniform a priori estimates. Using the main ingredients of [7], we prove that every solution of (P_{Ω_R}) is bounded and the bound is uniform if λ is bounded, which is an extension of the result in [7]. Precisely, concerning (P_{Ω_R}), we show the following main results:

Proposition 1.1. *Suppose that* (H1), (H2) *are satisfied, that* $1 < p < (N+2)/(N-2)$ *and that* Ω_R *is large enough that* $\Gamma \subset \Omega_R$. *Let* $\lambda_1(\Omega^+)$ *be the first eigenvalue to* $-\Delta$ *in* Ω^+. *Then,*

(i) *if* $\lambda \geq \lambda_1(\Omega^+)$, *there are no non trivial solutions of* (P_{Ω_R}),

(ii) *for any* $\lambda_0 \leq \lambda_1(\Omega^+)$, *there is a constant* C $(= C(\lambda_0))$ *such that if* (λ, u) *is a solution of* (P_{Ω_R}) *and* $\lambda \geq \lambda_0$ *then*

$$\|u\|_{H^1, L^\infty} \leq C$$

and C *depends only on* λ_0, Ω_R *and* h.

Consider $\phi_R > 0$ an eigenfunction associated to the first eigenvalue $\lambda_1(\Omega_R)$ which satisfies:

$$\begin{cases} -\Delta\phi_R = \lambda_1(\Omega_R)\phi_R & \text{in } \Omega_R, \\ \phi \geq 0, \end{cases}$$

and let $\Pi_{\mathbb{R}}$ denote the projection onto $\mathbb{R}$.

Theorem 1.2. *Assume that the conditions of Proposition* 1.1 *are satisfied. Then, there is a global branch of nontrivial solutions of* (P_{Ω_R}), $\mathcal{C}_R$, *connected in* $\mathbb{R} \times H^1 \cap L^\infty(\Omega_R)$, *bifurcating from* $(\lambda_1(\Omega_R), 0)$ *such that*

(i) $\Pi_{\mathbb{R}}\mathcal{C}_R = \,]-\infty, \lambda_0]$ *with* $\lambda_1(\Omega^+) > \lambda_0 \geq \lambda^1(\Omega_R)$. *Moreover,*

$$\text{if } \int_{\Omega_R} h(x){\phi_R}^p < 0, \text{ then } \lambda_0 > \lambda_1(\Omega_R).$$

(ii) *Let* $(\lambda_n, u_n) \in \mathcal{C}_R$ *such that* $\lambda_n \to -\infty$ *as* $n \to \infty$. *Then, up to subsequences,* $\|u_n\|_{H^1, L^\infty} \to \infty$.

Passing to the limit the branches $\mathcal{C}_{R_n}$, with $\lim_{n\to\infty} R_n = \infty$, converge to $\mathcal{C}$ a global branch of nontrivial solutions of (P) connected in $\mathbb{R}^- \times H^1(\mathbb{R}^N)$. This process uses the results of Whyburn (see [21]) which ensure that the connectedness of the branches $\mathcal{C}_{R_n}$ are preserved at the limit when $R_n \to \infty$:

Definition 1.1 (Whyburn). Let be G any infinite collection of point sets. The set of all points x such that every neighbourhood of x contains points of infinitely many sets of G is called the *superior limit* of G ($\limsup G$).

The set of all points y such that every neighbourhood of y contains points of all but a finite number of sets of G is called the *inferior limit* of G ($\liminf G$).

Theorem 1.3 (Whyburn). *Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of connected closed sets such that*

$$\liminf\{A_n\} \not\equiv \emptyset.$$

Then, if the set $\bigcup_{n\in\mathbb{N}} A_n$ is relatively compact, $\limsup\{A_n\}$ is a closed, connected set.

We apply Theorem 1.3 as follows: Set $\Lambda < 0$ and let A_n be the connected component (not necessary unique) in

$$\{\Lambda \le \lambda \le 1/\Lambda\} \times H^1(\mathbb{R}^N) \cap \mathcal{C}_{R_n}$$

such that $\Pi_{\mathbb{R}} A_n = [\Lambda, 1/\Lambda]$.

Proving that $\bigcup_{n\in\mathbb{N}} A_n$ is relatively compact in $\mathbb{R} \times H^1(\mathbb{R}^N)$ and applying Theorem 1.3, we obtain that $\limsup_{n\to\infty} A_n = \mathcal{C}_\Lambda$ is a connected set of nontrivial solutions of (P) in $\mathbb{R} \times H^1(\mathbb{R}^N)$. Passing to the limit $\Lambda \to -\infty$, we prove that $\mathcal{C} := \lim_{\Lambda\to-\infty} \mathcal{C}_\Lambda$ is a global branch of nontrivial solutions of (P).

The important step in this process is to prove that the a priori bound, proved in Proposition 1.2, for solutions of $(\mathrm{P}_{\Omega_{R_n}})$ does not depend on Ω_{R_n}.

The main results are the following:

Theorem 1.4. *Assume that* (H1), (H2) *are satisfied. Then, there exists $\mathcal{C}$, a global branch of nontrivial solutions of* (P)*, connected in $\mathbb{R}^- \times H^1(\mathbb{R}^N)$ such that*

- (i) $\Pi_{\mathbb{R}}\mathcal{C} = \,]-\infty, 0[$, $\mathcal{C} \subset \mathbb{R}^- \times L^\infty(\mathbb{R}^N)$.
- (ii) *Taking $(\lambda_n, u_n) \in \mathcal{C}$ such that $\lambda_n \to 0$, then, up to subsequences, $u_n \to u$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ where $(0, u)$ is a solution of* (P)*. Consequently, $\mathcal{C}$ is connected and closed in $\mathbb{R} \times \mathcal{D}^{1,2}(\mathbb{R}^N)$ and it is still imbedded in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$.*
- (iii) *If $(\lambda, u_\lambda) \in \mathcal{C}$ and $\lambda \to -\infty$ then $\|u_\lambda\|_{H^1, L^\infty} \to \infty$.*

Working in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$, we add the following assumption concerning the asymptotic behaviour of h:

(H3) $h(x) \to -\infty$ when $|x| \to \infty$.

Then, we prove the existence of a global branch unbounded and connected in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$, bifurcating to the right from the bottom of the essential spectrum:

Theorem 1.5. *Assume that* (H1)–(H3) *are satisfied. Then, there exists $\mathcal{C}$, a global branch of nontrivial solutions of* (P)*, connected in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$ and bifurcating from the bottom of the essential spectrum* (*i.e. from* $(0, 0)$) *such that*

- (i) $\Pi_{\mathbb{R}}\mathcal{C} = \,]-\infty, \lambda_0]$ *with* $0 < \lambda_0 \le \lambda_1(\Omega^+)$.
- (ii) *If $(\lambda, u_\lambda) \in \mathcal{C}$ with $\lambda < 0$ (resp. $\lambda \le 0$) then $u_\lambda \in H^1(\mathbb{R}^N)$ (resp. $u_\lambda \in \mathcal{D}^{1,2}(\mathbb{R}^N)$).*
- (iii) $\|u_\lambda\|_{H^1, L^\infty} \to \infty$ *when $\lambda \to -\infty$.*

Remark. The previous results still hold true if we replace the non linear term $h(x)u^p$ of equations (P) and (P_{Ω_R}) with a more general nonlinearity such as $h(x)g(u)$ under the following hypothesis on $g(u)$

- (G1) g is in $C^1(\mathbb{R},\mathbb{R}^+)$.
- (G2) There exists $1 < p < (N+2)/(N-2)$ such that $\lim_{u\to\infty} g(u)/u^p = C > 0$.
- (G3) There exists a constant C_1 such that $\liminf_{u\to 0} g(u)/u^p = C_1 > 0$.
- (G4) $0 \le g'(s) \le pg(s)$ for any $s \in \mathbb{R}^+$.

The proofs need only be slightly modified, but for the sake of simplicity we have decided to write them for u^p (which is the standard example of a function satisfying (G1)–(G4)).

The outline of the paper is the following: In Section 2, we study the local problem and prove Proposition 1.1 and Theorem 1.2. In Section 3, we deal with Problem (P), passing to the limit the branches $\mathcal{C}_{R_n}$ in $\mathbb{R}^- \times H^1(\mathbb{R}^N)$ (resp. $\mathbb{R}^- \times L^\infty(\mathbb{R}^N)$) we prove Theorem 1.4 (resp. Theorem 1.5).

2. Local problem

In this section, we are dealing with the following local problem:

$$(\mathrm{P}_{\Omega_R}) \qquad \begin{cases} -\Delta u = \lambda u + h(x)u^p & \text{in } \Omega_R, \\ u \in H_0^1(\Omega_R) & u \ge 0, \end{cases}$$

where h satisfies (H1)–(H2).

Remark. The result of this section holds for any regular bounded domain Ω, we have called it Ω_R (with the hypothesis that it contains a ball of radius R) to emphasize the fact that the result in the local problem will be used in the next section to obtain result in the global problem (P).

Our goal is to prove the existence of a global branch of nontrivial solutions of (P_{Ω_R}) connected in $\mathbb{R}^\times H_0^1 \cap L^\infty(\Omega_R)$ and bifurcating from the first eigenvalue $\lambda_1(\Omega_R) := \inf_{v\in H_0^1(\Omega_R)} \int |\nabla v|^2/\int |v|^2$ and give the global behaviour of the branch. For this, we use the global bifurcation theory of Rabinowitz recalled below, which ensures that the branch of positive solutions bifurcates from the first eigenvalue $\lambda_1(\Omega_R)$ and is unbounded:

Theorem 2.1 (Rabinowitz, 1971). *Let E be a real Banach space with norm $\|\cdot\|$ and consider $G(\lambda,\cdot) = \lambda L\,\cdot + H(\lambda,\cdot)$ where L is a compact linear map on E and $H(\lambda,\cdot)$ is compact and it satisfies $\lim_{\|u\|\to 0} \|H(\lambda,u)\|/\|u\| = 0$. If $r(L) = \{\mu \in \mathbb{R} \mid 1/\mu$ is an eigenvalue of L with odd multiplicity$\}$ and $\mu \in r(L)$, then*

$$S = \overline{\{(\lambda,u) \in \mathbb{R}\times E \mid (\lambda,u) \text{ is a nontrivial solution of } u = G(\lambda,u)\}}$$

possesses a maximal continuum (i.e. connected branch) of solutions, $\mathcal{C}_\mu$, such that $(\mu, 0) \in \mathcal{C}_\mu$ and either

(i) *$\mathcal{C}_\mu$ meets infinity in $\mathbb{R} \times E$, or*
(ii) *$\mathcal{C}_\mu$ meets $(\widehat{\mu}, 0)$ where $\mu \neq \widehat{\mu} \in r(L)$.*

To give information concerning the global behaviour of the branch, we need some a priori estimates about solutions of (P_{Ω_R}). This is done in Proposition 1.1. The following proof follows the main steps of [7]:

PROOF OF PROPOSITION 1.1. First, we prove (i). We use a standard argument for superlinear elliptic problems. Consider ϕ_{Ω^+} a positive eigenfunction associated to $\lambda_1(\Omega^+)$: Multiply (P_{Ω_R}) by ϕ_{Ω^+} and integrate by parts in Ω^+, we obtain:

$$\lambda_1(\Omega^+) \int_{\Omega^+} u\phi_{\Omega^+} + \int_{\partial\Omega^+} \frac{\partial \phi_{\Omega^+}}{\partial n} u = \int_{\Omega^+} h(x) u^p \phi_{\Omega^+} + \lambda \int_{\Omega^+} u\phi_{\Omega^+}. \tag{2.1}$$

From (2.1) and Hopf lemma:

$$(\lambda_1(\Omega^+) - \lambda) \int_{\Omega^+} u\phi_{\Omega^+} \geq \int_{\Omega^+} h(x) u^p \phi_{\Omega^+} > 0$$

which implies that $\lambda < \lambda_1(\Omega^+)$.

Now, let us prove the (ii). Since from standard regularity results the L^∞ bound implies an H^1 bound, it is enough to prove that the L^∞ bound is independent of u. As in [7], we divide the domain Ω_R in three regions:

(1) $\Omega^-_{\delta,R} := \Omega^-_R \cap \{x, \mathrm{dist}(x, \Gamma) \geq \delta > 0\}$.
(2) $\{x / \mathrm{dist}(x, \Gamma) \leq \delta\}$.
(3) $\Omega^+_\delta := \Omega^+ \cap \{x, \mathrm{dist}(x, \Gamma) \geq \delta > 0\}$.

To prove (ii), we will show that in each region the solution is uniformly bounded.

Step 1. A priori bound in $\Omega^-_{\delta,R}$.

As in [7], the argument is based upon an integral estimate. First consider $1 \geq \phi > 0$ an eigenfunction associated to the first eigenvalue $\lambda_1(\Omega_R)$ which satisfies:

$$\begin{cases} -\Delta\phi = \lambda_1(\Omega_R)\phi & \text{in } \Omega_R, \\ \phi \geq 0. \end{cases}$$

Then, multiplying the equation in (P_{Ω_R}) by $\phi^\alpha |h(x)|^{\alpha-1} h(x)$, we obtain:

$$\begin{aligned} \int_{\Omega_R} (-\Delta u)\phi^\alpha |h|^{\alpha-1} h &= \int_{\Omega_R} \nabla u \nabla(\phi^\alpha |h|^{\alpha-1} h) = -\int_{\Omega_R} u\Delta(\phi^\alpha |h|^{\alpha-1} h) \\ &\leq C(\alpha, \phi, \nabla^2 h) \int_{\Omega_R} |h|^{\alpha-2} \phi^{\alpha-2} u. \end{aligned} \tag{2.2}$$

From (2.2) we have

$$\int_{\Omega_R} (\lambda u + h(x) u^p) \phi^\alpha |h|^{\alpha-1} h \leq C \int_{\Omega_R} |h|^{\alpha-2} \phi^{\alpha-2} u. \tag{2.3}$$

From which it follows:

$$\begin{aligned}\int_{\Omega_R} |h|^{\alpha+1}\phi^\alpha u^p &\le C(\lambda,h)\int_{\Omega_R} |h|^{\alpha-2}\phi^{\alpha-2}u \\ &\le C(\lambda,h)\bigg(\int_{\Omega_R} |h|^{p(\alpha-2)}\phi^\alpha u^p)^{1/p}\bigg(\int_{\Omega_R}\phi^{\alpha-2p/(1-p)}\bigg)^{p-1/p}.\end{aligned}$$

Now, choosing $\alpha = 1+2p/(p-1) > 2$, i.e. $p(\alpha-2) = \alpha+1$ we obtain

$$\int_{\Omega_R} |h|^{\alpha+1}\phi^\alpha u^p \le C(\lambda,h,\phi).$$

For any $y \in \mathbb{R}^N$ such that $B_{2\varepsilon}(y) \subset \Omega^-_{\delta,R}$ for a $\varepsilon > 0$, then we obtain

$$\int_{B_{2\varepsilon}(y)} u^p \le C := C(\delta,h,\phi,\lambda). \tag{2.4}$$

Observe that (2.4) can be obtained similarly for $B_{2\varepsilon}(y) \subset \Omega^+_\delta$.

At this point, we have two possibilities either $-\Delta u(x) \ge 0$ or $-\Delta u(x) \le 0$. In the first case, we have just to remark that we have

$$-\Delta u = \lambda u + h(x)u^p \ge 0 \Rightarrow u \le \bigg(\frac{|\lambda|}{\inf|h|}\bigg)^{1/(p-1)} \tag{2.5}$$

and since $\inf|h| > 0$ in $\Omega^-_{\delta,R}$, we get the a priori bound.

Now, if the second case occurs, we use the following lemma (Lemma 9.20 in [15]).

Lemma 2.2. *Let $u \in W^{2,n}(\Omega)$ with $Lu \ge f$ where L is a strictly elliptic second order operator and $f \in L^n(\Omega)$. For all $B = B_{2\varepsilon}(y) \subset \Omega$ and $p > 0$, we have:*

$$\sup_{B_\varepsilon(y)} u \le C(n,p)\bigg(\bigg(\frac{1}{|B|}\int_B (u^+)^p\bigg) + \frac{\varepsilon}{\lambda_L}\|f\|_{L^n(B)}\bigg). \tag{2.6}$$

Now, combining (2.4) and (2.6) with $f = 0$ and $L = \Delta$ we get the a priori bound in the second case. Finally, by (2.4)–(2.6), we have:

$$\sup u \le m := m(\delta,h,\lambda,\Omega_R) \quad \text{in } \Omega^-_{\delta,R}.$$

Step 2. A priori bound in a neighbourhood of Γ.

First, fix $x_0 \in \Gamma$. Since Γ is compact, it is sufficient to give an a priori bound in a neighbourhood of x_0. The sketch of the proof is the following:

(1) First, through some transformations letting the equation invariant, we construct a convex neighbourhood of x_0.
(2) Applying in this domain the moving plane method to an auxiliary function (similar to [7]), we show a "Harnack inequality" satisfied by u in a cone with x_0 as vertex. Combining this inequality with the same integral estimate (2.4), we get the a priori bound.

A strict convex neighbourhood of x_0. Up to some rotation or translation, we can suppose that $x_0 = 0$ and that Γ is tangent to the hyperplane $x_1 = 0$. Doing a Kelvin transform (take the center of the inversion on the x_1-axis such that the sphere is tangent to $x_1 = 0$), we can suppose that Γ is strictly convex in a neighbourhood of x_0 and Ω^+ is at the left of Γ. Let

$$x_1 = \Phi(y), \quad y \in \mathbb{R}^{N-1}$$

be an equation of Γ in a neighbourhood of 0. Consider D the domain (containing x_0) enclosed by the surfaces $\partial^1 D := \{x \mid x_1 = \Phi(y) + \varepsilon\}$ and $\partial^2 D := \{x \mid x_1 = -2\varepsilon\}$. We choose ε small enough to ensure

(a) $\frac{\partial h}{\partial x_1}(x) \leq -\beta_0$ for all $x \in D$,
(b) $-\Delta\phi(y) \leq -\beta_0$.

for some positive constant β_0. But, contrary to the case of [6], the equation is not preserved by Kelvin transform. Indeed, define $\overline{u}(x) = |x - y_0|^{N-2} u(y_0 + |y_0|(x - y_0)/|x - y_0|^2)$ where y_0 is the center of the inversion. Then $\overline{u}$ satisfies the following equation

$$-\Delta\overline{u} = \frac{\lambda\overline{u}}{|x - y_0|^4} + \widetilde{h}(x)\overline{u}^p \tag{2.7}$$

where $\widetilde{h}(x) = |x - y_0|^{N+2-p(N-2)} h(y_0 + |y_0|(x - y_0)/|x - y_0|^2)$. Since $y_0 > 0$ and the origin is invariant by this Kelvin transform, for notation convenience, we will denote $\widetilde{h}$ by h. Observe that in a neighbourhood of 0, the equation in (2.7) is not singular.

Using the results in Step 1, and since $\partial^1 D \subset \Omega^-_{\varepsilon,R}$ let

$$m = m(\varepsilon, \Omega_R, h, \lambda) := \sup_{\partial^1 D} u \tag{2.8}$$

and let $\widetilde{u}$ be a continuation of u on all ∂D such that $\widetilde{u} \leq Cm$.

We are now ready to apply the moving plane method to some auxiliary function.

Moving plane method and Harnack Inequality. We consider the function w solution of

$$\begin{cases} -\Delta w = \dfrac{\lambda w}{|x - y_0|^4} + \lambda C_0 m \dfrac{(-x_1 + \varepsilon + \Phi(y))}{|x - y_0|^4} & \text{in } D, \\ w = \widetilde{u} & \text{for } x \in \partial D, \end{cases} \tag{2.9}$$

where C_0 is a constant to be fixed later and m is defined in (2.8). We introduce the auxiliary function v:

$$v = u - w + C_0 m(\varepsilon + \Phi(y) - x_1) + m(\varepsilon + \Phi(y) - x_1)^2. \tag{2.10}$$

From (2.10), one can see that v satisfies:

$$\begin{cases} \Delta v + \dfrac{\lambda v}{|x-y_0|^4} + \psi(y) + f(x,v) = 0 & \text{in } D, \\ v(x) = 0 & \text{on } \partial^1 D, \end{cases} \tag{2.11}$$

where

$$\psi(y) = C_0 m \Delta\Phi(y) - \Delta(m(\varepsilon+\Phi(y))^2) - 2m$$

and

$$\begin{aligned} f(x,v) = m\bigg(&\frac{\lambda(\varepsilon+\Phi(y)-x_1)^2}{|x-y_0|^4} + 2x_1\Delta\Phi(y)\bigg) \\ &+ h(x)(v + w - C_0 m(\varepsilon+\Phi(y)-x_1) - m(\varepsilon+\Phi(y)-x_1)^2)^p. \end{aligned}$$

We claim that $v \geq 0$. From (2.11), it is sufficient to prove that $\partial v/\partial x_1 \leq 0$ in D. Clearly, by the definition of v,

$$\frac{\partial v}{\partial x_1} = \frac{\partial u}{\partial x_1} - \frac{\partial w}{\partial x_1} - C_0 m - 2m(\varepsilon+\Phi(y)-x_1). \tag{2.12}$$

We are going to estimate $\partial w/\partial x_1$ in D. For this, using Theorem I.3 in [4] which extends the Alexandrov–Bakelman estimate (see also [5]) for narrow domains and since $0 \leq -x_1 + \Phi(y) + \varepsilon \leq \varepsilon$, we can prove that

$$\|w\|_{L^\infty} \leq \sup_{\partial D} w + K|D|C_0 m \leq C(m + C_0 m\varepsilon^2) \tag{2.13}$$

In order to apply Theorem I.3 in [4] we will choose ε small enough that $\lambda_1(D) > \sup_{x\in D} \lambda/|x-y_0|^4$. From (2.13) and Theorem 8.33 in [15] and eventually taking ε smaller, we have:

$$\bigg\|\frac{\partial w}{\partial x_1}\bigg\|_{L^\infty(D)} \leq \sup_{\partial D}\bigg|\frac{\partial \widetilde{u}}{\partial x_1}\bigg| + C(m + C_0 m\varepsilon^2) \leq Cm(C_0\varepsilon^2 + 1). \tag{2.14}$$

Let us first suppose that $x \in D_\varepsilon := \{x \mid -x_1 + \Phi(y) + \varepsilon \leq \varepsilon/2\}$ then $x \in \Omega^-_{\varepsilon/2,R}$. Therefore, by the estimates obtained in the previous step and by standard elliptic estimates, we have:

$$\sup_{x\in D_\varepsilon}\bigg|\frac{\partial u}{\partial x_1}\bigg| \leq Cm. \tag{2.15}$$

From (2.14) and (2.15), it follows that:

$$\frac{\partial v}{\partial x_1} \leq Cm + Cm(C_0\varepsilon^2 + 1) - C_0 m \leq 0 \tag{2.16}$$

for C_0 large enough. Combining (2.16) with $v = 0$ for x in $\partial^1 D$, we obtain $v \geq 0$.

On the other hand, if $-x_1 + \Phi(y) + \varepsilon \geq \varepsilon/2$, then

$$v \geq -w + C_0 m\varepsilon/2 \geq -(Cm + \varepsilon^2 C_0 m) + C_0 m\varepsilon/2 \geq -Cm + C_0 m(\varepsilon/2 - \varepsilon^2) \geq 0,$$

choosing $\varepsilon < 1/2$ and C_0 large enough. This proves that $v \geq 0$ in D.

Since $v \geq 0$ and $v = 0$ in $\partial^1 D$, we can apply the moving plane method. Let us define

$$\Sigma_\mu = \{x \in D \mid x_1 \geq \mu\}, \quad T_\mu = \{x \in D \mid x_1 = \mu\},$$

and let x_μ be the reflected point by T_μ of x and $v_\mu(x) = v(x_\mu)$. We want to show that $v(x_\mu) \geq v(x)$ for $x \in \Sigma_\mu$ and $\mu \geq -\varepsilon_1$ with $0 < \varepsilon_1 < \varepsilon$. This is obviously satisfied for $\mu = \varepsilon$. Therefore, we decrease μ and we move the plane T_μ towards the left.

A standard argument (see for instance [16]) can prove that this moving planes procedure can be carried on provided

$$f(x, v(x)) \leq f(x^\mu, v(x)) \quad \text{for } x = (x_1, y) \in D,\ x_1 > \mu > -\varepsilon_1. \tag{2.17}$$

It is easy to see that (2.17) holds if

$$\frac{\partial f}{\partial x_1}(x, v) \leq 0 \quad \text{for all } x \in \{x \mid x_1 \geq -2\varepsilon_1\} \cap D.$$

A simple computation yields:

$$\frac{\partial f}{\partial x_1} = 2m\Delta\Phi(y) + \lambda m \frac{\partial}{\partial x_1}(\varepsilon + \Phi(y) - x_1)^2 + |u|^{p-1}\left(\frac{\partial h}{\partial x_1} u + ph(x)\frac{\partial u}{\partial x_1}\right).$$

Choosing ε small enough, we have that $\Delta\Phi(y) + \lambda\partial(\varepsilon + \Phi(y) - x_1)^2/\partial x_1 \leq -\beta_0/2$. We consider now two cases.

First, $h(x) \leq 0$. In this case, since $\partial h/\partial x_1 \leq 0$ in D, it suffices to prove that $\partial u/\partial x_1 \geq 0$. Choosing C_0 large enough and taking into account (2.14), we have that

$$\frac{\partial u}{\partial x_1} = \frac{\partial w}{\partial x_1} + C_0 m + \varepsilon m(\varepsilon + \Phi(y) - x_1) \geq 0.$$

Now, let us consider the case where $h(x) > 0$. Suppose, that $u \geq 1$. To control $\partial f/\partial x_1$, we use the term $\partial h/\partial x_1 u \leq -\beta_0 u$ and the fact:

$$h(x) \leq C\varepsilon_1 \quad \text{for } -\varepsilon_1 < x < 0. \tag{2.18}$$

Indeed, from (2.18) and (2.14), one can prove

$$\begin{aligned}
\frac{\partial f}{\partial x_1} &\leq \left(\frac{\partial h}{\partial x_1} u + ph(x)\frac{\partial h}{\partial x_1} u\right) u^{p-1} \\
&\leq u^{p-1}\left(-\beta_0 + pC\varepsilon_1\left(\frac{\partial w}{\partial x_1} + C_0 m + 2m\varepsilon C\right)\right) \\
&\leq u^{p-1}(-\beta_0 + pC\varepsilon_1(Cm(C_0\varepsilon^2 + 1) + C_0 m + 2m\varepsilon C)) \leq 0
\end{aligned}$$

for ε_1 small enough. Finally, if $u \leq 1$, we use

$$\lambda m\left(\frac{\partial}{\partial x_1}(\varepsilon + \Phi(y) - x_1)^2\right) + m\Delta\Phi(y) \leq -\frac{\beta_0 m}{2}$$

from which, together with (2.18),

$$\frac{\partial f}{\partial x_1} \le -\frac{\beta_0 m}{2} + ph(x)u^{p-1}\frac{\partial u}{\partial x_1} \le -\frac{\beta_0 m}{2} + pC(\varepsilon, C_0)\varepsilon_1 \le 0$$

for ε_1 small enough. Therefore,

$$v(x_\mu) \ge v(x) \quad \text{for } -\varepsilon_1 \le \mu \le \varepsilon. \tag{2.19}$$

At this point, we conclude as in [7] (Section 3, Step 4: deriving the a priori bound). Let us just sketch the proof.

Inequality (2.19) implies that the function v is monotone decreasing in the x_1 direction. Clearly this is still true if we rotate the x_1-axis by a small angle. Therefore, for any $x_0 \in \Gamma$, there exists, Δ_{x_0}, a cone of vertex x_0 and staying to the left of x_0 such that

$$v(x) \ge v(x_0) \quad \text{for } x \in \Delta_{x_0}. \tag{2.20}$$

From (2.20) we obtain

$$u(x) + C \ge u(x_0) \quad \text{for } x \in \Delta_{x_0}. \tag{2.21}$$

By a similar argument, one can prove that (2.21) is true for any point x in a small neighbourhood of Γ. Remarking that the intersection of Δ_{x_0} with the set $\{x \mid h(x) \ge \delta_0 > 0\}$ has a positive measure, and combining with the integral estimate (2.4) we get the a priori bound in the neighbourhood of Γ.

Step 3. The a priori bound in the region where $h(x) > \delta > 0$. In this region, the a priori bound is obtained by a technique of blow up introduced in [14] and used in [3], [7]. Since the linear term (i.e. λu) vanishes in the blow-up analysis (See [3] for more details), the proof is as in [7] (see particularly p. 339–340). The proof of Proposition 1.1 is now completed. □

Remark. The a priori bound for solutions of (P_{Ω_R}), obtained in Proposition 1.1, depends only on λ, Ω_R, h.

We give now the proof of Theorem 1.2 which follows from Proposition 1.1 and Theorem 2.1.

Proof of Theorem 1.2. The existence of $\mathcal{C}_{\Omega_R}$ follows immediately from Theorem 2.1 and assertion (i) of Proposition 1.1. The fact that there is only one bifurcation point for positive solutions excludes (ii) of Theorem 2.1.(ii) on the other hand implies that $\mathcal{C}_{\Omega_R}$ meets infinity only for λ going to minus infinity.

The fact that $\int_{\Omega_R} h(x)\phi_R{}^p < 0$ implies that $\lambda_0 > \lambda_1(\Omega_R)$ is proved in [17], see also [1].

Let us prove (ii). Letting $\lambda_n \to -\infty$, we have

$$\int_{\Omega_R} |\nabla u_n|^2 + |\lambda_n| \int_{\Omega_R} |u_n|^2 = \int_{\Omega_R} h(x)u_n^{p+1}. \tag{2.22}$$

By Sobolev embedding and the boundedness of Ω^+, it follows that

$$C\|u_n\|^2_{L^{p+1}(\Omega_R)} \le \int_{\Omega_R} |\nabla u_n|^2 \le \|h\|_{L^\infty(\Omega^+)} \|u_n\|^{p+1}_{L^{p+1}(\Omega_R)},$$

which implies that

$$\|u_n\|_{L^{p+1}} \ge C > 0. \tag{2.23}$$

Therefore, if $\|u_n\|_{L^\infty, H^1} \le C$ which implies $\|u_n\|_{L^{p+1}} \le C$, then, from (2.22) and by Hölder inequality,

$$\|u_n\|_{L^2}, \|u_n\|_{L^{p+1}} \to 0 \quad \text{when } n \to \infty$$

which contradicts (2.23). The proof of Theorem 1.2 is now completed. □

3. Global bifurcation for problem (P)

3.1. A priori pound independent of Ω_R. In this section, we prove Theorems 1.4 and 1.5. For this, we need uniform a priori estimates which do not depend on the domain Ω_R. This is proved in the following result for $\lambda \le 0$:

Proposition 3.1. *Assume the* (H1)–(H3) *are satisfied. Let*

$$\Omega_n := \Omega_{R_n} \quad \textit{a sequence of domains such that } R_n \to \infty \textit{ when } n \to \infty.$$

Let $\{(\lambda_n, u_n)\}_{n\in\mathbb{N}}$ *solutions respectively of* (P_{Ω_n}) *such that* $\Lambda \le \lambda_n \le 0$ *for some* $\Lambda < 0$*. Then,*

$$\|u_n\|_{L^\infty(\mathbb{R}^N)} \le C(\Lambda)$$

where C *does not depend on* Ω_n*.*

Proof of Proposition 3.1. Let M_n such that $\|u_n\|_{L^\infty} = M_n$. Then, there exists x_n such that $u_n(x_n) = M_n$ and $-\Delta u_n(x_n) \ge 0$. Since $\lambda_n \le 0$,

$$x_n \in \Omega^+ \cup \Gamma. \tag{3.24}$$

Since the bound in $\Omega^+_{R_n,\delta}$ can be obtained exactly as in the proof of Proposition 1.1 we just have to prove that u_n remains bounded in a neighbourhood of Γ when $n \to \infty$ this can be done using an approach similar to Proposition 1.1 since Γ is compact.

But observe that the uniform bound in a neighbourhood of Γ requires to have an a priori bound on $\Omega^-_{n,\delta} := \Omega^-_n \cap \{x \mid \operatorname{dist}(x,\Gamma) \ge \delta\}$ since it will imply that m and hence ε_1 constructed in the second step are bounded. To estimate u_n in $\Omega^-_{n,\delta}$, let us consider $\varepsilon > 0$ and ϕ_ε non trivial solution of:

$$\begin{cases} -\Delta\phi = \lambda_1(\varepsilon)\phi & \text{in } B_{4\varepsilon}, \\ \phi \equiv 0 & \text{in } \partial B_{4\varepsilon},\ \phi \ge 0, \end{cases}$$

where $B_{4\varepsilon}$ is the ball centered in 0 with radius 4ε. Consider now x_0 such that $B_{4\varepsilon}(x_0) \subset \{x \mid h(x) \leq -\delta_0\}$ for a $\delta_0 > 0$. Let $\phi_{x_0,\varepsilon} = \tau_{-x_0}\phi_\varepsilon$, we still denote ϕ_ε. Multiply the equation in (P_{Ω_n}) by $|h|^{\alpha-1}h\phi_\varepsilon^\alpha$ for some $\alpha > 2$ that will be fixed later, we obtain (since $\alpha > 1$):

$$\begin{aligned}(3.25)\qquad -\int_{B_{4\varepsilon}(x_0)} \Delta u(|h|^{\alpha-1}h\phi_\varepsilon^\alpha) &= -\int_{B_{4\varepsilon}(x_0)} u\Delta(|h|^{\alpha-1}h\phi_\varepsilon^\alpha)\\ &\leq C\int_{B_{4\varepsilon}(x_0)} u|h|^{\alpha-2}\phi_\varepsilon^{\alpha-2}.\end{aligned}$$

Fixing in (3.25) $\alpha = 1 + 2p/(p-1) > 2$ for which $\alpha/p < \alpha - 2$, we have from (3.25) and the equation in (P_{Ω_n}):

$$(3.26)\qquad \int_{B_{4\varepsilon}(x_0)} \phi_\varepsilon^\alpha |h|^{\alpha+1}u^{p+1} \leq C\int_{B_{4\varepsilon}(x_0)} \phi_\varepsilon^{\alpha/p}|h|^{\alpha-2}u.$$

Therefore, from (3.25) and Hölder inequality, we have:

$$(3.27)\qquad \int_{B_{4\varepsilon}(x_0)} \phi_\varepsilon^\alpha |h|^{\alpha+1}u^{p+1} \leq C^{p/(p-1)}|B_{4\varepsilon}|.$$

Then, since $B_{4\varepsilon}(x_0) \subset \{h(x) \leq -\delta_0\}$,

$$(3.28)\qquad \int_{B_{2\varepsilon}(x_0)} u^p \leq C\frac{|B_{4\varepsilon}|}{\delta_0^{1+\alpha}}$$

where C depends on ϕ_ε, N, and Λ. From (3.28) and since $-\Delta u \leq 0$ we can apply Lemma 2.2 (i.e. Lemma 9.20 in [15]), we obtain:

$$(3.29)\qquad \sup_{B_\varepsilon(x_0)} u_n \leq C\bigg(\frac{1}{\varepsilon^N}\int_{B_{2\varepsilon}(x_0)} u^p\bigg) \leq C\frac{|B_{4\varepsilon}|^{1/p}}{\varepsilon^N(\delta_0)^{1+\alpha}}.$$

i.e.

$$(3.30)\qquad \sup_{\Omega_{n,\delta}} u_n \leq C(\delta,\Lambda).$$

Then, we can proceed as in Proposition 1.1 and the proof of Proposition 3.1 is now complete. □

Remark. Proposition 3.1 only concerns the case where $\lambda \leq 0$ (this ensures that the point where the maximum is attained can be chosen in a bounded domain: $\Omega^+ \cup \Gamma$).

In the case where $\lambda > 0$, some a priori estimates independent on the domain can be obtained if we impose some asymptotic behaviour to h. Precisely, we prove:

PROPOSITION 3.2. *Assume the same conditions of Proposition* 3.1 *and now suppose that* $\lambda_n \geq 0$. *Assume in addition that* h *satisfies:*

$$\limsup_{|x|\to\infty} h(x) < 0. \tag{3.31}$$

Then, $\|u_n\|_{L^\infty} \leq C(\Lambda)$.

PROOF OF PROPOSITION 3.2. In the case where $\lambda_n \geq 0$,

$$x_n \in \Omega^+ \cup \Gamma \cup \{x \in \Omega_n^- \mid \lambda_n u_n + h(x)u_n^p \geq 0\}.$$

If $\limsup_{n\to\infty} |x_n| < \infty$, then the proof is similar as for Proposition 3.1. Now, suppose that up to a subsequence,

$$|x_n| \to +\infty \quad \text{and} \quad \lambda_n \to \lambda > 0 \text{ when } n \to \infty.$$

Then, from (3.31),

$$\lambda_n u_n(x_n) + h(x_n)u_n(x_n)^p \geq 0 \Rightarrow u_n(x_n) \leq \left|\frac{\lambda_n}{h(x_n)}\right|^{1/(p-1)} < \infty$$

and the proof is now complete. □

3.2. Connected branch of solutions of (P) **in** $\mathbb{R}^- \times H^1(\mathbb{R}^N)$**.** We give now the proof of Theorem 1.4.

PROOF OF THEOREM 1.4. Let $\Omega_n := \Omega_{R_n}$ where $R_n \to \infty$ when $n \to \infty$ and $\mathcal{C}_n$ the corresponding branch see Theorem 1.2. For any large negative Λ, let A_n^Λ be a connected component of $\mathcal{C}_n$ in $\{\Lambda \leq \lambda \leq 1/\Lambda\} \times H^1(\mathbb{R}^N)$ such that

$$\Pi_{\mathbb{R}} A_n^\Lambda = [\Lambda, 1/\Lambda]$$

(the existence of A_n^Λ follows from the connectedness of $\mathcal{C}_n$ and $\Pi_{\mathbb{R}}\mathcal{C}_n \supset]-\infty, 0]$).

Using Theorem 1.3, we are going to prove that

$$\mathcal{C} := \lim_{\Lambda\to-\infty} \limsup_{n\to\infty} A_n^\Lambda \tag{3.32}$$

satisfies the assertions of Theorem 1.4. To do this, we have to prove that $\bigcup_{n\in\mathbb{N}} A_n^\Lambda$ is relatively compact. Therefore, take

$$(\lambda_n, u_n) \in A_n^\Lambda \quad \text{which implies } \Lambda \leq \lambda_n \leq 1/\Lambda < 0.$$

From Proposition 3.1 we have that

$$\|u_n\|_{L^\infty} \leq C(\Lambda). \tag{3.33}$$

Therefore, by a bootstrap argument and up to subsequences, when $n \to \infty$ the following holds:

$$\lambda_n \to \lambda < 0 \quad \text{and} \quad u_n \to u \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}^N) \tag{3.34}$$

where (λ, u) is a solution of (P). From (3.33), $u \in L^\infty(\mathbb{R}^N)$. Let us show that

$$(\lambda_n, u_n) \to (\lambda, u) \quad \text{in } \mathbb{R} \times H^1(\mathbb{R}^N) \text{ when } n \to \infty. \tag{3.35}$$

First, since (λ_n, u_n) is a solution of (P_{Ω_n}) and from (3.33), one can remark that

$$\int_{\Omega_n} |\nabla u_n|^2 + |\lambda_n| \int_{\Omega_n} |u_n|^2 = \int_{\Omega_n} h(x) u_n^{p+1} \leq \int_{\Omega^+} h(x) u_n^{p+1} \leq C. \tag{3.36}$$

Since $\lambda_n \to \lambda < 0$, it follows that

$$\|u_n\|_{H^1(\mathbb{R}^N)} \leq C \quad \text{and} \quad u_n \to u \text{ weakly in } H^1(R^N) \tag{3.37}$$

which implies (by Sobolev embedding):

$$\int_{\Omega^+} h(x) u_n^{p+1} \to \int_{\Omega^+} h(x) u^{p+1} \quad \text{when } n \to \infty. \tag{3.38}$$

Therefore,

$$\begin{aligned}
\int |\nabla u|^2 + |\lambda| \int |u|^2 &\leq \liminf_{n\to\infty} \int_{\Omega_n} |\nabla u_n|^2 + |\lambda_n| \int_{\Omega_n} |u_n|^2 \\
&\leq \limsup_{n\to\infty} \int_{\Omega_n} |\nabla u_n|^2 + |\lambda_n| \int_{\Omega_n} |u_n|^2 \\
&= \limsup_{n\to\infty} \int_{\Omega_n} h(x) u_n^{p+1} \\
&= \int_{\Omega^+} h(x) u^{p+1} + \limsup_{n\to\infty} \int_{\Omega_n/\Omega^+} h(x) u_n^{p+1} \\
&= \int_{\Omega^+} h(x) u^{p+1} - \liminf_{n\to\infty} \int_{\Omega_n/\Omega^+} |h(x)| u_n^{p+1} \\
&\leq \int_{\Omega^+} h(x) u^{p+1} - \int_{\mathbb{R}^N/\Omega^+} |h(x)| u^{p+1} \\
&= \int_{\mathbb{R}^N} h(x) u^{p+1} = \int |\nabla u|^2 + |\lambda| \int |u|^2.
\end{aligned} \tag{3.39}$$

From which, it follows

$$\lim_{n\to\infty} \int_{\Omega_n} |\nabla u_n|^2 + |\lambda_n| \int_{\Omega_n} |u_n|^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + |\lambda| \int_{\mathbb{R}^N} |u|^2. \tag{3.40}$$

Therefore, (3.35) is proved and $\bigcup_{n\in\mathbb{N}} A_n^\Lambda$ is relatively compact in $\mathbb{R}^- \times H^1(\mathbb{R}^N)$. Furthermore, u is not trivial. Indeed, by Theorem 1.2, since x_n (defined in Propositions 3.1 and 3.2) is uniformly bounded and from (3.33), we obtain:

$$0 < K \leq C(\lambda_n) = \left(\frac{|\lambda_n|}{\|h\|_{L^\infty(\Omega^+ \cup \Gamma)}} \right)^{1/(p-1)} \leq u_n(x_n) \to u(x) \text{ and } u \not\equiv 0. \tag{3.41}$$

Now, using Theorem 1.3, we see that

$$\mathcal{C}_\Lambda := \limsup_{n\to\infty} A_n^\Lambda$$

is connected. Furthermore, the connectedness of $\mathcal{C} := \lim_{\Lambda \to -\infty} \mathcal{C}_\Lambda$ is proved in the same way (i.e. the proof of the compactness of $\bigcup \mathcal{C}_\Lambda$ when $\Lambda \to -\infty$ can be proved as above). This completes the proof of (i).

Observe that since the convergence for $\lambda_n \to 0^-$ of the above sequence u_n is not established, $\mathcal{C}$ is not necessarily closed in $\mathbb{R}^- \times H^1(\mathbb{R}^N)$. But $\mathcal{C}$ is closed in $\mathbb{R}^- \times \mathcal{D}^{1,2}(\mathbb{R}^N)$. To prove this, let

$$(\lambda_n, u_n) \in \mathcal{C} \quad \text{such that} \quad \lambda_n \to 0^-.$$

Then, repeating the argument in (3.39) with $\lambda = 0$, we are done. Moreover u cannot be trivial. Indeed, let $1/q = 1 - (p+1)(N-2)/(2N)$ and remark that

$$\begin{aligned}(3.42)\qquad \int_{\mathbb{R}^N} |\nabla u_n|^2 + |\lambda_n| \int_{\mathbb{R}^N} |u_n|^2 &\le \int_{\Omega^+} h(x) u_n^{p+1} \\ &\le \left(\int_{\Omega^+} h(x)^q \right)^{1/q} \left(\int |u_n|^{2N/(N-2)} \right)^{(p+1)(N-2)/(2N)}.\end{aligned}$$

From (3.42), it follows:

$$\|u_n\|^2_{L^{2N/(N-2)}(\mathbb{R}^N)} \le C \|u_n\|^{p+1}_{L^{2N/(N-2)}(\mathbb{R}^N)}$$

from which we obtain (since $p > 1$):

$$\|u_n\|_{L^{2N/(N-2)}(\mathbb{R}^N)} \ge C. \tag{3.43}$$

This completes the proof of (ii).

Let us prove (iii). To this purpose, we argue by contradiction. Suppose that there exists a sequence $(\lambda_n, u_n) \in \mathcal{C}$ such that:

$$\lambda_n \to -\infty \quad \text{and} \quad \|u_n\|_{H^1(\mathbb{R}^N)} \le C.$$

From (3.36) and by interpolation, we obtain that passing to the limit as $n \to \infty$,

$$\|u_n\|_{L^2(\mathbb{R}^N)} \to 0, \qquad \int_{\Omega^+} h(x) u_n^{p+1} \to 0, \tag{3.44}$$

(3.42) and (3.44) imply that $\|u_n\|_{H^1(\mathbb{R}^N)} \to 0$, which contradicts (3.43). Hence, $\|u_n\|_{H^1(\mathbb{R}^N)} \to \infty$. Now, $\|u_n\|_{L^\infty(\mathbb{R}^N)} \to \infty$ follows from (3.43) and (3.44). This completes the proof of (iii) and Theorem 1.4. □

(1) $\mathcal{C}$ is connected in $\mathbb{R}^- \times H^1(\mathbb{R}^N)$ but not necessarily in $\mathbb{R}^- \times L^\infty(\mathbb{R}^N)$. Notice that we do not impose any asymptotic behaviour of h at infinity.

(2) From (ii), it follows that bifurcation from essential spectrum towards the left cannot occur.

3.3. Bifurcation from essential spectrum in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. In this subsection, assuming in addition (H4), we deal with the convergence of branches $\mathcal{C}_{\Omega_n}$ in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. To this purpose, we proceed as in the previous subsection. But, here the a priori bounds independent of Ω_n follow from Proposition 3.2.

Proof of Theorem 1.5. In view of Theorem 1.3, we define for some large negative Λ, the connected component A_n^Λ of

$$\{(\lambda, u) \in \mathcal{C}_{\Omega_n} \mid \lambda \geq \Lambda\} \tag{3.45}$$

which contains $(\lambda_1(\Omega_n), 0)$ and where Ω_n is as in the previous subsection. By Theorem 1.2 and Proposition 3.2 we have:

(a) $A_n^\Lambda \subset [\Lambda, \lambda_1(\Omega^+)] \times L^\infty(\mathbb{R}^N)$ and it is bounded,
(b) $(\lambda_1(\Omega_n), 0) \in A_n^\Lambda$ and $\Pi_{\mathbb{R}} A_n^\Lambda \supset [\Lambda, \lambda_1(\Omega_n)]$,
(c) $\lim_{n\to\infty}(\lambda_1(\Omega_n), 0) = (0,0) \in \liminf_{n\to\infty} A_n^\Lambda$.

Our next goal is to prove that $\bigcup_{n\in\mathbb{N}} A_n^\Lambda$ is relatively compact in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. For this, take a sequence $(\lambda_n, u_n) \in A_n^\Lambda$. By Proposition 3.2 and a bootstrap argument in the equation of (P_{Ω_n}), one can prove that up to a subsequence there exists (λ, u) solution of (P) such that $u \in L^\infty(\mathbb{R}^N)$ and (passing to the limit as $n \to \infty$)

$$\lambda_n \to \lambda \leq \lambda_1(\Omega^+) \quad \text{and} \quad u_n \to u \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}^N).$$

To prove that $u_n \to u$ in $L^\infty(\mathbb{R}^N)$, it is sufficient to show that

$$u_n(x) \to 0 \quad \text{uniformly when } |x| \to \infty. \tag{3.46}$$

Using similar arguments to those in Proposition 3.1 and choosing such that $B_{4\varepsilon}(x_0) \subset \{x \mid |x| \geq M\}$ with M large, we see that (3.27) holds (for ? this just take ε small enough such that $\lambda_1(B_{4\varepsilon}) > \lambda_1(\Omega^+)$). From (3.27) we obtain:

$$\int_{B_{2\varepsilon}} u_n^p \leq C \frac{|B_{4\varepsilon}|}{\inf_{B_{4\varepsilon}(x_0)} |h(x)|}. \tag{3.47}$$

From (3.47) and from (H4), for all $\delta > 0$, there exists M large enough such that

$$\int_{B_{2\varepsilon}(x_0)} u_n^p \leq \delta \quad \text{if } B_{4\varepsilon}(x_0) \subset \{x \mid |x| \geq M\}. \tag{3.48}$$

Using Lemma 9.20 in [15] together with (3.48) we obtain

$$\forall \delta > 0,\ \exists M > 0 \text{ such that } |x| \geq 2M \Rightarrow u_n(x) \leq \delta \quad \forall n$$

from which (3.46) follows. This proves that $\bigcup_{n\in\mathbb{N}} A_n^\Lambda$ is relatively compact in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. Now take

$$\mathcal{C} := \lim_{\Lambda\to-\infty} \limsup_{n\to\infty} A_n^\Lambda.$$

From Theorem 1.3 we see that $\mathcal{C}$ is connected in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$ and it bifurcates from the essential spectrum. So, to prove (i), we just have to prove that

$$\lambda_0 := \sup\{\lambda \mid (\lambda, u) \in \mathcal{C}\} > 0.$$

Recalling that from Theorem 1.4, for $\lambda \le 0$, $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ and that there's no bifurcation towards the left from the essential spectrum in $\mathbb{R} \times \mathcal{D}^{1,2}(\mathbb{R}^N)$, we are done.

Now, (ii) follows from the same arguments proving (ii) and (iii) of Theorem 1.4. This completes the proof of Theorem 1.5. $\square$

Remarks.

1. It would be interesting to understand what happens when
$$p = (N+2)/(N-2) \quad \text{(critical case)}.$$
2. It is worth to notice that Theorem 1.5 proves that the bifurcation from essential spectrum occurs towards the right (which implies the existence of nontrivial solutions for $\lambda > 0$ in $L^\infty(\mathbb{R}^N)$).
3. It would be interesting to extend Theorem 1.5 in the case where
$$\limsup_{|x|\to\infty} h(x) < 0.$$

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