Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 15, 2000, 359–368

CAUCHY PROBLEMS AND APPLICATIONS

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Abstract. Of concern is the Cauchy problem

$$\frac{du}{dt} \in Au, \quad u(0) = u_0, \quad t > 0,$$

where $u : [0, \infty) \to X$, X is a real Banach space, and $A : D(A) \subset X \to X$ is nonlinear and multi-valued. It is showed by the method of lines, combined with the Crandall–Liggett theorem that this problem has a limit solution, and that the limit solution is a unique strong one if A is what is called embeddedly quasi-demi-closed. In the case of linear, single-valued A, further results are given. An application to nonlinear partial differential equations in non-reflexive X is given.

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space with the norm $\|\cdot\|$. Let $A : D(A) \subset X \to X$ be a densely defined, linear operator. Consider the Cauchy problem

$$\frac{du}{dt} = Au, \quad u(0) = u_0, \quad t > 0 \tag{1}$$

on X. The fundamental Hille–Yosida theorem ([5], [10], [11], [15]) says that if A is m-dissipative, that is, if A satisfies:

(i) $\|u\| \leq \|u - \lambda Au\|$ for $u \in D(A)$ and $\lambda > 0$,
(ii) the range of $(I - \lambda A)$ equals X for $\lambda > 0$,

2000 *Mathematics Subject Classification.* 47D03, 47H20, 47H06.
Key words and phrases. Cauchy problems, operator semigroups.

then A generates a linear operator semigroup $S(t)$, and $S(t)u_0$ for $u_0 \in D(A)$ is the unique solution of (1). To obtain $S(t)$, Hille ([15]) proves that $\{(I - (t/n)A)^{-n}x\}_{n\in\mathbb{N}}$ is Cauchy for $x \in D(A^2)$, while Yosida ([11]) uses the so-called Yosida approximation $\lambda A(\lambda - A)^{-1}, \lambda > 0$.

Extend (1) to the nonlinear multi-valued case

$$\frac{du}{dt} \in Au, \quad u(0) = u_0,\ t > 0, \tag{2}$$

where $A : D(A) \subset X \to X$ is nonlinear and multi-valued. The fundamental Crandall–Liggett theorem (see [3], [1], [8], [9]) shows that if A satisfies the following conditions:

(iii) $\|u - v\| \le \|(u - v) - \lambda(x - y)\|$ for $\lambda > 0$, $u, v \in D(A)$, $x \in Au$, and $y \in Av$,

(iv) the range of $(I - \lambda A) \supset \overline{D(A)}$ for small enough $\lambda > 0$,

then A generates a nonlinear operator semigroup $S(t)$. Due to Benilan (see [1], [2], [8], [9]), $S(t)u_0$ for $u_0 \in D(A)$ satisfies some integral inequalities, associated with (2), and is called the unique generalized solution to (2). Crandall and Liggett prove the existence of $S(t)$ by showing that $\{(I - (t/n)A)^{-n}x\}_{n\in\mathbb{N}}$ is Cauchy for $x \in D(A)$.

For the existence of a strong solution in the nonlinear case, the known results include the cases where reflexive X (see [1], [3], [8], [9]) or demi-continuous A [9, p. 88] is assumed. In this paper, we will use the method of lines ([7], [12]), combined with the Crandall–Liggett theorem, to show that (2) has a limit solution, which is a unique strong one if A is what we call, embeddedly quasi-demi-closed. Here note that our assumption of embeddedly quasi-demi-closed (see Section 4) is weaker than that of demi-continuous. An application to nonlinear partial differential equations in non-reflexive X is given in Section 5.

In the case of linear, single-valued closed A which satisfies (iii) and (iv), the Hille–Yosida theorem applied to the section of A in the Banach space $\overline{D(A)}$ shows that the section is an m-dissipative operator on the Banach space $\overline{D(A)}$ and that for $u_0 \in D(A)$ with $Au_0 \in \overline{D(A)}$, (1) has a unique solution $u(t)$, and du/dt is differentiable in t for $u_0 \in D(A^2)$ with $A^2u_0 \in \overline{D(A)}$. In this paper, we will show, by making use of the Crandall–Liggett theorem that the same results hold true, together with the additional property that du/dt is Lipschitz continuous in t for $u_0 \in D(A^2)$.

In [6], the Crandall–Liggett theorem is applied to this nonlinear differential operator $B : D(B) \subset (C[0,1], \|\cdot\|_\infty) \to (C[0,1], \|\cdot\|_\infty)$, where $Bu \equiv \psi(x, u')u''$ for $u \in D(B) \equiv \{v \in C^2[0,1] : v'(j) \in (-1)^j\beta_j(v(j)),\ j = 0, 1\}$. It is showed that B satisfies (iii) and (iv) and then, there is a unique generalized solution to the nonlinear parabolic boundary value problem

$$\begin{aligned} \frac{\partial}{\partial} u(x,t) &= \psi(x,u_x)u_{xx}, \qquad (x,t) \in (0,1)\times(0,\infty),\\ u_x(j,t) &\in (-1)^j \beta_j(u(j,t)), \quad j=0,1,\\ u(x,0) &= u_0(x). \end{aligned}$$

In this paper, we extend this result to a more general nonlinear differential operator and obtain a strong solution (Section 5), stronger than a generalized solution; precisely, we consider this nonlinear differential operator $G : D(G) \subset C[0,1] \to C[0,1]$, where $Gu \equiv \psi(x,u')u'' + g(x,u,u')$ for $u \in D(G) \equiv D(B)$. We show that G satisfies (iii) and (iv) and is embeddedly quasi-demi-closed. That gives a strong solution to the corresponding nonlinear parabolic boundary value problem.

The rest of the paper is planned as follows. Section 2 gives a preliminary result. Section 3 deals with a limit solution and the case of linear A. Section 4 is concerned with a strong solution and Section 5 is about an application to nonlinear partial differential equations in non-reflexive X.

2. A preliminary result

As before, let $(X, \|\cdot\|)$ be a real Banach space with the norm $\|\cdot\|$, $A : D(A) \subset X \to X$ be a multi-valued nonlinear operator, and A satisfy the conditions (iii) and (iv) in the Introduction.

Let $T > 0$, $u_0 \in D(A)$, $n \in \mathbb{N}$ be large. Consider the discretization of (2)

$$u_i - \lambda A u_i \ni u_{i-1}, \quad u_i \in D(A), \tag{3}$$

where $\lambda = T/n$ and $i = 1$ to n. For the given $u_0 \in D(A) \subset X$, u_i exists for $i = 1,\dots,n$ by the condition (iv). Uniqueness of u_i follows from (iii). For convenience, define u_{-1} to be an element in $(u_0 - \lambda A u_0)$, so that $u_{-1} = u_0 - \lambda v_0$ for some $v_0 \in Au_0$.

By the condition (iii), $\|u_i - u_{i-1}\| \le \|u_{i-1} - u_{i-2}\|$. It follows that $\|u_i - u_{i-1}\| \le \|u_0 - u_{-1}\| \le \lambda\|v_0\|$. Thus we have proved

Proposition 1. *The u_i in* (3) *satisfies $\|u_i - u_{i-1}\| \le \lambda\|v_0\|$ for $i = 1,\dots,n$.*

3. A limit solution

From here on, let k be a generic constant which can vary with different occasions.

Consider the u_i in (3) and put $t_i = i\lambda$ for $i = 1,\dots,n$. Define $\chi^n(0) = u_0$, $\chi^n(t) = u_i$ for $t \in (t_{i-1}, t_i]$ and

$$u^n(t) = u_{i-1} + \frac{u_i - u_{i-1}}{\lambda}(t - t_{i-1}) \tag{4}$$

for $t \in [t_{i-1}, t_i]$. By the definitions of $\chi^n(t)$ and $u^n(t)$, we have

$$\limsup_{n\to\infty} \|u^n(t) - \chi^n(t)\| = 0$$

and

$$\frac{du^n(t)}{dt} \in A\chi^n(t), \quad u^n(0) = u_0 \tag{5}$$

for almost every t, where the last equation has values in $B([0,T];X)$, the real Banach space of bounded functions from $[0,T]$ to X.

PROPOSITION 2. *For each $t \in [0,T]$, $u^n(t)$ has a convergent subsequence in X and so, $u^n(t)$ is relatively compact in X.*

PROOF. Note that for each bounded $t \in (0,T)$, we have $t \in [t_{i-1}, t_i)$ for some i and so, $i-1 = [t/\lambda]$. Here for each $x \in \mathbb{R}$, $[x]$ is the greatest integer that is less than or equal to x. Also note from the definition of u_t^n that pointwise convergence of u_t^n is the same as that of u_{i-1} since $\|(u_u - u_{i-1})/\lambda\| \le k$, by Proposition 1.

Since $u_{i-1} = (I - \lambda A)^{-(i-1)}u_0$ and the convergence of u_{i-1} as $\lambda \to 0$ is the same as the convergence of $(I - (t/n)A)^{-n}u_0$ for each bounded t as $n \to 0$, the proof is completed by applying the Crandall–Liggett theorem.

Applying Proposition 1 to (4) we have

$$\|u^n(t) - u^n(\tau)\| \le k|t - \tau| \tag{6}$$

for $t, \tau \in [t_{i-1}, t_i]$, that is, $u^n(t)$ is equi-continuous in $C([0,T];X)$. Here the space $C([0,T];X)$ is the real Banach space of continuous functions from $[0,T]$ to X. Proposition 2 says that for each $t \in [0,T]$, $u^n(t)$ is relatively compact in X. Therefore, $u^n(t)$ converges to some $u(t)$ in $C([0,T];X)$ by the Ascoli–Arzela theorm [13]. Here we denote $u(t)$ by $S(t)u_0$. Since $u^n(t)$ satisfies (5) and converges to $u(t)$ in $C([0,T];X)$, we call $u(t)$ a limit solution of (2) on $[0,T]$ (and then on $[0,\infty)$ since T is arbitrary). Thus we have proved

PROPOSITION 3. *The equation* (2)*, where A is defined as in Section* 2*, has a limit solution for $u_0 \in D(A)$.*

Note that since for each $t \in [t_{i-1}, t_i)$, $[t/\lambda] = i-1$ and from (3) we have that

$$u_{i-1} = (I - \lambda A)^{-(i-1)}u_0 = (I - \lambda A)^{-[t/\lambda]}u_0 \to S(t)u_0 = u(t)$$

for $u_0 \in D(A)$, as $\lambda \to 0^+$. Since $u(t)$ is the uniform limit of $u^n(t)$, $S(t)u_0$ is continuous for $u_0 \in D(A)$. By (iii), $(I-\lambda A)^{-[t/\lambda]}$ are contractions and so, $S(t)u_0$ also exists for $u_0 \in \overline{D(A)}$ and is continuous in t. On the other hand, from (6), where k depends on $\{\|v\| : v \in Au_0,\ u_0 \in D(A)\}$, we see that $u(t) = S(t)u_0$ for $u_0 \in D(A)$ is Lipschitz continuous in t.

Now suppose additionally that A is linear, single-valued, and closed. Here by closedness of A, we mean that if $x_n \in D(A), y_n = Ax_n, x_n \to x$, and $y_n \to y$, then $x \in D(A)$ and $y = Ax$. Assume that $u_0 \in D(A)$ and $Au_0 \in \overline{D(A)}$.

From (3) we have

$$\begin{aligned} Au_i &= \frac{u_i - u_{i-1}}{\lambda} = \frac{(I-\lambda A)^{-i}u_0 - (I-\lambda A)^{-(i-1)}u_0}{\lambda} \\ &= (I-\lambda A)^{-(i-1)}\left(\frac{(I-\lambda A)^{-1} - I}{\lambda}\right)u_0 \\ &= (I-\lambda A)^{-(i-1)}(I-\lambda A)^{-1}Au_0 = (I-\lambda A)^{-i}(Au_0) \end{aligned}$$

since A is linear. Letting $\lambda \to 0^+$, we have Au_i or $A\chi^n(t) \to S(\tau)Au_0$ since $Au_0 \in \overline{D(A)}$ and so, $Au(\tau) = S(\tau)Au_0$ since A is closed; also $\int_0^t A\chi^n(\tau)\,d\tau \to \int_0^t Au(\tau)\,d\tau$ by the Lebesgue convergence theorem since $\|A\chi^n(\tau)\| \le k$. Thus the integrated (5)

$$u^n(t) - u_0 = \int_0^t A\chi^n(\tau)\,d\tau$$

converges to

$$u(t) - u_0 = \int_0^t Au(\tau)\,d\tau. \tag{7}$$

Since $Au(\tau) = S(\tau)Au_0$ is continuous in τ for $Au_0 \in \overline{D(A)}$ and is Lipschitz continuous in τ for $Au_0 \in D(A)$, we have that so is $Au(\tau)$. Thus by the fundamental theorem of calculus, (7) gives that

$$\frac{du(t)}{dt} = Au = S(t)Au_0, \quad u(0) = u_0. \tag{8}$$

Thus du/dt is continuous in t for $u_0 \in D(A)$ with $Au_0 \in \overline{D(A)}$ and Lipschitz continuous in t for $u_0 \in D(A)$ with $Au_0 \in D(A)$. This, together with (7), in turn shows that du/dt is differentiable in t for $u_0 \in D(A^2)$ with $A^2u_0 \in \overline{D(A)}$. More regularity of du/dt in t can be obtained iteratedly.

Uniqueness of solution in (8) is standard and follows from e.g. [9, Lemma 4.9, p. 88]. Thus, we have proved

Theorem 1. *If the operator A in Section 2 is linear and closed, then (8) has a unique solution $u(t)$ for $u_0 \in D(A)$ with $Au_0 \in \overline{D(A)}$, which has the property that $du(t)/dt$ is continuous in t. Furthermore, du/dt is Lipschitz continuous in t for $u_0 \in D(A)$ with $Au_0 \in D(A)$ and differentiable in t for $u_0 \in D(A^2)$ with $A^2u_0 \in \overline{D(A)}$. More regularity of du/dt in t can be obtained iteratedly.*

Remark 1. The above result, except for the case for the Lipschitz continuity of du/dt in t, can be obtained by applying the Hille–Yosida theorem to the section of A in the Banach space $\overline{D(A)}$, in which case, the section becomes an m-dissipative operator in $\overline{D(A)}$.

4. A strong solution

Let $(Y, \|\cdot\|_Y)$ be a real Banach space with $(X, \|\cdot\|)$ continuously embedded into it. Assume additionally that A is embeddedly quasi-demi-closed, that is, assume that if $x_n \in D(A) \to x$ and $\|y_n\| \leq k$ for some $y_n \in Ax_n$, then $x \in D(\phi \circ A)$ (that is, $\phi(Ax)$ exists.) and $|\phi(y_{n_k}) - z)| \to 0$ for some subsequence y_{n_k} of y_n, for some $z \in \phi(Ax)$ and for each $\phi \in Y^* \subset X^*$, the real dual spaces of Y and X, respectively.

Let $v_n(t) \in A\chi^n(t)$ for $t \in (t_{i-1}, t_i]$ be such that (5) gives

$$\frac{du^n(t)}{dt} = v_n(t)$$

for $t \in (t_{i-1}, t_i]$. Integrating (5) gives that for each $\phi \in Y^* \subset X^*$, $\phi(u^n(t) - u_0) = \int \phi(v^n(\tau)\, d\tau$ and

$$\phi(u^n(t) - u_0) \in \phi\biggl(\int_0^t A\chi^n(\tau)\, d\tau\biggr) = \int_0^t \phi(A\chi^n(\tau))\, d\tau,$$

where note that $\sup_{t\in[0,T]} \|v_n(t)\| \leq k$ by $v_n(t) \in A\chi^n(t)$ and $\|(u_i - u_{i-1})/\lambda\| \leq k$ from Proposition 1. Since $u^n(t) \to u(t)$ uniformly for bounded t and A is embeddedly quasi-demi-closed, we have that $\phi(v_n(t))$ converges to $\phi(v(t))$ through some subsequence for some $v(t) \in Au(t)$ and then, by the Lebesgue convergence theorem, we have

$$\phi(u(t) - u_0) = \int \phi(v(\tau))\, d\tau = \phi\biggl(\int v(\tau)\, d\tau\biggr).$$

Thus $u(t) - u_0 = \int v(\tau)\, d\tau$ in Y. Therefore we have by the Radon–Nikodym type theorem [9] that

$$\frac{du(t)}{dt} = v(t) \quad \text{in } Y \text{ for almost every } t,$$

and then

$$\begin{aligned} &\frac{du(t)}{dt} \in Au(t) \quad \text{in } Y \text{ for almost every } t, \\ &u(0) = u_0. \end{aligned} \tag{9}$$

Again, uniqueness of solution for (9) in X is standard [9]. Thus we have proved

THEOREM 2. *If the operator A in Section 2 is additionally embeddedly quasi-demi-closed, then (9) has a strong solution in Y for $u_0 \in D(A)$, which is unique if $Y \equiv X$.*

REMARK 2. Here note that X is not necessarily reflexive, and that the assumption of embeddedly quasi-demi-closedness is weaker than that of demicontinuity ([9, p. 88]).

5. An application

From here on, k denotes a generic constant, which can vary with different occasions. We make the following assumptions (5.1) to (5.3).

(5.1) $\beta_0, \beta_1 : \mathbb{R} \to \mathbb{R}$ are multi-valued maximal monotone functions with $0 \in \beta_0(0) \cap \beta_1(0)$.

(5.2) $\psi(x,p) \geq \delta_1 > 0$ holds for some constant δ_1 and is continuous.

(5.3) $g(x,z,p)$ is continuous and satisfies $zg(x,z,0) \leq 0$, and

$$\left| \frac{g(x,z,p)}{\psi(x,p)} \right| \leq N(x,z)(1+|p|),$$

where N is positive and continuous.

Define a nonlinear operator $E : D(E) \subset C[0,1] \to C[0,1]$ by $Eu = u''$ for $u \in D(E) \equiv \{u \in C^2[0,1] : u'(j) \in (-1)^j \beta_j(u(j)),\ j = 0,1\}$.

We have the following result in [6], and [14]:

PROPOSITION 4. *For $h \in C[0,1]$, there is a unique solution $u \in C^2[0,1]$ to*

$$u - u'' = h, \quad u'(j) \in (-1)^j \beta_j(u(j)), \quad j = 0,1,$$

and the operator $(I-E)^{-1} : C[0,1] \to C^2[0,1] \subset C[0,1]$ exists and is nonexpansive.

Define a nonlinear operator $G : D(G) \subset C[0,1] \to C[0,1]$ by

$$Gu = \psi(x,u')u'' + g(x,u,u')$$

for $u \in D(G) \equiv \{v \in C^2[0,1] : v'(j) \in (-1)^j \beta_j(v(j)),\ j = 0,1\}$.

PROPOSITION 5. *For each $h \in C[0,1]$, the equation*

$$u - \lambda(\psi(x,u,u')u'' + g(x,u,u')) = h, \quad u'(j) \in (-1)^j \beta_j(u(j)), \quad j = 0,1, \tag{10}$$

has a solution for small enough $\lambda > 0$. And so G satisfies (iv).

PROOF. As in [6] and [14], consider the operator equation equation $u = (u - \lambda E)^{-1} W u$, where

$$W : C^1[0,1] \to C[0,1], \quad Wu = u + \frac{h - u + \lambda g(x,u,u')}{\psi(x,u')},$$

and $(I - \lambda E)^{-1} : C[0,1] \to C^2[0,1]$ is from Proposition 4 and continuous. Solvability of this operator equation will complete the proof.

We truncate W by defining, for each $m \in \mathbb{N}$,

$$W_m u = \begin{cases} Wu & \text{if } \|u\|_{C^1[0,1]} \leq m, \\ W\left(\dfrac{mu}{\|u\|_{C^1[0,1]}}\right) & \text{if } \|u\|_{C^1[0,1]} > m. \end{cases}$$

It follows that $(I-\lambda E)^{-1}W_m : C^1[0,1] \to C^1[0,1]$ is continuous, compact, and uniformly bounded for each m; the compactness follows from the Ascoli–Arzela theorem. By the Schauder fixed point theorem [4],

$$(I-\lambda E)^{-1}W_m u_m = u_m$$

holds for some u_m. We complete the proof by showing $\|u_{m_0}\|_{C^1} \le u_{m_0}$ for some m_0 since $(I-\lambda E)^{-1}Wu_{m_0} = u_{m_0}$ in this case.

Assume $\|u_m\|_{C^1} > m$ for all m and we seek a contradiction. By the definition of W_m, we have

$$u_m - \lambda u_m'' = v_m + \frac{h - v_m + \lambda(g(x, v_m, v_m'))}{\psi(x, v_m')}, \tag{11}$$

where $u_m \in D(E)$ and $v_m = m u_m/\|u_m\|_{C^1}$. We have from the first and second derivative tests that

$$\|u_m\|_\infty = |u_m(x_0)|, \quad u_m'(x_0) = 0, \quad \text{and} \quad u_m(x_0)u_m''(x_0) \le 0$$

for some $x_0 \in [0,1]$. Multiplying (11) by u_m and evaluating it at x_0, we have

$$\begin{aligned} 0 &\le ((1 - \frac{m}{\|u_m\|_{C^1}})u_m^2(x_0) - \lambda u_m(x_0)u_m''(x_0))\psi(x_0, 0) \\ &= (h - v_m)(x_0)u_m(x_0) + \lambda u_m(x_0)(g(x_0, v_m, 0)) \\ &\le (h - v_m)(x_0)u_m(x_0) \end{aligned}$$

and so, $\|v_m\|_\infty \le \|h\|_\infty$. It follows from (11) and (5.3) that

$$\|v_m''\|_\infty \le \lambda^{-1}k + k(1 + \|v_m'\|_\infty).$$

Using the interpolation inequality [4]:

$$\|v_m'\|_\infty \le \varepsilon\|v_m''\|_\infty + \eta(\varepsilon)\|v_m\|_\infty$$

for all $\varepsilon > 0$, we have $\|v_m\|_{C^2[0,1]} \le k$, which is a contradiction to $m = \|v_m\|_{C^1} \le \|v_m\|_{C^2} \le k$ as $m \to \infty$. □

Proposition 6. *G satisfies* (iii).

Proof. Let $u, v \in D(B)$. As in [6], applying the first and second derivative tests gives $\|u-v\|_\infty = |(u-v)(x_0)|$, $(u-v)'(x_0) = 0$, and $(u-v)(x_0)(u-v)''(x_0) \le 0$ for some $x_0 \in [0,1]$. Since

$$\begin{aligned} &(u-v)(x_0)(Bu - Bv)(x_0) \\ &= \psi(x_0, u, u')(u-v)(x_0)(u-v)''(x_0) + (u-v)(x_0)(g(x_0, u, u') - g(x_0, v, u')) \le 0, \end{aligned}$$

we have

$$\|u-v\|_\infty^2 = (u-v)^2(x_0) \le (u-v)(x_0)((u-v)(x_0) - \lambda(Gu - Gv)(x_0))$$

for all $\lambda > 0$, and so

$$\|u - v\|_\infty \leq \|(u - v) - \lambda(Gu - Gv)\|_\infty.$$

Thus (iv) is proved. $\square$

We now show that G is embeddedly quasi-demi-closed, so that Theorem 2 applies.

Let $(Y, \|\cdot\|) = (L^2(0,1), \|\cdot\|)$, which has $(C[0,1], \|\cdot\|_\infty)$ continuously embedded into it. Let $u_n \in D(G) \to u$ and $\|Gu_n\|_\infty \leq k$. As in prove of Proposition 5, we have $\|u_n\|_{C^2[0,1]} \leq k$. It follows from the Ascoli–Arzela theorem that $u_n \to u$ in $C^1[0,1]$ through some subsequence. Apply this to the following.

Let $\eta \in L^2(0,1) = (L^2(0,1))^*$. We have to show that $u \in D(\eta \circ G)$, that is, $\eta(Gu)$ exists, and that $|\eta(Gu_n) - \eta(Gu)| \to 0$. Formally, we have

$$\begin{aligned}
\eta(Gu_n) &= \int \eta(\psi(x, u_n')u_n'' + g(x, u_n, u_n'))\,dx \\
&= \int \eta\psi(x, u')(u_n - u)''\,dx + \int \eta(\psi(x, u_n') - \psi(x, u'))u_n''\,dx \\
&\quad + \int \eta(g(x, u_n, u_n') - g(x, u, u'))\,dx + \int \eta(Gu)\,dx \equiv \sum_i^4 I_i.
\end{aligned}$$

Here the integration range $[0,1]$ is omitted.

It follows that I_1 converges to 0 since $\|u_n\|_{C^2} \leq k$, $W^{2,2}(0,1)$ is a Hilbert space, $\eta\psi(x,u') \in L^2(0,1)$, and u_n'' converges weakly to u'' through some subsequence (this also shows $\eta(Gu)$ exists), that I_2 converges to 0 by $|I_2| \leq \|\psi(x, u_n') - \psi(x, u')\|_\infty \|\eta\| \|u_n''\|$, and that I_3 converges to 0 by the uniform convergence theorem. Thus $\eta(Gu)$ exists and $\eta(Gu_n) \to \eta(Gu)$ and so, G is embeddedly quasi-demi-closed. By Theorem 2, we have that

Theorem 3. *In $L^2(0,1)$, there is a strong solution u to the nonlinear parabolic boundary value problem*

$$\begin{aligned}
\frac{\partial}{\partial t} u(x,t) &= \psi(x, u, u_x)u_{xx} + g(x, u, u_x), \quad (x,t) \in (0,1) \times (0,\infty), \\
u_x(j,t) &\in (-1)^j \beta_j(u(j,t)), \quad j = 0, 1, \\
u(x,0) &= u_0(x),
\end{aligned}$$

for almost every t and for $u_0 \in D(G)$.

Remark 3. Theorem 2 with a more general equation, obtains a strong solution and so, is stronger than [4], [14]. More applications to partial differential equations can be done through Theorem 2.

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Manuscript received April 27, 1999

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