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ON SOME PROPERTIES OF DISSIPATIVE FUNCTIONAL DIFFERENTIAL INCLUSIONS IN A BANACH SPACE

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Abstract. For a semilinear functional differential inclusion of the form

$$y'(t) \in Ay(t0 + f(t, x_t)$$

satisfying a dissipativity condition in a separable Banach space we prove the existence of a periodic solution and a global compact attractor.

0. Introduction

In this paper we prove the existence of a periodic solution and a global attractor for a semilinear functional differential inclusion satisfying a dissipativity condition in a separable Banach space. Usually in the investigation of the solutions set for dissipative differential equations and functional differential equations it is assumed that the solution determined by the initial value is unique and hence the translation operator along the trajectories of the equation is single-valued (see, for example, [11], [10], [19], [16], [17]). We can overcome this restriction using recent results on the structure of the integral funnel [8] and developing the method of the translation multioperator along solutions of

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the functional differential inclusion. In particular, improving the recent result of the paper of M. Kamenskiĭ, V. Obukhovskiĭ and P. Zecca [18] we give sufficient conditions under which this multioperator is condensing with respect to the Hausdorff measure of noncompactness in the space of initial functions.

Let us note that the results of the work are new, even in the case of functional differential equations as the restrictive condition of the uniqueness of a solution is not required.

1. Preliminaries

Let X and Y be Hausdorff topological spaces; a multivalued map (multimap) $\mathcal{F} : X \multimap Y$ is said to be upper semicontinuous (u.s.c.) if

$$\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$$

is an open subset of X for every open $V \subseteq Y$. If a multimap $\mathcal{F}$ is u.s.c. and compact-valued then:

(i) it is closed, i.e. its graph

$$\mathcal{G}_{\mathcal{F}} = \{(x, y) \in X \times Y : y \in \mathcal{F}(x)\}$$

is a closed subset of $X \times Y$;

(ii) the image $\mathcal{F}(\mathcal{R})$ of every compact subset $\mathcal{R} \subset X$ is compact in Y (see for example [5]).

For a multimap $\mathcal{F} : X \multimap X$ and $Q \subset X$, the set

$$\Gamma(Q) = \bigcup_{n \geq 0} \mathcal{F}^n(Q)$$

is said to be the orbit of Q under $\mathcal{F}$.

In this section $\mathcal{E}$ will denote a real separable Banach space. A real valued function χ defined on bounded subsets of $\mathcal{E}$ is said to be the Hausdorff measure of noncompactness provided

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

Let us recall the following properties of χ (see, e.g. [1]).

(i) χ is regular, i.e. $\chi(\Omega) = 0$ implies that Ω is relatively compact,
(ii) χ is monotone, i.e. $\Omega_0 \subseteq \Omega_1$ implies $\chi(\Omega_0) \leq \chi(\Omega_1)$,
(iii) χ is algebraically semiadditive, i.e. $\chi(\Omega_0 + \Omega_1) \leq \chi(\Omega_0) + \chi(\Omega_1)$,
(iv) χ is non singular, i.e. $\chi(\Omega \cup \{a\}) = \chi(\Omega)$.

If L is a bounded linear operator in $\mathcal{E}$ then its χ-norm may be defined as

$$\|L\|^{(\chi)} = \chi(LS)$$

where S denotes the unit sphere in $\mathcal{E}$. It is known that, for any bounded $\Omega \subset \mathcal{E}$

$$\chi(L\Omega) \leq \|L\|^{(\chi)} \chi(\Omega).$$

For a bounded equicontinuous set $D \subset C([a,b];\mathcal{E})$ the function $\chi(D(t))$ (where $D(t) = \{(y(t) : y \in D\})$ is continuous and the characteristic

$$\chi_C(D) = \sup_{t\in[a,b]} \chi(D(t))$$

coincides with the Hausdorff measure of noncompactness of D in $C([a,b];\mathcal{E})$.

Let $\mathcal{B} \subseteq \mathcal{E}$ be a closed set; an u.s.c. multimap $\mathcal{F} : \mathcal{B} \multimap \mathcal{E}$ is said to be (k,χ)-condensing (or simply χ-condensing) if there exists $k,\ 0 \leq k < 1$ such that

$$\chi(\mathcal{F}(\Omega)) \leq k\chi(\Omega)$$

for every bounded $\Omega \subset \mathcal{B}$. It is known (see, for example [4]) that, provided $\mathcal{B}$ is bounded, there exists a nonempty convex compact set $M \subseteq \overline{\text{co}}\mathcal{F}(\mathcal{B})$ (a fundamental set of $\mathcal{F}$) with the property that $\mathcal{B} \cap M \neq \emptyset$ and $\mathcal{F}(\mathcal{B} \cap M) \subseteq M$. Moreover, the fundamental set can be chosen so that it will contain any prescribed point $a \in \mathcal{B}$.

In what follows if $\mathcal{L}$ is a nonempty subset of a normed space then

$$\|\mathcal{L}\| := \sup\{\|l\| : l \in \mathcal{L}\}.$$

A multifunction $G : [a,b] \multimap \mathcal{E}$ with compact values is said to be measurable if it satisfies any of the following two equivalent conditions:

(i) the set $G^{-1}(V) = \{t \in [a,b] : G(t) \subset V\}$ is measurable for every open $V \subseteq \mathcal{E}$;

(ii) there exists the sequence $\{g_n\}_{n=1}^{\infty}$ of measurable functions $g_n : [a,b] \to \mathcal{E}$ such that $G(t) = \overline{\{g_n(t)\}_{n=1}^{\infty}}$ for all $t \in [a,b]$ (see, for example, [7]).

By the symbol of S_G^1 we will denote the set of all Bochner integrable selectors of the multifunction $G : [a,b] \multimap \mathcal{E}$, i.e.

$$S_G^1 = \{g \in L^1([a,b],\mathcal{E}) : g(t) \in G(t) \text{ a.e. } t \in [a,b]\}.$$

If $S_G^1 \neq \emptyset$ then the multifunction G is called integrable and

$$\int_{\mathcal{I}} G(s)\,ds := \left\{ \int_{\mathcal{I}} g(s)\,ds : g \in S_G^1 \right\}$$

for every measurable set $\mathcal{I} \subseteq [a,b]$. Clearly, if G is measurable and integrably bounded (i.e. there exists $\alpha \in L_+^1([a,b])$ such that $\|G(t)\| \leq \alpha(t)$ a.e.) then G is integrable.

We will need also the following property (see [22]).

Lemma 1.1. *Let the multifunction* $G : [a,b] \multimap \mathcal{E}$ *with bounded values be measurable, integrably bounded and*

$$\chi(G(t)) \le \gamma(t) \quad \text{a.e. on } [a,b],$$

where $\gamma \in L^1_+([a,b])$. *Then*

$$\chi\bigg(\int_I G(s)\,ds\bigg) \le \int_I \gamma(s)\,ds$$

for every measurable set $\mathcal{I} \subseteq [a,b]$. *In particular, if* G *is measurable and integrably bounded then*

$$\chi(G(\,\cdot\,)) \in L^1_+([a,b])$$

and

$$\chi\bigg(\int_I G(s)\,ds\bigg) \le \int_I \chi(G(s))\,ds.$$

We recall that a nonempty space $\mathcal{R}$ is said to be an R_δ if $\mathcal{R}$ is the intersection of a decreasing sequence of compact, contractible sets (see, e.g. [13]).

An u.s.c. multimap $\mathcal{G} : X \multimap Y$ of normed spaces is said to be R_δ provided every value $\mathcal{G}(x)$, $x \in X$ is an R_δ set.

Let $\mathcal{B}$ be a closed subset of $\mathcal{E}$. An u.s.c. multimap $\mathcal{F} : \mathcal{B} \multimap \mathcal{E}$ is said to be quasi-R_δ provided there exists a normed linear space Y, a continuous linear map $f : Y \to \mathcal{E}$ and an R_δ multimap $\mathcal{G} : \mathcal{B} \multimap Y$ such that $\mathcal{F} = f \circ \mathcal{G}$.

Using the method of single-valued approximations of R_δ multimaps (see [13]) we can prove the following generalization of the known F. E. Browder's asymptotic fixed point theorem ([6], see also [3], [4], et al.).

Theorem 1.1. *Let* B_0, B_1, B *be balls in a Banach space* $\mathcal{E}$ *with* $\overline{B}_0 \subset B_1 \subset \overline{B}_1 \subset B$. *Let* $M \subset \mathcal{E}$ *be a convex compact set such that* $B_{0M} = B_0 \cap M \neq \emptyset$ *and* $\mathcal{F} : \overline{B} \cap M \multimap M$ *be a quasi-*R_δ *multimap such that:*

(i) $\bigcup_{1\le j\le m-1} \mathcal{F}^j(\overline{B}_{1M}) \subset B_M$,
(ii) $\bigcup_{1\le j\le m-1} \mathcal{F}^j(\overline{B}_{0M}) \subset B_{1M}$,
(iii) $\mathcal{F}^m(\overline{B}_{1M}) \subset B_{0M}$ *for some integer* $m \ge 1$.

Then $\mathcal{F}$ *has a fixed point* $x_* \in B_{0M}$, $x_* \in \mathcal{F}(x_*)$.

2. The translation multioperator along the solutions of a functional differential inclusion

Let E be a separable Banach space; for $\tau > 0$ let us denote by $\mathcal{C}$ the space $C([-\tau,0];E)$ endowed with the usual topology of uniform convergence. The norm in the space $\mathcal{C}$ will be denoted as $\|\cdot\|_0$. For a continuous function $y : [-\tau,a) \to E$, $0 < a \le \infty$ and $0 \le t < a$ the function $y_t \in \mathcal{C}$ is defined by the relation $y_t(\theta) = y(t+\theta)$. For a family Y of such functions we will denote $Y_t \subset \mathcal{C}$, $Y_t = \{y_t : y \in Y\}$.

Consider the following problem for the semilinear functional differential inclusion in E of the form

$$y'(t) \in Ay(t) + F(t, y_t), \quad t \geq 0, \tag{1}$$

$$y(t) = x(t), \quad t \in [-\tau, 0], \tag{2}$$

for a given initial function $x \in \mathcal{C}$ under the following assumptions:

(A) A is a closed linear, not necessarily bounded operator in E generating an analytic semigroup e^{At}.

Let $Kv(E)$ denote the collection of all nonempty compact convex subsets of E. We will suppose that the multimap $F : R_+ \times \mathcal{C} \to Kv(E)$ satisfies the following conditions:

(F_1) for $T > 0$ the multimap F is T-periodic in the first argument, i.e.

$$F(t+T, x) = F(t, x) \quad \text{for all } (t, x) \in R_+ \times \mathcal{C}.$$

Condition (F_1) obviously implies that it is enough to work with the restriction $F : [0, T] \times \mathcal{C} \to Kv(E)$.

Further it is assumed that:

(F_2) for every $x \in \mathcal{C}$ the multifunction $F(\,\cdot\,, x) : [0, T] \to Kv(E)$ admits a measurable selection,

(F_3) for a.e. $t \in [0, T]$ the multimap $F(t, \,\cdot\,) : \mathcal{C} \to Kv(E)$ is u.s.c.,

(F_4) $\|F(t, x)\| \leq \alpha(t) + \beta(t)\|x\|_0$ for every $x \in \mathcal{C}$ and a.e. $t \in [0, T]$ where $\alpha, \beta \in L^1_+([0, T])$,

(F_5) for every nonempty bounded equicontinuous set $D \subset \mathcal{C}$ we have

$$\chi(F(t, D)) \leq g(t, \xi(D)) \quad \text{a.e. } t \in [0, T]$$

where $\xi(D) \in C([-\tau, 0]; R_+)$, $\xi(D)(\theta) = \chi(D(\theta))$ and $g : [0, T] \times C([-\tau, 0], R_+) \to R_+$ is a Carathéodory type function such that:

(i) $g(t, \,\cdot\,) : C([-\tau, 0]; R_+) \to R_+$ is nondecreasing for a.e. $t \in [0, T]$ in the following sense: $\varphi, \psi \in C([-\tau, 0], R_+)$; $\varphi(\theta) < \psi(\theta)$ for all $\theta \in [-\tau, 0]$ implies $g(t, \varphi) \leq g(t, \psi)$,

(ii) $|g(t, \phi) - g(t, \psi)| \leq k(t)\|\phi - \psi\|_1$ a.e. $t \in [0, T]$ for every ϕ, $\psi \in C([-\tau, 0]; R_+)$, where $k \in L^+_1([0, T])$ and $\|\cdot\|_1$ denotes the norm in the space $C([-\tau, 0]; R_+)$ and

(iii) $g(t, 0) = 0$ for a.e. $t \in [0, T]$.

It is clear that condition (F_2) is fulfilled if the multifunction $F(\,\cdot\,, x)$ is measurable for every $x \in \mathcal{C}$. For a function $y(\,\cdot\,) \in C([-\tau, T]; E)$ let us denote by H_y the multifunction $H_y : [0, T] \to Kv(E)$ given by $H_y(t) = F(t, y_t)$. From conditions (F_2)–(F_4) it follows that $S^1_{H_y} \neq \emptyset$ for every $y(\,\cdot\,) \in C([-\tau, T]; E)$ (see, for example, [23]).

Definition 2.1. A function $y(\,\cdot\,) \in C([-\tau, T]; E)$ is said to be a mild solution of the problem (1), (2) provided

$$\begin{aligned} y(t) &= x(t), && t \in [-\tau, 0],\\ y(t) &= e^{At} y(0) + \int_0^t e^{A(t-s)} f(s)\, ds, && t \in [0, T], \end{aligned}$$

where $f \in S^1_{H_y}$.

The following theorem is the combination of the results proved by V. Obukhovskiĭ [22] and G. Conti, V. Obukhovskiĭ and P. Zecca [8].

Theorem 2.1. *Under assumptions* (A), (F_2)–(F_5) *the set* $\Sigma(x)$ *of all mild solutions of the problem* (1)–(2) *is an* R_δ *subset of* $C([-\tau, T]; E)$ *and the multimap* $\Sigma : \mathcal{C} \multimap C([-\tau, T]; E)$, $x \multimap \Sigma(x)$ *is* u.s.c. *and hence* R_δ.

It is easy to see that the periodicity condition (F_1) allows to extend every mild solution of (1), (2) on the whole R_+.

Definition 2.2. A multimap $P_T : \mathcal{C} \multimap \mathcal{C}$ defined as

$$P_T(x) = (\Sigma(x))_T$$

is said to be the translation multioperator along mild solutions of the functional differential inclusion (1).

Note that since P_T can be represented as the composition of the R_δ multimap Σ and the continuous linear map $e : C([-\tau, T]; E) \to \mathcal{C}$, $e(y) = y_T$ the multimap P_T is quasi-R_δ. Our aim now is to investigate the further properties of this multioperator.

Lemma 2.1. *For every bounded set* $D \subset \mathcal{C}$ *the set* $P_T(D)$ *is bounded and, provided* $T > \tau$, *equicontinuous.*

Proof. To prove the first statement, let us demonstrate that the set $\Sigma(D)$ is bounded.

Let $\|x\|_0 \leq N$ for all $x \in D$ and take $y \in \Sigma(D)$. Then we have

$$\begin{aligned} y(t) &= e^{At} x(0) + \int_0^t e^{A(t-s)} f(s)\, ds, && t \in [0, T],\\ y(t) &= x(t), && t \in [-\tau, 0], \end{aligned}$$

where $x \in D$, $f \in S^1_{H_y}$. From the property (F_4) we have the following estimation

$$\|f(s)\| \le \alpha(s) + \beta(s)(\|x\|_0 + \sup_{0\le\eta\le s}\|y(\eta)\|), \quad 0 \le s \le t,$$

and denoting $R = \sup_{0\le t\le T}\|e^{At}\|$ we obtain that

$$\|y(t)\| \le R\|x(0)\| + R\int_0^t \big(\alpha(s) + \beta(s)\big(\|x\|_0 + \sup_{0\le\eta\le s}\|y(\eta)\|\big)\big)\, ds.$$

Since the right-hand part of the above inequality does not decrease, we have

$$\sup_{0\le\eta\le t}\|y(\eta)\| \le R(\|\alpha\| + N(1+\|\beta\|)) + R\int_0^t \beta(s)\sup_{0\le\eta\le s}\|y(\eta)\|\, ds.$$

Applying to the function $w(t) = \sup_{0\le\eta\le t}\|y(\eta)\|$ the Gronwall–Bellman inequality we conclude that

$$w(t) \le C\exp\left(R\int_0^t \beta(s)\, ds\right)$$

where $C = R(\|\alpha\| + N(1+\|\beta\|))$ implying the desired boundedness.

Now, again let $y \in \Sigma(D)$,

$$y(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}f(s)\, ds, \quad t \in [0,T],$$

where $x \in D$, $f \in S^1_{H_y}$. From the above, there exists $N_1 > 0$ such that $\|\Sigma(D)\| \le N_1$ and hence we have the following estimate

$$\|f(t)\| \le \gamma(t) = \alpha(t) + \beta(t)N_1, \quad t \in [0,T].$$

Now, for $T > \tau$, consider $y_T \in \mathcal{C}$. Take $\theta_1, \theta_2 \in [-\tau, 0]$, $\theta_1 < \theta_2$, then

$$\begin{aligned}
&\|y_T(\theta_2) - y_T(\theta_1)\| = \|y(T+\theta_2) - y(T+\theta_1)\| \\
&\le \|(e^{A(T+\theta_2)} - e^{A(T+\theta_1)})x(0)\| + \int_0^{T+\theta_1} \|(e^{A(T+\theta_2-s)} - e^{A(T+\theta_1-s)})\| \cdot \|f(s)\|\, ds \\
&\quad + \int_{T+\theta_1}^{T+\theta_2} \|e^{A(T+\theta_2-s)}\| \cdot \|f(s)\|\, ds \le \|e^{A(T+\theta_2)} - e^{A(T+\theta_1)}\|N \\
&\quad + \int_0^{T+\theta_1} \|e^{A(T+\theta_2-s)} - e^{A(T+\theta_1-s)}\|\gamma(s)\, ds + R\int_{T+\theta_1}^{T+\theta_2} \gamma(s)\, ds
\end{aligned}$$

concluding the proof. □

Remark 2.1. From the proof it is clear that the set $P_T(D)$ is equicontinuous for every $T \geq 0$ provided D is bounded and equicontinuous.

Now, improving the idea of [18] we will formulate the conditions under which the translation multioperator P_T is condensing. At first note that from the periodicity of the multimap F it follows that we may extend the functions $g(t, x)$ and $k(t)$ appearing in the condition (F_5) on the whole half-axis R_+ in t.

Let us suppose that

(H_1) there exists a continuous bounded function $h : R_+ \to R_+$ such that

$$\|e^{At}\|^{(\chi)} \leq h(t), \quad t \in R_+ \tag{3}$$

and

$$\sup_{t>0} \int_0^t h(t-s)k(s)\,ds < 1 \tag{4}$$

(it is clear that $h(0) \geq 1$).

Using the method of successive approximations (see, e.g. [9, §6.5]) we may prove that under the hypothesis (4) the initial problem for the scalar Volterra delay integral equation

$$w(t) = v(t), \qquad t \in [-\tau, 0], \tag{5}$$

$$w(t) = \frac{1}{h(0)} h(t)w(0) + \int_0^t h(t-s)g(s, w_s)\,ds, \quad t \geq 0, \tag{6}$$

has a unique bounded solution $w(t, v)$ on R_+ for every initial function $v \in C([-\tau, 0]; R_+)$.

Our second assumption is that

(H_2) the solutions of the problem (5), (6) are uniformly asymptotically bounded in the following sense: there exists a function $\sigma : R_+ \to R_+$ such that $\limsup_{t\to\infty} \sigma(t) < 1/h(0)$ and for every solution $w(t, v)$ of (5), (6) we have

$$\|w_t\|_1 \leq \sigma(t)\|v\|_1, \quad t \geq 0. \tag{7}$$

Let us denote

$$t_* = \inf\left\{t' : \sigma(t) < \frac{1}{h(0)} \text{ for all } t \geq t'\right\}.$$

Remark 2.2. From the estimate (7) it follows that $\sigma(\tau) \geq 1$ and hence $t_* \geq \tau$.

Example 2.1 below will illustrate applications of the assumption (H_2).

Now we are in position to prove the principal result on the condensivity of the translation multioperator P_T.

Theorem 2.2. *Let the conditions* (A), (F_1)–(F_5), (H_1), (H_2) *be satisfied. Then for every bounded equicontinuous set* $D \subset \mathcal{C}$ *we have that*

$$\chi_C(P_T(D)) \le \sigma(T)h(0)\chi_C(D)$$

and hence, P_T *is* χ_C*-condensing* (*on equicontinuous sets*), *provided*

$$T > t_*. \tag{8}$$

Proof. Let us estimate $\chi(\Sigma(D)(t))$ for $t \in [0,T]$. From the properties of the Hausdorff measure of noncompactness χ and the hypotheses (F_5) and (H_1) we have that

$$\begin{aligned}\chi(\Sigma(D)(t)) &\le \chi\bigg(e^{At}\Sigma(D)(0) + \int_0^t e^{A(t-s)}F(s,(\Sigma(D))_s)\,ds\bigg)\\ &\le \|e^{At}\|^{(\chi)}\chi\,(\Sigma(D)(0)) + \int_0^t \|e^{A(t-s)}\|^{(\chi)}\chi(F(s,(\Sigma(D))_s))\,ds\\ &\le h(t)\chi(\Sigma(D)(0)) + \int_0^t h(t-s)g(s,\xi((\Sigma(D))_s))\,ds.\end{aligned}$$

Therefore, if we denote $u(t) = \chi(\Sigma(D)(t))$, we see that this function whose initial value is $r(t) = \xi(D)(t)$ for $t \in [-\tau, 0]$ satisfies the integral inequality

$$u(t) \le h(t)u(0) + \int_0^t h(t-s)g(s,u_s)\,ds.$$

Now the standard technique of comparison theorems for integral inequalities (see [21, Chapter 5]) can be applied to show that $u(t) \le w(t)$, where $w(t)$ is the solution of the integral equation

$$w(t) = \frac{1}{h(0)}h(t)w(0) + \int_0^t h(t-s)g(s,w_s)\,ds, \quad t \ge 0$$

with the initial value $v(t) = h(0)r(t)$, $t \in [-\tau, 0]$.

From the assumption (H_2) we obtain that

$$\begin{aligned}\chi_C(P_T(D)) &= \|u_T\|_1 \le \|w_T\|_1 \le \sigma(T)\|v\|_1 = \sigma(T)h(0)\|r\|_1\\ &= \sigma(T)h(0)\|\xi(D)\|_1 \le \sigma(T)h(0)\chi_C(D)\end{aligned}$$

concluding the proof. □

Before considering an example illustrating the above hypotheses we will formulate the following statement which we will need also in the sequel. It can be easily verified on the basis of the Lemma of [15, §4.5].

Lemma 2.2. *For positive γ, δ, τ let the function $m : [t_0 - \tau, a) \to R_+$, $0 \le t_0 < a \le \infty$, satisfies the functional differential inequality*

$$m'(t) \le -\gamma m(t) + \delta \|m_t\|_1 \quad \text{for a.e. } t \in [t_0, a)$$

with the initial value $m(t) = n(t)$, $t \in [t_0 - \tau, t_0]$, where $\delta < \gamma$. Then

$$\|m_t\|_1 \le \|n\|_1 \cdot e^{\lambda\tau} e^{-\lambda(t-t_0)}, \quad t \in [t_0, a)$$

where λ, $0 < \lambda < \gamma$, is the solution of the equation $\gamma = \lambda + \delta e^{\lambda\tau}$.

Example 2.1. Consider the case when the solutions of the problem (5), (6) are exponentially decreasing, i.e.

$$\sigma(t) = Ce^{-\mu t}. \tag{9}$$

In this situation we have that

$$t_* = \frac{1}{\mu} \ln \mathrm{Ch}(0).$$

In order to get (9) we may assume that

$$h(t) = Le^{-\gamma t}, \quad t \in R_+, \ L, \gamma > 0 \tag{10}$$

and the function $k(\,\cdot\,)$ from (F_5)(ii) is constant $k(t) \equiv k$ with

$$k < \frac{\gamma}{L}. \tag{11}$$

In fact, the condition (11) implies inequality (4) and the problems (5), (6) take the form

$$w(t) = v(t), \qquad t \in [-\tau, 0], \tag{5'}$$

$$w(t) = e^{-\gamma t} w(0) + L \int_0^t e^{-\gamma(t-s)} g(s, w_s)\, ds, \quad t \ge 0, \tag{6'}$$

and hence the function $w(t)$ for a.e. $t \ge 0$ satisfies

$$w'(t) = -\gamma w(t) + Lg(t, w_t) \le -\gamma w(t) + Lk\|w_t\|_1.$$

Applying Lemma 2.2 we obtain that

$$\|w_t\|_1 \le \|v\|_1 e^{\lambda\tau} e^{-\lambda t}, \quad t \ge 0,$$

where λ is the solution of the equation $\gamma = \lambda + Lke^{\lambda\tau}$ and therefore the condition (9) of the exponential decay of solutions of (5), (6) is fulfilled with $C = e^{\lambda\tau}$ and $\mu = \lambda$.

So in this case the translation multioperator would be condensing for

$$T > \tau + \frac{1}{\lambda} \ln L.$$

Let us note also that the estimation (3) with the gauge function (10) holds, for example, as it was shown in [18], in the case when $A = A_1 + A_2$ where A_1 and A_2 are linear operators such that A_1 is an infinitesimal generator of the exponentially decreasing semigroup $e^{A_1 t}$, i.e.

$$\|e^{A_1 t}\| \leq K e^{-dt}, \quad \text{where } K > 0,\ d > 0,$$

and the χ-norm of A_2 satisfies the following estimation $\|A_2\|^{(\chi)} \leq b$, where $0 \leq b < d/K$ (in particular, A_2 can be compact).

3. Dissipative functional differential inclusions: periodic solutions and attractors

Everywhere in this section we will consider the functional differential inclusion (1) under assumptions (A), (F_1)–(F_5), and (H_1), (H_2). It will be supposed that the condition (8) which guarantees the χ_C-condensivity of the translation multioperator P_T on bounded equicontinuous sets is also fulfilled.

Let $Y(t,x)$ denote the set of all mild solutions of the problem (1), (2) on R_+.

Definition 3.1. The inclusion (1) is said to be dissipative if there exists $d > 0$ such that

$$\limsup_{t\to\infty} \|Y_t\|_0 < d.$$

In other words, the inclusion (1) is dissipative provided for every initial function $x \in \mathcal{C}$ there exists $t(x) \geq 0$ such that each mild solution $y(t,x)$ of the problem (1), (2) satisfies

$$\|y_t\|_0 < d \quad \text{for } t \geq t(x).$$

In this paragraph, extending some results known for dissipative ordinary differential equations and functional differential equations satisfying the condition of the uniqueness of solution (see, for example, [11], [10], [19], [16], [17]) we will discuss some properties of dissipative inclusions. But at first let us consider the following condition for dissipativity.

Theorem 3.1. *Let us assume that*

(i) *the operator A is an infinitesimal generator of an exponentially decreasing semigroup:*

$$\|e^{At}\| \leq N e^{-\rho t}, \quad N \geq 1,\ \rho > 0;$$

(ii) *the functions $\alpha(\,\cdot\,)$, $\beta(\,\cdot\,)$ from the condition* (F_4) *are constant: $\alpha(t) \equiv \alpha$, $\beta(t) \equiv \beta$ and $\beta < \rho/N$.*

Then the functional differential inclusion (1) *is dissipative and the set of mild solutions emanating from every bounded set $D \subset \mathcal{C}$ is bounded.*

Before proving this statement, let us note that the condition (i) is fullfilled, for example, if the operator $-A$ is strongly positive (see [20]).

Proof. Take $R > 0$ large enough to provide $\alpha/R + \beta = \delta < \rho/N$.

Now let $y(t)$ be any mild solution of the problem (1), (2) with the initial value $x \in \mathcal{C}$. If we suppose that $\|y_s\|_0 > R$ for all s on some interval $[t_0, a)$, $t_0 \geq 0$ we will have, for $t \in [t_0, a)$, the following estimation:

$$\begin{aligned}\|y(t)\| &\leq Ne^{-\rho t}\|y(t_0)\| + \int_{t_0}^{t} Ne^{-\rho(t-s)}(\alpha + \beta\|y_s\|_0)\,ds \\ &< Ne^{-\rho t}\bigg(\|y(t_0)\| + \int_{t_0}^{t} e^{\rho s}\delta\|y_s\|_0\,ds\bigg).\end{aligned}$$

Denoting the last expression by $z(t)$ and defining this function on the interval $[t_0 - \tau, t_0]$ as $z(t) = N\|y(t)\|$ we have that for a.e. $t \in [t_0, a)$

$$z'(t) = -\rho z(t) + N\delta\|y_t\|_0 \leq -\rho z(t) + N\delta\|z_t\|_1.$$

Now applying Lemma 2.2 we obtain

$$\|y_t\|_0 < N\|y_{t_0}\|_0 e^{\lambda\tau}e^{-\lambda(t-t_0)}, \quad t \in [t_0, a)$$

where $0 < \lambda < \rho$, $\rho = \lambda + \delta \cdot e^{\lambda\tau}$ and hence for every mild solution $y(t, x)$ emanating from the initial value $x \in \mathcal{C}$ there exists a moment $t_1 = t_1(x) \geq 0$ before which y_t will reach the ball $\{\|x\|_0 \leq R\} \subset \mathcal{C}$. Moreover, from the above estimation we can see that, if $\|y_{t'}\|_0 \leq R$ for any $t' \geq 0$, then y_t will never leave the ball $\{\|x\|_0 \leq \mathrm{NRe}^{\lambda\tau}\} \subset \mathcal{C}$ for $t \geq t'$.

The second statement of the theorem follows now from the fact that we have

$$\|Y_t\|_0 \leq Ne^{\lambda\tau}\max\{R, \|x\|_0\}, \quad t \geq 0. \qquad \square$$

Now we can prove the following proposition on the existence of a periodic solution for a dissipative inclusion.

Theorem 3.2. *Assume that the functional differential inclusion* (1) *is dissipative and the orbits* Γ *of bounded sets under* P_T *are bounded. Then the inclusion* (1) *has a* T*-periodic solution.*

Proof. Let $\overline{B}_0 \subset \mathcal{C}$ be the ball $\{x : \|x\|_0 \leq d\}$ where d is the number appearing in Definition 3.1. Take balls $B_1, B \subset \mathcal{C}$ such that $\overline{B}_0 \subset B_1 \subset \overline{B}_1 \subset B$ and $\Gamma(\overline{B}_0) \subset B_1$, $\Gamma(\overline{B}_1) \subset B$.

Consider now the translation multioperator P_T on the ball $\overline{B}$. From Lemma 2.1 (see also Remark 2.2) it follows that the set $\mathcal{N} = P_T(\overline{B})$ is equicontinuous and the dissipativity condition yields $\mathcal{N} \cap B_0 \neq \emptyset$. Therefore P_T is condensing with respect to the Hausdorff measure of noncompactness as the multioperator

$$P_T : \overline{B} \cap \overline{\text{co}}\,\mathcal{N} \multimap \overline{\text{co}}\,\mathcal{N}$$

and hence there exists a compact fundamental set M of P_T with the property that $M \subseteq \overline{\text{co}}\,\mathcal{N}$ and $M \cap B_0 \neq \emptyset$.

Applying again the dissipativity condition we obtain that for every point $x \in \overline{B}_1 \cap M$ there exists a number $m(x)$ such that $P_T^n(x) \subset B_0$ for all $n \geq m(x)$. From the upper semicontinuity of the multioperator P_T it follows that there exists a neighbourhood $V(x)$ of x such that $P_T^n(x') \subset B_0$ for all $x' \in V(x)$ and $n \geq m(x)$. The neighbourhoods $V(x)$ cover the compact set $\overline{B}_1 \cap M$, so we may select a finite covering $V(x_1), \ldots, V(x_k)$. For $m = \max\{m(x_1), \ldots, m(x_k)\}$ we will have $P_T^m(\overline{B}_1 \cap M) \subset B_0$.

Now we can see that for the multimap P_T all conditions of the Theorem 1.1 are fullfilled and hence there exists a fixed point $x \in B_0$ of P_T. This fixed point is the initial value from which the mild T-periodic solution emanates. □

Collorary 3.1. *Under conditions* (i) *and* (ii) *of Theorem* 3.1 *there exists a mild T-periodic solution of the functional differential inclusion* (1).

Before to proceed we want to recall the recent contributions of L. Górniewicz et al. ([2], [12], [14]) to the periodic problemsfor differential inclusions.

Our aim, now, is to describe the structure of the set of mild solutions of a dissipative functional differential inclusion (1). We prove that this set has a global compact attractor.

To be more precise we need the following definition which extends the notion known for differential equations (see, e.g. [11], [10], [19]).

For bounded sets $\mathcal{A}, \mathcal{B} \subset E$ let us denote

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{a \in \mathcal{A}} \text{dist}\,(a, \mathcal{B})$$

the deviation of the set $\mathcal{A}$ from the set $\mathcal{B}$.

Definition 3.2. A set $\mathcal{M} \subset C([-\tau, \infty); E)$ of bounded mild solutions of the inclusion (1) is said to be a Λ-center of the inclusion (1) provided:

(i) the set $\mathcal{M}_0 \subset \mathcal{C}$ is closed and P_T-invariant in the following sense: $P_T(x) \cap \mathcal{M}_0 \neq \emptyset$ for every $x \in \mathcal{M}_0$ and $P_T(\mathcal{M}_0) \supseteq \mathcal{M}_0$;

(ii) for every collection of solutions $Y(t, x)$ of the problem (1), (2) we have

$$\lim_{n \to \infty} \rho(Y_{nT}, \mathcal{M}_0) = 0 \quad \text{and} \quad \lim_{t \to \infty} \rho(Y_t, \mathcal{M}_t) = 0.$$

We will prove the following main statement.

Theorem 3.3. *Every dissipative functional differential inclusion* (1) *has a* Λ*-center which is compact in the sense of the uniform convergence on every finite interval.*

Lemma 3.1. *If the inclusion* (1) *is dissipative then the orbit* $\Gamma(x)$ *of every point* $x \in \mathcal{C}$ *under* P_T *is relatively compact.*

Proof. From the dissipativity condition and Lemma 2.1 it follows that the set $\Gamma(x)$ is bounded. Applying the same lemma to the equality

$$\Gamma(x) = P_T(\Gamma(x)) \cup \{x\}$$

we obtain that the set $\Gamma(x)$ is equicontinuous and the condensivity of P_T implies $\chi_C(\Gamma(x)) = 0$. □

Definition 3.3. A point $z \in \mathcal{C}$ is said to be an ω-limit point of the orbit $\Gamma(x)$ if there exist a sequence of numbers $n_k \to \infty$ and a sequence $\{p_{n_k}\} \subset \Gamma(x)$, $p_{n_k} \in P_T^{n_k}(x)$ such that $p_{n_k} \to z$. The set of all ω-limit points of $\Gamma(x)$ is said to be an ω-limit set of $\Gamma(x)$ and it is denoted as $\Omega(x)$.

It is easy to see that

$$\Omega(x) = \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} P_T^k(x)},$$

and hence Lemma 3.1 yields that $\Omega(x)$ is a nonempty compact subset of $\overline{\Gamma(x)}$ provided the inclusion (1) is dissipative. It is clear also that $\Omega(x)$ is contained in the ball $B_0 = \{x : \|x\|_0 < d\}$.

Lemma 3.2. *If the inclusion* (1) *is dissipative then the set* $\Omega(x)$ *has the following properties:*

(i) $\Omega(x)$ *is* P_T*-invariant in the sense that* $P_T(z) \cap \Omega(x) \neq \emptyset$ *for every* $z \in \Omega(x)$ *and* $P_T(\Omega(x)) \supseteq \Omega(x)$,

(ii) *for every thickened solution* $Y(t,x)$ *of the problem* (1), (2) *we have*

$$\lim_{n\to\infty} \rho(Y_{nT}, \Omega(x)) = 0.$$

Proof. Let $z \in \Omega(x)$ and $\{p_{n_k}\} \subset \Gamma(x)$ be the sequence such that $p_{n_k} \in P_T^{n_k}(x)$ and $p_{n_k} \to z$. Take arbitrary points $w_{n_k} \in P_T(p_{n_k}) \subseteq P_T^{n_k+1}(x)$. From Lemma 3.1 it follows that the sequence $\{w_{n_k}\}$ is relatively compact hence we may find a subsequence $\{w_{n_{k_j}}\}$ convergent to $w \in \Omega(x)$. The u.s.c. multioperator P_T is closed therefore the relations $w_{n_{k_j}} \in P_T(p_{n_{k_j}})$, $p_{n_{k_j}} \to z$, $w_{n_{k_j}} \to w$ imply $w \in P_T(z)$ yielding $P_T(z) \cap \Omega(x) \neq \emptyset$.

From the other side, for the same point $z \in \Omega(x)$ we have that $p_{n_k} \in P_T^{n_k}(x) = P_T(P_T^{n_k-1}(x))$ and hence we may choose a sequence $v_{n_k} \in P_T^{n_k-1}(x)$ such that $p_{n_k} \in P_T(v_{n_k})$. Again using the relative compactness of the sequence

$\{v_{n_k}\}$ we may choose a subsequence $v_{n_{k_j}}$ converging to $v \in \Omega(x)$. The same argument as above implies that $z \in P_T(v)$ and therefore $P_T(\Omega(x)) \supseteq \Omega(x)$.

The property (ii) of $\Omega(x)$ evidently follows from Lemma 3.1. □

Proof of Theorem 3.3. Consider now the set $M = \bigcup_{x\in\mathcal{C}}\Omega(x)$. It is clear that $M \subset B_0$ and it is also P_T-invariant in the above sense. The relation $P_T(M) \supseteq M$ implies its relative compactness. It is easy to verify that the compact set $\overline{M}$ is also P_T-invariant.

Now let $\mathcal{M} \subset C([-\tau,\infty);E)$ be the set of all mild solutions of the inclusion (1) emanating from the set $\overline{M}$. This set is the desirable one. In fact, its compactness in the topology of the uniform convergence on every finite interval follows from the compactness of $\overline{M}$ and u.s.c. dependence of the solutions set on initial data, see Theorem 2.1.

Further, from the same u.s.c. dependence it follows that, since the set $\overline{M}$ is compact, for given $\varepsilon > 0$ there exists $\delta > 0$ such that $\operatorname{dist}(x,\overline{M}) < \delta$, $x \in \mathcal{C}$ implies $\rho(\Sigma(x)_t, \Sigma(\overline{M})_t) < \varepsilon$ for $t \in [0,T]$. Therefore for any collection of solutions $Y(t,x)$ we have that the relation

$$\rho(Y_{nT},\overline{M}) \leq \rho(Y_{nT},\Omega(x)) \underset{n\to\infty}{\longrightarrow} 0$$

implies

$$\lim_{t\to\infty}\rho(Y_t,\mathcal{M}_t) = 0. \qquad \square$$

Remark 3.1. It is easy to see that the Λ-center constructed is minimal in the sense that it is contained in any other Λ-center of the dissipative functional differential inclusion (1).

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