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MULTIPLE NONTRIVIAL SOLUTIONS OF ELLIPTIC SEMILINEAR EQUATIONS

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Abstract. We find multiple solutions for semilinear boundary value problems when the corresponding functional exhibits local splitting at zero.

1. Introduction

In his studies of semilinear elliptic problems with jumping nonlinearities, Các [2] proved the following

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$. Let $0 < \lambda_0 < \ldots < \lambda_k < \ldots$ be the sequence of distinct eigenvalues of the eigenvalue problem*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Let $p(t)$ be a continuous function such that $p(0) = 0$ and

$$p(t)/t \to a \quad \text{as } t \to -\infty \quad \text{and} \quad p(t)/t \to b \quad \text{as } t \to \infty.$$

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Assume that for some $k \geq 1$, we have $a \in (\lambda_{k-1}, \lambda_k)$, $b \in (\lambda_k, \lambda_{k+1})$, and the only solution of

$$\begin{cases} -\Delta u = bu^+ - au^- & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

is $u \equiv 0$, where $u^\pm = max[\pm u, 0]$. Assume further that

$$\frac{p(s) - p(t)}{s - t} \leq \nu < \lambda_{k+1}, \quad s, t \in \mathbb{R}, \ s \neq t. \tag{1.3}$$

Assume also that $p'(0)$exists and satisfies $p'(0) \in (\lambda_{j-1}, \lambda_j)$ for some $j \leq k$. Then

$$\begin{cases} -\Delta u = p(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

has at least two nontrivial solutions.

This theorem generalizes the work of Gallouët and Kavian [7] which required λ_k to be a simple eigenvalue and the left hand side of (1.3) to be sandwiched in between λ_{k-1} and λ_{k+1} and bounded away from both of them. Các proves a counterpart of the theorem in which the inequalities are reversed.

In the present paper we generalize this theorem and its reverse inequality counterpart by not requiring $p(t)/t$ to converge to limits at either $\pm\infty$ or 0. Rather, we work with the primitive

$$F(x,t) := \int_0^t f(x,s)\, ds$$

and bound $2F(x,t)/t^2$ near $\pm\infty$ and 0 (we replace $p(t)$ with a function $f(x,t)$ depending on x as well). Our main assumptions are

$$t[f(x,t_1) - f(x,t_0)] \leq a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R}, \ t = t_1 - t_0, \tag{1.5}$$

$$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x,t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| < \delta \tag{1.6}$$

for some $\delta > 0$,

$$a_2(t^-)^2 + b_2(t^+)^2 - W_1(x) \leq 2F(x,t), \quad |t| > K \tag{1.7}$$

for some $K > 0$ and $W_1 \in L^1(\Omega)$, where the constants $a, a_0, a_1, a_2, b, b_0, b_1, b_2$ are suitably chosen (they include the cases considered by Các). The advantage of such inequalities is that they do not restrict the expression $2F(x,t)/t^2$ or $f(x,t)/t$ to any particular interval. Special cases of our theorems were proved by Li–Willem [9]

2. Statement of the theorems

Let Ω be a smooth, bounded domain in $\mathbb{R}^n$, and let A be a selfadjoint operator on $L^2(\Omega)$. We assume that

$$C_0^\infty(\Omega) \subset D := D(|A|^{1/2}) \subset H^{T,2}(\Omega) \tag{2.1}$$

holds for some $T > 0$ (T need not be an integer), and the eigenvalues of A satisfy

$$0 < \lambda_0 < \ldots < \lambda_k < \ldots .$$

We use the notation

$$a(u,v) = (Au, v), \quad a(u) = a(u,u), \quad u, v \in D.$$

Let $f(x,t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$. This means that $f(x,t)$ is continuous in t for a.e. $x \in \Omega$ and measurable in x for every $t \in \mathbb{R}$. We assume that the function $f(x,t)$ satisfies

$$|f(x,t)| \le C(|t| + 1), \quad x \in \Omega,\ t \in \mathbb{R}. \tag{2.2}$$

We define

$$\begin{aligned} \|u\|_D &:= \|A^{1/2}u\|, \\ F(x,t) &:= \int_0^t f(x,s)\,ds, \\ G(u) &:= \|u\|_D^2 - 2\int_\Omega F(x,u)\,dx. \end{aligned} \tag{2.3}$$

It is known that G is a continuously differentiable functional on the whole of D (cf. [17, p. 57]) and

$$(G'(u), v)_D = 2(u,v)_D - 2(f(u), v),$$

where we write $f(u)$ in place of $f(x, u(x))$. In connection with the operator A, the following quantities are very useful. For each fixed positive integer ℓ we let N_ℓ denote the subspace of D spanned by the eigenfunctions corresponding to $\lambda_0, \ldots, \lambda_\ell$, and let $M_\ell = N_\ell^\perp \cap D$. Then $D = M_\ell \oplus N_\ell$. For real a, b we define

$$I(u,a,b) = (Au,u) - a\|u^-\|^2 - b\|u^+\|^2,$$

where $u^\pm(x) = \max\{\pm u(x), 0\}$.

$$\begin{aligned} \gamma_\ell(a) &= \sup\{I(v,a,0) : v \in N_\ell, \|v^+\| = 1\}, \\ \Gamma_\ell(a) &= \inf\{I(w,a,0) : w \in M_\ell, \|w^+\| = 1\}, \\ F_{1\ell}(w,a,b) &= \sup\{I(v+w,a,b) : v \in N_\ell\}, \\ F_{2\ell}(v,a,b) &= \inf\{I(v+w,a,b) : w \in M_\ell\}, \\ M_\ell(a,b) &= \inf\{F_{1\ell}(w,a,b) : w \in M_\ell, \|w\|_D = 1\}, \\ m_\ell(a,b) &= \sup\{F_{2\ell}(v,a,b) : v \in N_\ell, \|v\|_D = 1\}, \end{aligned}$$

$$\begin{aligned}\nu_\ell(a) &= \sup\{b : M_\ell(a,b) \geq 0\},\\ \mu_\ell(a) &= \inf\{b : m_\ell(a,b) \leq 0\}.\end{aligned}$$

Our first result is

Theorem 2.1. *Assume that for some integers $l < m$ the following inequalities hold.*

$$t[f(x,t_1) - f(x,t_0)] \leq a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0, \tag{2.4}$$

where $b < \Gamma_m(a)$.

$$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x,t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| < \delta, \tag{2.5}$$

for some $\delta > 0$, with $a_0, b_0 < \lambda_{l+1}$, $a_1, b_1 > \lambda_l$, $b_0 > \mu_l(a_0)$, and $b_1 < \nu_l(a_1)$.

$$a_2(t^-)^2 + b_2(t^+)^2 - W_1(x) \leq 2F(x,t), \quad |t| > K, \tag{2.6}$$

for some $K \geq 0$, where $a_2, b_2 < \lambda_{m+1}$, $b_2 > \mu_m(a_2)$, and $W_1 \in L^1(\Omega)$. Then the equation

$$Au = f(x,u), \quad u \in D \tag{2.7}$$

has at least two nontrivial solutions.

In contrast to this we have

Theorem 2.2. *Equation* (2.7) *will have at least two nontrivial solutions if we assume that for some integers $l > m$ the following inequalities hold:*

$$t[f(x,t_1) - f(x,t_0)] \geq a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0, \tag{2.8}$$

where $b > \gamma_m(a)$,

$$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x,t) \leq a_1(t^-)^2 + b_1(t^+)^2, \quad |t| < \delta, \tag{2.9}$$

for some $\delta > 0$, with $a_0, b_0 < \lambda_{l+1}$, $b_0 > \mu_l(a_0)$ and $a_1, b_1 > \lambda_l$, $b_1 < \nu_l(a_1)$,

$$2F(x,t) \leq a_2(t^-)^2 + b_2(t^+)^2 + W_2(x), \quad |t| > K, \tag{2.10}$$

for some $K \geq 0$, where $a_2, b_2 > \lambda_m$, $b_2 < \nu_m(a_2)$ and $W_2 \in L^1(\Omega)$.

Immediate consequences of these theorems are

Corollary 2.1. *Assume that for some integers $l < m$ the following inequalities hold:*

$$t[f(x,t_1) - f(x,t_0)] \leq at^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0, \tag{2.11}$$

where $a < \lambda_{m+1}$,

$$a_0 t^2 \leq 2F(x,t) \leq a_1 t^2, \quad |t| < \delta, \tag{2.12}$$

for some $\delta > 0$*, with* $\lambda_l < a_0 \le a_1 < \lambda_{l+1}$*,*

$$a_2 t^2 - W_1(x) \le 2F(x,t), \quad |t| > K, \tag{2.13}$$

for some $K \ge 0$*, where* $a_2 > \lambda_m$ *and* $W_1 \in L^1(\Omega)$*. Then the equation* (2.7) *has at least two nontrivial solutions.*

Corollary 2.2. *Equation* (2.7) *will have at least two nontrivial solutions if we assume that for some integers* $l > m$ *the following inequalities hold:*

$$t[f(x,t_1) - f(x,t_0)] \ge at^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0, \tag{2.14}$$

where $a > \lambda_m$*,*

$$a_0 t^2 \le 2F(x,t) \le a_1 t^2, \quad |t| < \delta, \tag{2.15}$$

for some $\delta > 0$*, with* $\lambda_l < a_0 \le a_1 < \lambda_{l+1}$*,*

$$2F(x,t) \le a_2 t^2 + W_2(x), \quad |t| > K, \tag{2.16}$$

for some $K \ge 0$*, where* $a_2 < \lambda_{m+1}$ *and* $W_2 \in L^1(\Omega)$*.*

It was shown in [15] that the functions γ_l, μ_l, ν_{l-1}, Γ_{l-1} all emanate from the point (λ_l, λ_l) and satisfy

$$\Gamma_{l-1}(a) \le \nu_{l-1}(a) \le \mu_l(a) \le \gamma_l(a)$$

on their common domains. It would therefore give a weaker result if we assumed in Theorems 2.1 and 2.2 that $b_0 > \gamma_l(a_0)$ and $b_1 < \Gamma_l(a_1)$. However, the functions γ_l, Γ_l are defined on the whole of $\mathbb{R}$, while the others are not. For cases in which the other functions are not defined we state the following

Tjeorem 2.3. *Theorems* 2.1 *and* 2.2 *remain true if we assume that* (2.5) *holds with* $b_0 > \gamma_l(a_0)$*, and* $b_1 < \Gamma_l(a_1)$ *for some* $a_0, a_1 \in \mathbb{R}$*.*

3. Some lemmas

The proofs of the theorems of Section 2 will be based on a series of lemmas.

Lemma 3.1. *If* $b < \Gamma_l(a)$*, then there is an* $\varepsilon > 0$ *such that*

$$I(w,a,b) \ge \varepsilon \|w\|_D^2, \quad w \in M_l. \tag{3.1}$$

Proof. By the continuity of Γ_l, there is a $t < 1$ such that $b/t < \Gamma_l(a/t)$. Then,

$$I(w, a/t, b/t) = \|w\|_D^2 - \frac{a}{t}\|w^-\|^2 - \frac{b}{t}\|w^+\|^2 \ge 0, \quad w \in M_l.$$

Therefore,

$$I(w,a,b) = t\left[\|w\|_D^2 - \frac{a}{t}\|w^-\|^2 - \frac{b}{t}\|w^+\|^2\right] + (1-t)\|w\|_D^2 \ge (1-t)\|w\|_D^2. \quad \square$$

Lemma 3.2. *If $b > \gamma_l(a)$, then there is an $\varepsilon > 0$ such that*

$$I(v,a,b) \le -\varepsilon \|v\|_D^2, \quad v \in N_l. \tag{3.2}$$

Proof. By the continuity of γ_l, there is a $t > 1$ such that $b/t > \gamma_l(a/t)$. Hence,

$$I(v,a/t,b/t) = \|v\|_D^2 - \frac{a}{t}\|v^-\|^2 - \frac{b}{t}\|v^+\|^2 \le 0, \quad v \in N_l,$$

and

$$I(v,a,b) = t\left[\|v\|_D^2 - \frac{a}{t}\|v^-\|^2 - \frac{b}{t}\|v^+\|^2\right] + (1-t)\|v\|_D^2 \le (1-t)\|v\|_D^2. \quad \square$$

Lemma 3.3. *If*

$$t[f(x,t_1) - f(x,t_0)] \le a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0, \tag{3.3}$$

then

$$(G'(v+w_1) - G'(v+w_0), w) \ge 2I(w,a,b), \quad v, w_j \in D,\ w = w_1 - w_0. \tag{3.4}$$

Proof. We have

$$(f(x,v+w_1) - f(x,v+w_0), w) \le a\|w^-\|^2 + b\|w^+\|^2.$$

Hence,

$$\begin{aligned}(G'(v+w_1) &- G'(v+w_0), w)/2 \\ &= \|w\|_D^2 - (f(x,v+w_1) - f(x,v+w_0), w) \ge I(w,a,b). \quad \square\end{aligned}$$

Lemma 3.4. *If $f(x,t)$ satisfies* (3.3), *and $b < \Gamma_m(a)$, then there is a continuous map φ from N_m into M_m such that*

$$J(v) \equiv G(v+\varphi(v)) = \min_{w \in M_m} G(v+w) \in C^1(N_m,\mathbb{R}), \quad v \in N_m, \tag{3.5}$$

and

$$J'(v) = G'(v+\varphi(v)), \quad v \in N_m. \tag{3.6}$$

Proof. In view of Lemmas 3.1 and 3.3, we have

$$(G'(v+w_1) - G'(v+w_0), w) \ge \varepsilon\|w\|_D^2, \quad w \in M_m.$$

We can now apply a well known theorem of Castro [3] to arrive at the conclusion. $\square$

Lemma 3.5. *If, in addition,*

$$a_0(t^-)^2 + b_0(t^+)^2 \le 2F(x,t), \quad |t| < \delta, \tag{3.7}$$

for some $\delta > 0$*, with* $a_0, b_0 < \lambda_{l+1}$*,* $b_0 > \mu_l(a_0)$*,* $l \le m$*, then there are* $\varepsilon > 0$*,* $r > 0$ *such that*

$$J(v) \le -\varepsilon \|v\|_D^2, \quad v \in N_l \cap B_r, \tag{3.8}$$

where $B_r = \{u \in D : \|u\|_D \le r\}$*.*

Proof. Let q be any number satisfying

$$\begin{aligned} 2 &< q \le 2n/(n-2T), && 2T < n, \\ 2 &< q < \infty, && n \le 2T. \end{aligned}$$

It was shown in Schechter [16] that there is a continuous map $\tau : N_l \to M_l$ such that

$$\tau(s\,v) = s\,\tau(v), \quad s \ge 0, \tag{3.9}$$

$$I(v + \tau(v), a_0, b_0) = \inf_{w \in M_l} I(v + w, a_0, b_0), \quad v \in N_l, \tag{3.10}$$

$$\|\tau(v)\|_D \le C \|v\|_D, \quad v \in N_l. \tag{3.11}$$

Then, for $u = v + \tau(v)$, we have by (2.2)

$$\begin{aligned} J(v) \le G(u) &\le I(u, a_0, b_0) + \int_{|u|>\delta} [a_0(u^-)^2 + b_0(u^+)^2 - 2F(x,u)]\,dx \\ &\le F_{2l}(v, a_0, b_0) + C \int_{|u|>\delta} |u|^q\,dx \\ &\le m_l(a_0, b_0)\|v\|_D^2 + o(\|v\|_D^2) \le -\varepsilon \|v\|_D^2 \end{aligned}$$

for r sufficiently small (cf. [17], p. 159–160). □

Lemma 3.6. *Assume that*

$$a(t^-)^2 + b(t^+)^2 - W_1(x) \le 2F(x,t), \quad |t| > K, \tag{3.12}$$

for some $K \ge 0$*, where* $a, b < \lambda_{m+1}$*,* $b \ge \mu_m(a)$*,* $l \le m$*, and* $W_1 \in L^1(\Omega)$*. Then there is a* $K_1 < \infty$ *such that*

$$J(v) \le K_1. \tag{3.13}$$

If $b > \mu_m(a)$*, then*

$$J(v) \to -\infty \quad \text{as } \|v\|_D \to \infty. \tag{3.14}$$

Proof. For $u = v + w$, $v \in N_m$, $w \in M_m$, we have

$$G(u) \le I(u, a, b) + C \int_{|u|<K} |u|^q\,dx + \int_\Omega W_1(x)\,dx \le I(u, a, b) + K'.$$

Thus,

$$\begin{aligned} J(v) &= \inf_{w\in M_m} G(v+w) \le \inf_{w\in M_m} I(v+w,a,b)+K' \\ &= F_{2m}(v,a,b)+K' \le m(a,b)\|v\|_D^2+K'. \end{aligned}$$

If $b \ge \mu_m(a)$, then $m(a,b) \le 0$. This proves (3.13). If $b > \mu_m(a)$, then $m(a,b) < 0$. This proves (3.14). □

Lemma 3.7. *If $l < m$, and $\lambda_l < a, b < \lambda_{m+1}$, then there are continuous functions $\xi : N_m \cap M_l \to N_l$, $\eta : N_m \cap M_l \to M_m$ homogeneous of degree one and such that, for $y \in N_m \cap M_l$,*

$$\begin{aligned} I(\xi(y)+\eta(y)+y,a,b) &= \sup_{v\in N_l} \inf_{w\in M_m} I(v+w+y,a,b) \\ &= \inf_{w\in M_m} \sup_{v\in N_l} I(v+w+y,a,b). \end{aligned} \tag{3.15}$$

Proof. Let $L_y(v,w) = I(v+w+y,a,b)$. Then L_y is a strictly convex lower semicontinuous functional in $w \in M_m$, and strictly concave and continuous in $v \in N_l$. By a theorem of Ky-Fan (cf. [6]), for each $y_0 \in N_m \cap M_l$ there are unique elements $v_0 = \xi(y_0) \in N_l$, $w_0 = \eta(y_0) \in M_m$ such that (3.15) holds, i.e., that

$$L_{y_0}(v,w_0) \le L_{y_0}(v_0,w_0) \le L_{y_0}(v_0,w), \quad v \in N_l,\ w \in M_m.$$

The functions ξ, η are clearly homogeneous of degree one. To prove continuity, let $y_j \to y_0$ in $N_l \cap M_m$, and let $v_j = \xi(y_j)$, $w_j = \eta(y_j)$. We note that the functions v_j and w_j are bounded in D. For otherwise, it is easy to show that

$$\begin{aligned} I(v+w_j+y_j,a,b) &\to \infty \quad \text{as } j\to\infty, \quad \text{for any } v \in N_l, \\ I(v_j+w+y_j,a,b) &\to -\infty \quad \text{as } j\to\infty, \quad \text{for any } w \in M_m. \end{aligned}$$

This would contradict (3.15). Thus there are renamed subsequences such that $v_j \to v_1$, $w_j \rightharpoonup w_1$ in D. Since

$$I(v+w_j+y_j,a,b) \le I(v_j+w_j+y_j,a,b) \le I(v_j+w+y_j,a,b),$$

for $v \in N_l$, $w \in M_m$, we have in the limit

$$I(v+w_1+y_0,a,b) \le I(v_1+w_1+y_0,a,b) \le I(v_1+w+y_0,a,b),$$

for $v \in N_l$, $w \in M_m$, showing that $v_1 = v_0$, $w_1 = w_0$. Since this is true for any subsequence, the result follows. □

Lemma 3.8. *If*

$$2F(x,t) \le a_1(t^-)^2 + b_1(t^+)^2, \quad |t| \le \delta, \tag{3.16}$$

for some $\delta > 0$*, with* $a_1, b_1 > \lambda_l$*,* $b_1 < \nu_l(a_1)$*,* $l < m$*, then there are* $\varepsilon > 0$*,* $r > 0$ *such that*

$$J(y + \xi(y)) \ge \varepsilon \|y\|_D^2, \quad y \in N_m \cap M_l \cap B_r. \tag{3.17}$$

Proof. By Lemma 3.7 we have

$$\inf_{w \in M_m} I(\xi(y) + y + w, a_1, b_1) = \inf_{w \in M_m} \sup_{v \in N_l} I(v + y + w, a_1, b_1), \tag{3.18}$$

for $y \in N_m \cap M_l$. Then for $y \in (N_m \cap M_l \cap B_r) \setminus \{0\}$,

$$\begin{aligned} J(\xi(y) + y) &= G(\xi(y) + y + \varphi(\xi(y) + y)) \\ &\ge I(\xi(y) + y + \varphi(\xi(y) + y), a_1, b_1) - o(\|y\|_D^2) \\ &\ge \inf_{w \in M_m} I(\xi(y) + y + w, a_1, b_1) - o(\|y\|_D^2) \\ &= \inf_{w \in M_m} \sup_{v \in N_l} I(v + y + w, a_1, b_1) - o(\|y\|_D^2) \\ &\ge \inf_{w \in M_m} M_l(a,b)\|y + w\|_D^2 - o(\|y\|_D^2) \\ &= M_l(a,b)\|y\|_D^2 - o(\|y\|_D^2) \ge \varepsilon \|y\|_D^2. \end{aligned} \tag{3.19}$$

□

Lemma 3.9. *Assume*

$$t[f(x,t_1) - f(x,t_0)] \ge a(t^-)^2 + b(t^+)^2, \quad t_j \in \mathbb{R},\ t = t_1 - t_0. \tag{3.20}$$

Then

$$(G'(v_1 + w) - G'(v_0 + w), v) \le 2I(v, a, b), \quad v_j, w \in D,\ v = v_1 - v_0. \tag{3.21}$$

Proof. We have

$$(f(x, v_1 + w) - f(x, v_0 + w), v) \ge a\|v^-\|^2 + b\|v^+\|^2.$$

Hence

$$(G'(v_1+w) - G'(v_0+w), v)/2 = \|v\|_D^2 - (f(x, v_1+w) - f(x, v_0+w), v) \le I(v, a, b).$$

□

Lemma 3.10. *If* $f(x,t)$ *satisfies* (3.20)*, and* $b > \gamma_m(a)$*, then there is a continuous map* ψ *from* $M_m \to N_m$ *such that*

$$J(w) \equiv G(w + \psi(w)) = \max_{v \in N_m} G(v + w) \in C^1(M_m, \mathbb{R}), \quad w \in M_m, \tag{3.22}$$

and

$$J'(w) = G'(w + \psi(w)), \quad w \in M_m. \tag{3.23}$$

Proof. In view of Lemmas 3.2 and 3.9 we have

$$(G'(v_1 + w) - G'(v_0 + w), v) \le -\varepsilon \|v\|_D^2, \quad v \in N_m.$$

We can now apply the theorem of Castro [3] to obtain the conclusion. □

Lemma 3.11. *If, in addition,*

$$a_0(t^-)^2 + b_0(t^+)^2 \le 2F(x,t), \quad |t| < \delta, \tag{2.24}$$

for some $\delta > 0$, with $a_0, b_0 < \lambda_{l+1}$, $b_0 > \mu_l(a_0)$, $l > m$, then there are $\varepsilon > 0$, $r > 0$ such that

$$J(y + \eta(y)) \le -\varepsilon \|y\|_D^2, \quad y \in N_l \cap M_m \cap B_r. \tag{3.25}$$

Proof. For $y \in M_m \cap N_l$, let $u = y + \eta(y) \in M_m$. By (2.2),

$$\begin{aligned} J(u) &= G(u + \psi(u)) \le I(u + \psi(u), a_0, b_0) + o(\|u\|_D^2) \\ &\le \sup_{v \in N_m} I(u + v, a_0, b_0) + o(\|u\|_D^2) \\ &= I(y + \eta(y) + \xi(y), a_0, b_0) + o(\|u\|_D^2) \\ &= \sup_{v \in N_m} \inf_{w \in M_l} I(y + v + w, a_0, b_0) + o(\|u\|_D^2) \\ &= \sup_{v \in N_m} F_{2l}(y + v, a_0, b_0) + o(\|u\|_D^2) \\ &\le \sup_{v \in N_m} m_l(a_0, b_0)\|y + v\|_D^2 + o(\|u\|_D^2) \le -\varepsilon \|y\|_D^2 \end{aligned} \tag{3.26}$$

for r sufficiently small (cf. [17, p. 159]). □

Lemma 3.12. *If*

$$2F(x,t) \le a_1(t^-)^2 + b_1(t^+)^2, \quad |t| \le \delta, \tag{3.27}$$

for some $\delta > 0$, with $a_1, b_1 > \lambda_l$, $b_1 < \nu_l(a_1)$, $l > m$, then there are $\varepsilon > 0$, $r > 0$ such that

$$J(w) \ge \varepsilon \|w\|_D^2, \quad w \in M_l \cap B_r. \tag{3.28}$$

Proof. We recall from Schechter [16] that there is a continuous map $\theta : M_l \to N_l$ such that

$$\theta(s\,w) = s\,\theta(w), \quad s \ge 0, \tag{3.29}$$

$$I(\theta(w) + w, a_1, b_1) = \sup_{v \in N_l} I(v + w, a_1, b_1), \quad w \in M_l. \tag{3.30}$$

Thus,

$$\begin{aligned} J(w) &\geq G(w+\theta(w),a_1,b_1) \geq I(w+\theta(w),a_1,b_1) - o(\|w\|_D^2) \\ &= \sup_{v\in N_l} I(v+w,a_1,b_1) - o(\|w\|_D^2) \\ &= F_{1l}(w,a_1,b_1) - o(\|w\|_D^2) \\ &\geq M_l(a_1,b_1)\|w\|_D^2 - o(\|w\|_D^2) \geq \varepsilon\|w\|_D^2 \end{aligned}$$

for r sufficiently small. □

LEMMA 3.13. *Assume that*

$$2F(x,t) \leq a(t^-)^2 + b(t^+)^2 + W_1(x), \quad |t| > K \tag{3.31}$$

for some $K \geq 0$*, where* $a, b > \lambda_m$*,* $b \leq \nu_m(a)$*,* $l \geq m$*, and* $W_1 \in L^1(\Omega)$*. Then there is a* $K_1 < \infty$ *such that*

$$J(w) \geq -K_1, \quad w \in M_m. \tag{3.32}$$

If $b < \nu_m(a)$*, then*

$$J(w) \to \infty \quad \text{as } \|w\|_D \to \infty. \tag{3.33}$$

PROOF. For $u = v + w$, $v \in N_m$, $w \in M_m$, we have

$$G(u) \geq I(u,a,b) - C\int_{|u|<K} |u|^q\,dx - \int_\Omega W_1(x)dx \geq I(u,a,b) - K'.$$

Thus,

$$\begin{aligned} J(w) &= \sup_{v\in N_m} G(v+w) \geq \sup_{v\in N_m} I(v+w,a,b) - K' \\ &= F_{1m}(w,a,b) - K' \geq M_m(a,b)\|w\|_D^2 - K'. \end{aligned}$$

If $b \leq \nu_m(a)$, then $M_m(a,b) \geq 0$. This proves (3.32). If $b < \nu_m(a)$, then $M_m(a,b) > 0$. This proves (3.33). □

LEMMA 3.14. *If*

$$a_0(t^-)^2 + b_0(t^+)^2 \leq 2F(x,t), \quad |t| < \delta \tag{3.34}$$

for some $\delta > 0$*, with* $b_0 > \gamma_l(a_0)$*,* $l \leq m$*, then there are* $\varepsilon > 0$*,* $r > 0$ *such that*

$$J(v) \leq -\varepsilon\|v\|_D^2, \quad v \in N_l \cap B_r, \tag{3.35}$$

where $B_r = \{u \in D : \|u\|_D \leq r\}$*.*

PROOF. Let q be any number satisfying

$$\begin{aligned} &2 < q \leq 2n/(n-2T), \quad 2T < n, \\ &2 < q < \infty, \qquad\qquad n \leq 2T. \end{aligned}$$

By (2.2),

$$\begin{aligned} J(v) &\le G(v) \le I(v, a_0, b_0) + \int_{|v|>\delta} [a_0(v^-)^2 + b_0(v^+)^2 - 2F(x,v)]\,dx \\ &\le -\varepsilon\|v\|_D^2 + C\int_{|v|>\delta} |v|^q\,dx \le -\varepsilon\|v\|_D^2 + o(\|v\|_D^2) \le -\varepsilon\|v\|_D^2 \end{aligned}$$

for r sufficiently small (cf. [17, p. 60]). □

Lemma 3.15. *If*

$$2F(x,t) \le a_1(t^-)^2 + b_1(t^+)^2, \quad |t| \le \delta \tag{3.36}$$

for some $\delta > 0$, with $b_1 < \Gamma_l(a_1)$, $l < m$, then there are $\varepsilon > 0$, $r > 0$ such that

$$J(v) \ge \varepsilon\|v\|_D^2, \quad v \in N_m \cap M_l \cap B_r. \tag{3.37}$$

Proof. Let $u = v + \varphi(v) \in M_l$. Then

$$\begin{aligned} J(v) = G(u) &\ge I(u, a_1, b_1) + \int_{|u|>\delta} [a_0(u^-)^2 + b_0(u^+)^2 - 2F(x,u)]\,dx \\ &\ge \varepsilon\|u\|_D^2 - C\int_{|u|>\delta} |u|^q\,dx \ge \varepsilon\|u\|_D^2 - o(\|u\|_D^2) \\ &\ge \varepsilon\|v\|_D^2 - o(\|v\|_D^2) \ge \varepsilon\|v\|_D^2 \end{aligned}$$

for r sufficiently small, since $\|v\|_D \le \|u\|_D \le C\|v\|_D$. □

Lemma 3.16. *If*

$$a_0(t^-)^2 + b_0(t^+)^2 \le 2F(x,t), \quad |t| < \delta, \tag{3.38}$$

for some $\delta > 0$ with $b_0 > \gamma_l(a_0)$, $l \ge m$, then there are $\varepsilon > 0$, $r > 0$ such that

$$J(w) \le -\varepsilon\|w\|_D^2, \quad w \in N_l \cap M_m \cap B_r. \tag{3.39}$$

Proof. For $w \in M_m \cap N_l$, let $u = w + \psi(w) \in N_l$. By (2.2),

$$\begin{aligned} J(w) &= G(w + \psi(w)) = G(u) \\ &\le I(u, a_0, b_0) + \int_{|u|>\delta} [a_0(v^-)^2 + b_0(u^+)^2 - 2F(x,u)]\,dx \\ &\le -\varepsilon\|u\|_D^2 + C\int_{|u|>\delta} |u|^q\,dx \le -\varepsilon\|u\|_D^2 + o(\|u\|_D^2) \le -\varepsilon\|u\|_D^2 \end{aligned}$$

for r sufficiently small (cf. [17, p. 60]). Since $\|w\|_D \le \|u\|_D \le C\|w\|_D$, the result follows. □

Lemma 3.17. *If*

$$2F(x,t) \le a_1(t^-)^2 + b_1(t^+)^2, \quad |t| \le \delta, \tag{3.40}$$

for some $\delta > 0$*, with* $b_1 < \Gamma_l(a_1)$*,* $l > m$*, then there are* $\varepsilon > 0$*,* $r > 0$ *such that*

$$J(w) \ge \varepsilon \|w\|_D^2, \quad w \in M_l \cap B_r. \tag{3.41}$$

Proof. We have

$$\begin{aligned} G(w) &\ge I(w,a_1,b_1) + \int_{|w|>\delta} [a_0(u^-)^2 + b_0(w^+)^2 - 2F(x,w)]\,dx \\ &\ge \varepsilon\|w\|_D^2 - C\int_{|w|>\delta} |w|^q\,dx \ge \varepsilon\|w\|_D^2 - o(\|w\|_D^2) \\ &\ge \varepsilon\|w\|_D^2 - o(\|w\|_D^2) \ge \varepsilon\|w\|_D^2 \end{aligned}$$

for r sufficiently small. Since $J(w) = \sup_{v\in N_l} G(v+w) \ge G(w)$, the result follows. □

4. The proofs

We prove the theorems of Section 2.

Proof of Theorem 2.1. By Lemma 3.4, it suffices to show that $J(v)$ has two nontrivial solutions. Now J is bounded from above by Lemma 3.6 and it satisfies (PS) by (3.14). Moreover,

$$J(v) < 0, \quad v \in N_l \cap B_r \setminus \{0\}, \tag{4.1}$$

by Lemma 3.5, and

$$J(\xi(y)+y) > 0, \quad y \in N_m \cap M_l \cap B_r \setminus \{0\}, \tag{4.2}$$

by Lemma 3.8. Thus J has a positive maximum on N_m. We can now apply a theorem of Perera [11] to obtain the desired conclusion. □

Proof of Theorem 2.2. By Lemma 3.10, it suffices to show that $J(w)$ given by (3.22) has two nontrivial solutions. Now J is bounded from below by Lemma 3.13 and it satisfies (PS) by (3.33). Moreover,

$$J(w+\eta(w)) < 0, \quad w \in N_l \cap M_m \cap B_r \setminus \{0\}, \tag{4.3}$$

by Lemma 3.11, and

$$J(w) > 0, \quad w \in M_l \cap B_r \setminus \{0\}, \tag{4.4}$$

by Lemma 3.12. Thus J has a negative minimum on M_m. We can now apply the theorem of Perera [11] to obtain the desired conclusion. □

Proof of Theorem 2.3. With reference to Theorem 2.1, we note that by Lemma 3.4, it suffices to show that $J(v)$ has two nontrivial solutions. Now J is bounded from above by Lemma 3.6 and it satisfies (PS) by (3.14). Moreover,

$$J(v) < 0, \quad v \in N_l \cap B_r \setminus \{0\}, \tag{4.5}$$

by Lemma 3.14, and

$$J(v) > 0, \quad v \in N_m \cap M_l \cap B_r \setminus \{0\}, \tag{4.6}$$

by Lemma 3.15. Thus J has a positive maximum on N_m. We can now apply a theorem of Brézis–Nirenberg [1] to obtain the desired conclusion. With respect to Theorem 2.2, we note that by Lemma 3.10, it suffices to show that $J(w)$ given by (3.22) has two nontrivial solutions. Now J is bounded from below by Lemma 3.13 and it satisfies (PS) by (3.33). Moreover,

$$J(w) < 0, \quad w \in N_l \cap M_m \cap B_r \setminus \{0\}, \tag{4.7}$$

by Lemma 3.16, and

$$J(w) > 0, \quad w \in M_l \cap B_r \setminus \{0\}, \tag{4.8}$$

by Lemma 3.17. Thus J has a negative minimum on M_m. We can now apply the theorem of Brézis–Nirenberg [1] to obtain the desired conclusion. □

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