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A GLOBAL BIFURCATION RESULT FOR QUASILINEAR ELLIPTIC EQUATIONS IN ORLICZ–SOBOLEV SPACES

Vy Khoi Le

Abstract. The paper is concerned with a global bifurcation result for the equation

$$-\mathrm{div}(A(|\nabla u|)\nabla u) = g(x,u,\lambda)$$

in a general domain Ω with non necessarily radial solutions. Using a variational inequality formulation together with calculations of the Leray–Schauder degrees for mappings in Orlicz–Sobolev spaces, we show a global behavior (the Rabinowitz alternative) of the bifurcating branches.

1. Introduction

This paper is concerned with global bifurcation for boundary value problems of the form:

$$\begin{cases} -\mathrm{div}\,(A(|\nabla u|)\nabla u) = g(x,u,\lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, Ω is a bounded open set in $\mathbb{R}^N (N \geq 1)$ with smooth boundary, $g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and

$$\phi(s) = A(|s|)s \tag{2}$$

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is an odd, continuous, increasing function from $\mathbb{R}$ to $\mathbb{R}$. The operator

$$-\mathrm{div}\,(A(|\nabla u|)\nabla u)$$

is a natural nonlinear extension of the Laplacian, called p-Laplacian like or ϕ-Laplacian operator. If $\Omega = B(0, R)$ is a ball and u is a radial solution (i.e., u depends only on the radius), then (1) can be reduced to an ordinary diffential equation. In this case, the problem was studied in [14], using methods of ODE's theory. The ordinary differential equation is converted to an integral equation. Since the functions are with one variable, the class of continuous functions is appropriate for the problem.

The goal of this paper is to study the bifurcation of (1) in a general domain Ω and the solutions are not assumed to be symmetric. In this general setting, the usual appropriate function space is a Sobolev space, which is not generally embedded into continuous fucntions. The case where ϕ has polynomial growth has been studied in [12] and [24]. Local bifurcation of (1) was considered in [12] and [24] was about global bifurcation of that equation. A main difficulty in investigating the bifurcation of (1) is that in several interesting cases the operator ϕ has different rates of growth at 0 and at infinity. As a result, the "linearized" problem is formulated appropriately only in a function space different from the original function space. In this paper, we concentrate our attention on cases where the principal function $\phi(s)$ is not equivalent to the powers s^p. Our suitable function spaces here would therefore be Orlicz–Sobolev spaces. The pioneering works of Donaldson and Gossez for equations in Orlicz–Sobolev spaces [10], [15] (and the references therein, also the recent papers [16], [4], [5]) were based mostly on pseudo-monotone operator methods. Some related issues have been revisited recently by other approaches in [13] (an eigenvalue problem for operators in Orlicz–Sobolev spaces), [7] (Mountain pass type theorem for equations in Orlicz–Sobolev spaces), and [23] (Existence results for equations with fast growth rates), etc. We are interested here in bifurcation problems for quasilinear elliptic equations in Orlicz–Sobolev spaces.

As indicated previously, we need to change the working space to different auxiliary spaces, where the Leray–Schauder degree are more conveniently computed. Another feature worth to point out is that we convert equation (1) to a variational inequality. This permits us to use convergence of Γ-type to study the stability of solutions. Another advantage of formulating the problem as an inequality is the flexibilty in working with a proper subspace of the original function space by restricting to the effective domains of the convex functionals in the inequality. Our discussion on bifurcation of variational inequalities is motivated by general arguments and results in [22] and [24] (see also [21], [27], [8] and the references therein for other approaches to bifurcation problems in inequalities). The main bifurcation result here is based on a general, abstract theorem proved in [24]. However, the justification of that abstract result for equations in Orlicz–Sobolev spaces requires nontrivial adaptation and new arguments as compared to the case of equations with principal operators having polynomial growths.

The paper is organized as follows. In Section 2, we present the basic, general bifurcation result that is needed for the discussion in Section 3, together with related assumptions and notation. The main result is discussed in Section 3. In 3.1, we formulate (the weak form of) equation (1) as a variational inequality. The assumptions for the "homogenization" process are verified in 3.2, which results in our main bifurcation result (Theorem 2). An example is presented in 3.3 to illustrate the bifurcation theorem in 3.2. An appendix is included for the somewhat technical proof of a stability property concerning quasilinear equations in Orlicz–Sobolev spaces. This property is needed in the proof of our main result in 3.2 and can also be used to prove the Palais–Smale property of quasilinear operators in Orlicz–Sobolev spaces.

2. Preliminaries – abstract bifurcation result

In this section, we recall certain concepts and abstract results in [24] that will be used to prove our main results in Section 3. The interested reader is referred to that paper for more details and complete proofs.

In [24], we consider variational inequalities of the form:

$$\begin{cases} j(v) - j(u) \geq \langle B(u,\lambda), v-u \rangle \quad \text{for all } v \in X, \\ u \in X. \end{cases} \tag{3}$$

Here, X is a (real) reflexive Banach space with norm $\|\cdot\| = \|\cdot\|_X$, dual X^* and dual pairing $\langle\,\cdot\,,\,\cdot\,\rangle = \langle\,\cdot\,,\,\cdot\,\rangle_X$. We assume furthermore that X is compactly embedded in another Banach space Z, i.e., $X \subset Z$ and the mapping $i : X \to Z$, $i(x) = x$, is compact. Let $\|\cdot\|_Z$, Z^*, and $\langle\,\cdot\,,\,\cdot\,\rangle_Z$ denote the norm, dual, and dual pairing of Z. Assume that j is a convex, lower semicontinuous functional from X to $[0,\infty]$ such that $j(0) = 0$ and j is coercive on X, i.e.,

$$\lim_{\substack{v \in X \\ \|v\| \to \infty}} \frac{j(v)}{\|v\|} = \infty. \tag{4}$$

For simplicity, we assume here that j is strictly convex. This assumption allows us to used the Leray–Schauder degree of single-valued instead of multi-valued compact fields.

Suppose B is a continuous, bounded mapping from $Z \times \mathbb{R}$ to Z^* such that $B(0,\lambda) = 0$, for all $\lambda \in \mathbb{R}$. Since $B(u,\lambda) \in Z^* \subset X^*$, (3) is well defined. In case there is no confusion, we still denote by B the restriction of B on $X \times \mathbb{R}$, i.e., $B = B|_{X\times\mathbb{R}} = B \circ (i \times I)$ and $B(\,\cdot\,,\lambda) = B(\,\cdot\,,\lambda)|_X = B(\,\cdot\,,\lambda) \circ i$. Here, I is the identity mapping on $\mathbb{R}$ and $i \times I : X \times \mathbb{R} \to Z \times \mathbb{R}$ is the embedding of $X \times \mathbb{R}$ into $Z \times \mathbb{R}$. The inequality (3) is, in fact,

$$\begin{cases} j(v) - j(u) \geq \langle B|_{X\times\mathbb{R}}(u, \lambda), v - u\rangle & \text{for all } v \in X \\ u \in X. \end{cases}$$

We now present a homogenization for the inequality (3) which is useful for studying global behaviors of its bifurcation branches. Let $p > 1$ be a fixed constant. Let A be a fixed positive number and $\alpha_0 : (0, A) \to (0, \infty)$ be a continuous function. Put $\alpha(s) = s\alpha_0(s)$ for $s \in (0, A)$. Assume there exists a continuous mapping f from $Z \times \mathbb{R}$ to Z^* such that for all sequences $\{v_n\} \subset Z$, $\{\sigma_n\} \subset \mathbb{R}^+ \setminus \{0\}$, $\{\lambda_n\} \subset \mathbb{R}$ satisfying

$$v_n \to v \quad \text{in } Z, \quad \lambda_n \to \lambda, \quad \sigma_n \to 0^+ \quad \text{in } \mathbb{R}, \tag{5}$$

we have

$$\frac{1}{\alpha_0(\sigma_n)} B(\sigma_n v_n, \lambda_n) \to f(v, \lambda) \quad \text{in } Z^*. \tag{6}$$

Suppose furthermore that if $\{u_n\}$ is a bounded sequence in Z, $\lambda_n \to \lambda$ and $\sigma_n \to 0^+$, then the sequence

$$\left\{\frac{B(\sigma_n u_n, \lambda_n)}{\alpha_0(\sigma_n)}\right\} \quad \text{is bounded in } Z^*. \tag{7}$$

For each $\sigma > 0$, we define the functional $j_\sigma : X \to [0, \infty]$ by

$$j_\sigma(v) = \frac{1}{\alpha(\sigma)} j(\sigma v), \quad \text{for all } v \in X.$$

Assume that there exists a coercive, strictly convex, lower semicontinuous functional $j_0 : X \to [0, \infty]$ with $j_0(0) = 0$ such that for any sequence $\{\sigma_n\} \subset \mathbb{R}^+ \setminus \{0\}$, $\sigma_n \to 0$, the sequence $\{j_{\sigma_n}\}$ converges to j_0 in the following sense.

(H1) If $u_n \rightharpoonup u$ in X ("$\rightharpoonup$" denotes the weak convergence), then

$$j(u) \leq \liminf j_{\sigma_n}(u_n). \tag{8}$$

(H2) For each $u \in X$, there exists a sequence $\{u_n\}$ in X such that

$$u_n \rightharpoonup u \quad \text{in } X \tag{9}$$

and

$$j_{\sigma_n}(u_n) \to j(u) \quad \text{in } [0, \infty]. \tag{10}$$

The (H1)–(H2) condition is usually used in the literature in the name of variational or Mosco convergence (cf. [2]).

(H3) (Equi-coercivity) $j_{\sigma_n}(u)/\|u\| \to \infty$ as $\|u\| \to \infty$, $n \to \infty$ in the following sense: for each $M > 0$, there exist $n_0 \in \mathbb{N}$ and $R > 0$ such that

$$j_{\sigma_n}(u)/\|u\| \geq M, \tag{11}$$

whenever $n \geq n_0$ and $\|u\| \geq R$.

(H4) (Nondegeneracy condition) If $u_n \rightharpoonup 0$ in X and $j_{\sigma_n}(u_n) \to 0$, then $u_n \to 0$ in X.

Note that (H4) is equivalent to the following condition: If $u_n \rightharpoonup u$ in X, $\inf\{\|u_n\| : n \in \mathbb{N}\} > 0$, and $j_0(u) = \lim j_{\sigma_n}(u_n)$, then $u \neq 0$. Also, (H4) is a form of the Palais–Smale condition for a family of variational inequalities at the level $c = 0$. We also assume that j and j_0 satisfy the following compactness condition:

(H5) Assume $\{u_n\} \subset X$ and $\{f_n\} \subset X^*$ satisfy:

$$\begin{cases} \text{(i)} & u_n \rightharpoonup u \quad \text{in } X, \\ \text{(ii)} & f_n \to f \quad \text{in } X^*, \quad \text{and} \\ \text{(iii)} & f_n \in \partial j(u_n) (\text{or } f_n \in \partial j_0(u_n)), \quad \text{for all } n, \end{cases} \tag{12}$$

then

$$u_n \to u \quad \text{in } X. \tag{13}$$

It is easy to prove that j_0 and f that satisfy the above assumptions are uniquely determined. Moreover, if $\alpha(\sigma) = \sigma^p$ then j_0 is homogeneous of degree p in X and $f(\,\cdot\,, \lambda)$ is homogeneous of degree $(p-1)$ in Z, i.e.,

$$\begin{aligned} j_0(\sigma v) &= \sigma^p j_0(v) && \text{for all } v \in X,\ \sigma \geq 0, \\ f(\sigma v, \lambda) &= \sigma^{p-1} f(v, \lambda) && \text{for all } v \in Z,\ \sigma \geq 0. \end{aligned} \tag{14}$$

We consider the following homogenized variational inequality associated with (3):

$$\begin{cases} j_0(v) - j_0(u) \geq \langle f(u, \lambda), v - u \rangle \quad \text{for all } v \in X, \\ u \in X. \end{cases} \tag{15}$$

Using classical arguments in the theory of variational inequalities (cf. [25] or [18]), one can prove that for each $f \in X^*$, the inequality

$$\begin{cases} j_0(v) - j_0(u) \geq \langle f, v - u \rangle \quad \text{for all } v \in X, \\ u \in X, \end{cases} \tag{16}$$

has a unique solution $u = u_f$. We denote by P_0 the solution (resolvent) operator associated with (16):

$$P_0 : X^* \to X, \quad f \mapsto u_f,$$

where u_f is the unique solution of (16). P_0 is the inverse of the subgradient operator ∂j_0. Using P_0, we can write (15) equivalently as the fixed point equation

$$u = P_0[f(u,\lambda)], \quad u \in X.$$

We use usual definitions of bifurcation points of (3) and eigenvalues of (15) (cf. e.g. [24], [22]). It is proved in [24] the following result relating global bifurcation of (3) with eigenvalues and topological degrees of operators in (15). The presentation in [24] is for the particular case $\alpha(\sigma) = \sigma^p$. However, the proofs and arguments there are still applied for general functions α as defined above.

THEOREM 1. (I) *If* $(0,\lambda)$ *is a bifurcation point of* (3)*, then* λ *is an eigenvalue of* (15)*.*

(II) *If* a *and* b $(a < b)$ *are not eigenvalues of* (15) *and if*

$$d_X(I - P_0[f(\,\cdot\,,a)\circ i], B_r(0), 0) \neq d_X(I - P_0[f(\,\cdot\,,b)\circ i], B_r(0), 0),$$

for some $r > 0$ $(B_r(0)$ *is the open ball in* X *with center at 0 and radius* $r)$*. Let*

$$\mathcal{S} = \overline{\{(u,\lambda) : (u,\lambda) \text{ is a solution of } (17) \text{ with } u \neq 0\} \cup (\{0\}\times[a,b])}^{X},$$

and $\mathcal{C}$ *be the connected component of* $\mathcal{S}$ *that contains* $\{0\}\times[a,b]$*. Then , either*

(i) $\mathcal{C}$ *is unbounded in* $X \times \mathbb{R}$*, or*

(ii) $(0,\lambda_1) \in \mathcal{C}$ *for some eigenvalue* $\lambda_1 \notin [a,b]$ *of* (15)*.*

3. Global bifurcation of equations in Orlicz–Sobolev spaces

3.1. Variational inequality formulation. We note that (1) is written in weak form as the variational equation:

$$\int_\Omega A(|\nabla u|)\nabla u \cdot \nabla v\,dx = \int_\Omega g(x,u,\lambda)v\,dx \quad \text{for all } v \in X, \tag{17}$$

where X is an appropriate set of admissible functions. We assume that ϕ is strictly increasing, $\phi(s) \to \infty$ as $s \to \infty$, and that Φ defined by

$$\Phi(t) = \int_0^t \phi(s)\,ds, \quad s \in \mathbb{R},$$

is a Young function (cf. [19], [1]). We will choose $X = W_0^1 L_\Phi$, where $W^1 L_\Phi$ is the Orlicz–Sobolev space associated with Φ and $W_0^1 L_\Phi$ is the set of functions in $W^1 L_\Phi$ with zero boundary condition (cf. [19], [1], [15]). We denote by $\|\cdot\|_\Phi$ the Luxemburg norm of L_Φ and by $\|\cdot\|_X$ the norm $\|\cdot\|_{W_0^1 L_\Phi}$:

$$\|u\| = \|u\|_X = \|u\|_{W_0^1 L_\Phi} = \||\nabla u|\|_\Phi,$$

which, by Poincaré's inequality, is equivalent to the usual norm of $W_0^1 L_\Phi$, defined as the restriction of the norm

$$u \mapsto \|u\|_{W^1 L_\Phi} = \|u\|_\Phi + \sum_{i=1}^N \|\partial_i u\|_\Phi$$

of $W^1 L_\Phi$ to $W_0^1 L_\Phi$. We denote by $\overline{\Phi}$ the conjugate function of Φ:

$$\overline{\Phi}(t) = \sup\{ts - \Phi(s) : s \in \mathbb{R}\}$$

and by Φ^* the Sobolev conjugate of Φ. Φ^* is defined by

$$(\Phi^*)^{-1}(t) = \int_0^t \Phi^{-1}(s) s^{-(N+1)/N}\, ds.$$

We assume that $\int_0^\infty \Phi^{-1}(s) s^{-(N+1)/N}\, ds = \infty$, so that Φ^* is also a Young function. However, if $\int_0^\infty \Phi^{-1}(s) s^{-(N+1)/N}\, ds < \infty$, then $W_0^1 L_\Phi \hookrightarrow C(\overline{\Omega})$ and all arguments used in the sequel hold for this case without any significant modifications. For simplicity of calculations, we assume that:

$$\text{both } \Phi \text{ and } \overline{\Phi} \text{ satisfy the } \Delta_2 \text{ condition} \tag{18}$$

(cf. [19], [1], [20]). This assumption implies the following properties:

(a) $L_\Phi = E_\Phi = \widetilde{L}_\Phi$, where $\widetilde{L}_\Phi$ is the Orlicz class and E_Φ is the closure of $L^\infty(\Omega)$ in L_Φ. In particular, L_Φ and $L_{\overline{\Phi}}$ are both separable and reflexive and $L_{\overline{\Phi}} = (L_\Phi)^*$,

(b) $W^1 L_\Phi$ and $W_0^1 L_\Phi$ are reflexive (and separable),

(c) the norm convergence in L_Φ is equivalent to the mean (modular) convergence, i.e.,

$$\|u_n - u\|_\Phi \to 0 \quad \text{if and only if} \quad \int_\Omega \Phi(u_n - u) \to 0,$$

(d) the functional $u \mapsto \int_\Omega \Phi(u)$ is coercive in the sense that

$$\frac{1}{\|u\|_\Phi} \int_\Omega \Phi(u) \to \infty \quad \text{when } \|u\|_\Phi \to \infty. \tag{19}$$

For more detailed discussion on these properties, together with other properties of Orlicz–Sobolev spaces with the Δ_2 condition, we refer to [19], [10], [15], [1], [20]. (17) is now formulated as:

$$\begin{cases} \displaystyle\int_\Omega A(|\nabla u|)\nabla u \cdot \nabla v = \int_\Omega g(x,u,\lambda) v \quad \text{for all } v \in W_0^1 L_\Phi, \\ u \in W_0^1 L_\Phi. \end{cases} \tag{20}$$

We assume that g has the growth condition

$$|g(x,u,\lambda)| \le C(\lambda)[A(x) + |\Psi'(u)|], \tag{21}$$

for a.e. $x \in \Omega$, all $\lambda, u \in \mathbb{R}$. Here, Ψ is a Young function of class C^1 such that

$$\Psi \ll \Phi^*, \tag{22}$$

$$A \in L_{\overline{\Psi}}, \tag{23}$$

and $C \in L^\infty_{\mathrm{loc}}(\mathbb{R})$. Using arguments in [13], we can prove from (22) and (23) that the integral in the right hand side of (20) is well defined for all $u, v \in L_\Psi$. Moreover, the mapping $(u, \lambda) \mapsto B(u, \lambda)$, with

$$\langle B(u,\lambda), v\rangle = \int_\Omega g(x,u,\lambda)v \quad \text{for all } v \in L_\Psi, \tag{24}$$

is continuous and bounded from L_Ψ to $(L_\Psi)^* (= L_{\overline{\Psi}})$.

On the other hand, for $u \in W_0^1 L_\Phi$, $|\nabla u| \in L_\Phi$ and thus

$$\phi(|\nabla u|) = A(|\nabla u|)|\nabla u| \in L_{\overline{\Phi}}.$$

This shows that for all $v \in W_0^1 L_\Phi$, the integral in the left hand side of (20) is well defined. Hence, under the above conditions, (20) is defined. Now, let us convert (20) into a variational inequality. For $u \in X (= W_0^1 L_\Phi)$, put

$$j(u) = \int_\Omega \Phi(|\nabla u|)\, dx.$$

Then, j is a strictly convex, $j \geq 0$ on X, and $j(0) = 0$. Moreover, j is differentiable on X and

$$\langle j'(u), v\rangle = \int_\Omega A(|\nabla u|)\nabla u \cdot \nabla v\, dx. \tag{25}$$

In fact, that j is Gateaux differentiable follows from the arguments in [13]. Since the mapping $u \mapsto A(|\nabla u|)\nabla u$ is continuous from $W_0^1 L_\Phi$ to $(L_{\overline{\Phi}})^N$ (as a consequence of the Lebesgue dominated convergence theorem), j is Fréchet differentiable on X and we have (25). Now, (20) is equivalent to the following variational inequality

$$\begin{cases} j(v) - j(u) \geq \displaystyle\int_\Omega g(\,\cdot\,, u, \lambda)(v-u), & \text{for all } v \in X, \\ u \in X. \end{cases} \tag{26}$$

In the next section, we shall study the global bifurcation of this inequality. Let us conclude this section with the verification of some properties of the functional j. It is clear that j is convex, continuous, and thus weakly lower semicontinuous in X. Moreover, (19) implies that j is coercive in the sense of (4). Now, we check that j satisfies (H5). Let $u_n \rightharpoonup u$ in X, $f_n \to f$ in X^*, and $f_n \in \partial j(u_n)$, i.e.,

$$j(v) - j(u_n) \geq \langle f_n, v - u_n\rangle \quad \text{for all } v \in X.$$

This inequality is equivalent to the equation

$$\int_\Omega A(|\nabla u_n|)\nabla u_n \cdot \nabla v = \langle f_n, v\rangle \quad \text{for all } v \in X.$$

Hence, $\int_\Omega A(|\nabla u_n|)\nabla u_n\cdot(\nabla u_n-\nabla v) = \langle f_n, u_n-v\rangle \to 0$. Since $A(|\nabla u|)\nabla u \in L_{\overline{\Phi}}$,

$$\int_\Omega A(|\nabla u|)\nabla u \cdot (\nabla u_n - \nabla v) \to 0.$$

Thus,

$$\int_\Omega [A(|\nabla u_n|)\nabla u_n - A(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla v) \to 0.$$

It follows from this convergence and Theorem 4 in the appendix about the stability of solutions of quasilinear elliptic equations in Orlicz–Sobolev spaces that $u_n \to u$ in $W_0^1 L_\Phi$. We have checked (H5) for j.

3.2. Homogenization – main bifurcation result. Now, we consider the homogenization of (26), as featured in Section 2. First, let us consider certain assumptions on the behaviors of Φ near 0. Assume that for any $s \geq 0$, any sequences $\{\sigma_n\} \subset (0,1]$, $\{s_n\} \subset [0,\infty)$ such that

$$\sigma_n \to 0^+, \quad s_n \to s, \tag{27}$$

we have

$$\frac{\Phi(\sigma_n s_n)}{\alpha(\sigma_n)} \to s^p. \tag{28}$$

Here, $p \geq 1$ is a fixed constant. Some remarks are in order. If (28) is satisfied then

$$\lim_{\sigma\to 0^+} \frac{\Phi(\sigma s)}{\alpha(\sigma)} = s^p \quad \text{for all } s > 0. \tag{29}$$

Hence, we can replace α by Φ, i.e.,

$$\lim_{\sigma\to 0^+} \frac{\Phi(\sigma s)}{\Phi(\sigma)} = s^p \quad \text{for all } s > 0. \tag{30}$$

Also, the function s^p in (30) is a natural choice since if Q satisfies

$$\lim_{\sigma\to 0^+} \frac{\Phi(\sigma s)}{\Phi(\sigma)} = Q(s),$$

then, for $s_1, s_2 > 0$,

$$Q(s_1 s_2) = \lim_{\sigma\to 0^+} \frac{\Phi(\sigma s_1 s_2)}{\Phi(\sigma)} = \lim_{\sigma\to 0^+} \frac{\Phi(\sigma s_1 s_2)}{\Phi(\sigma s_1)} \lim_{\sigma\to 0^+} \frac{\Phi(\sigma s_1)}{\Phi(\sigma)} = Q(s_2)Q(s_1).$$

Hence, $Q(s)$ is a multiplicative function; we have (under an assumption on the continuity of Q) that $Q(s) = s^p$ for some $p \in \mathbb{R}$. We also need some conditions on the growth of $\Phi(\sigma s)/\Phi(\sigma)$ for small values of σ. Assume that there exist constants $A_1 \in (0, A)$, $B_1, B_2, B_3 > 0$, and $s_1 > 0$ such that

$$\frac{\Phi(\sigma s)}{\alpha(\sigma)} \geq B_1 \Phi(s) \quad \text{for all } \sigma \in (0, A_1) \text{ and } s \geq s_1, \tag{31}$$

$$\frac{\Phi(\sigma s)}{\alpha(\sigma)} \geq B_3 s^p \quad \text{for all } \sigma \in (0, A_1) \text{ and } s \in (0, s_1), \tag{32}$$

and

$$\frac{\Phi(\sigma s)}{\alpha(\sigma)} \leq B_2(|s|^p + 1) \quad \text{for all } \sigma \in (0, A_1) \text{ and } s \in \mathbb{R}. \tag{33}$$

It follows from (29) that $s^p \geq B_1 \Phi(s)$ for all $s \geq s_1$. In particular,

$$\Phi < \phi_p, \tag{34}$$

where $\phi_p(s) = s^p$ (again, we refer to [1] or [19] for a definition of the ordering "$<$" in the class of Young functions). We define $j_0 : W_0^1 L_\Phi \to [0, \infty]$ by

$$j_0(u) = \int_\Omega |\nabla u|^p \, dx, \quad u \in W_0^1 L_\Phi, \tag{35}$$

and for $\sigma > 0$,

$$j_\sigma(u) = \frac{j(\sigma u)}{\alpha(\sigma)} = \int_\Omega \frac{\Phi(\sigma |\nabla u|)}{\alpha(\sigma)} \, dx, \quad u \in W_0^1 L_\Phi. \tag{36}$$

By (34) and (35), the effective domain $D(j_0)$ of j_0 is $W_0^{1,p}(\Omega)$. It is easy to check that j_0 is convex and lower semicontinuous, and thus weakly lower semicontinuous in $W_0^1 L_\Phi$. Now, let us prove that j_0 is coercive in $W_0^1 L_\Phi$ (in the sense of (4)). From (34), there are positive constants c_1, c_2 such that

$$|s|^p \geq c_1 \Phi(s) - c_2 \quad \text{for all } s \in \mathbb{R}.$$

Thus,

$$\frac{1}{\|u\|} \int_\Omega |\nabla u|^p \, dx \geq \frac{c_1}{\|u\|} \int_\Omega \Phi(|\nabla u|) - c_2 |\Omega|,$$

for $\|u\| \geq 1$. The coercivity of j_0 now follows from that of j. To prove that j_0 satisfies (H5), we assume that $u_n \rightharpoonup u$ in $W_0^1 L_\Phi$, $f_n \to f$ in $(W_0^1 L_\Phi)^*$, and

$$\int_\Omega |\nabla v|^p - \int_\Omega |\nabla u_n|^p \geq \langle f_n, v - u_n \rangle \quad \text{for all } v \in W_0^1 L_\Phi.$$

It follows that $u_n \in W_0^{1,p}(\Omega)$ and that

$$\int_\Omega |\nabla v|^p - \int_\Omega |\nabla u_n|^p \geq \langle f_n, v - u_n \rangle \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{37}$$

Since the embedding

$$(W_0^1 L_\Phi)^* \hookrightarrow [W_0^{1,p}(\Omega)]^* \tag{38}$$

is continuous, $f_n \to f$ in $[W_0^{1,p}(\Omega)]^*$. Moreover, (37) is equivalent to the p-Laplacian equation

$$p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla v = \langle f_n, v\rangle \quad \text{for all } v \in W_0^{1,p}(\Omega). \tag{39}$$

From classical results concerning the continuity of the inverse operator of the p-Laplacian (i.e., the stability of (39) (cf. e.g. [11] or [9]), we have from (38) and (39) that $u_n \to u$ in $W_0^{1,p}(\Omega)$. In particular, $u_n \to u$ in $W_0^1 L_\Phi$, because the embedding

$$W_0^{1,p}(\Omega) \hookrightarrow W_0^1 L_\Phi \tag{40}$$

is continuous, as a consequence of (34). We have shown that j_0 satisfies (H5). Now, let $\{\sigma_n\}$ be a sequence in $[0,\infty)$ such that $\sigma_n \to 0^+$. We show that

$$j_{\sigma_n} \text{ converges to } j_0 \text{ is the sense of (H1)–(H4).} \tag{41}$$

Let $u_n \rightharpoonup u$ in $W_0^1 L_\Phi$. We prove that

$$\int_\Omega |\nabla u|^p \le \liminf \int_\Omega \frac{\Phi(\sigma_n|\nabla u_n|)}{\alpha(\sigma_n)}. \tag{42}$$

First, let us extend α to a function on $\mathbb{R}$ such that $\alpha(0) = 0$, $\alpha(s) > 0$ for $s \neq 0$, and α is continuous on $\mathbb{R}$. Define the function $T : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ as follows:

$$T(\sigma,\xi) = \begin{cases} \Phi(\sigma|\xi|)/\alpha(\sigma) & \text{if } \sigma \neq 0, \\ |\xi|^p & \text{if } \sigma = 0. \end{cases}$$

(28) and the continuity of Φ and α show that T is continuous in both σ and ξ. Moreover, $T(\sigma,\xi) \ge 0$. On the other hand, since $u_n \rightharpoonup u$ in $W_0^1 L_\Phi$, $\nabla u_n \rightharpoonup \nabla u$ in $(L_\Phi)^N$. It follows from the continuous embedding $L_\Phi \hookrightarrow L^1(\Omega)$ that $\nabla u_n \rightharpoonup \nabla u$ in $[L^1(\Omega)]^N$. Now, since $\sigma_n \to 0$ in $\mathbb{R}$ (and thus pointwise in Ω if σ_n is seen as a constant function), one can apply Theorem 5.4 of [3] to get

$$\int_\Omega T(0,\nabla u) = \int_\Omega |\nabla u|^p \le \liminf \int_\Omega T(\sigma_n, \nabla u_n) = \int_\Omega \frac{\Phi(\sigma_n|\nabla u_n|)}{\alpha(\sigma_n)}.$$

(42) and then (H1) are verified. Now, let us check (H2). Let $u \in W_0^1 L_\Phi$. We choose $u_n = u$ for which (9) is clearly satisfied. If $\int_\Omega |\nabla u|^p = \infty$, then

$$\lim \int_\Omega \frac{\Phi(\sigma_n|\nabla u_n|)}{\alpha(\sigma_n)} = \infty$$

by (H1) and we have (10). Assume now that $\int_\Omega |\nabla u|^p dx < \infty$. We show that

$$\int_\Omega \frac{\Phi(\sigma_n|\nabla u|)}{\alpha(\sigma_n)} \to \int_\Omega |\nabla u|^p. \tag{43}$$

From (28), we have

$$\frac{\Phi(\sigma_n|\nabla u|)}{\alpha(\sigma_n)} \to |\nabla u|^p \quad \text{a.e. on } \Omega.$$

On the other hand, it follows from (33) that

$$\frac{\Phi(\sigma_n|\nabla u|)}{\alpha(\sigma_n)} \le B_2(|\nabla u|^p + 1) \quad \text{a.e. on } \Omega,$$

for all n. Since the function $B_2(|\nabla u|^p + 1)$ is in $L^1(\Omega)$, we can apply the dominated convergence theorem to obtain (43).

To check (H3), assume $\{\sigma_n\} \subset \mathbb{R}^+$, $\{u_n\} \subset W_0^{1,p}(\Omega)$ are sequences such that $\sigma_n \to 0^+$ and $\|u_n\|_{W_0^1 L_\Phi} \to \infty$. We show that

$$\frac{1}{\|u_n\|} \int_\Omega \frac{\Phi(\sigma_n|\nabla u|)}{\alpha(\sigma_n)} \to \infty. \tag{44}$$

For $n \in \mathbb{N}$, put

$$\Omega_{1n} = \{x \in \Omega : |\nabla u_n| \ge s_1\}, \quad \Omega_{2n} = \{x \in \Omega : |\nabla u_n| < s_1\}. \tag{45}$$

It follows from (31) and (32) that

$$\frac{\Phi(\sigma_n|\nabla u|)}{\alpha(\sigma_n)} \ge \begin{cases} B_1\Phi(|\nabla u_n|) & \text{for } x \in \Omega_{1n}, \\ B_3|\nabla u_n|^p & \text{for } x \in \Omega_{2n}. \end{cases}$$

Hence,

$$\int_\Omega \frac{\Phi(\sigma_n|\nabla u|)}{\alpha(\sigma_n)} \ge B_1 \int_{\Omega_{1n}} \Phi(|\nabla u_n|) + B_3 \int_{\Omega_{2n}} |\nabla u_n|^p. \tag{46}$$

On the other hand, since $s^p \ge B_1\Phi(s)$, for all $s \ge s_1$, we have

$$s^p \ge B_1(\Phi(s) - \overline{B_1}) \quad \text{for all } s \in \mathbb{R},$$

where $\overline{B_1} = \max\{\Phi(s) : s \in [0, s_1]\}$. Therefore,

$$\begin{aligned} \int_\Omega \frac{\Phi(\sigma_n|\nabla u|)}{\alpha(\sigma_n)} \ge& B_1 \int_{\Omega_{1n}} \Phi(|\nabla u_n|) + B_3B_1 \int_{\Omega_{2n}} \Phi(|\nabla u_n|) - B_3B_1\overline{B_1}|\Omega_{2n}| \\ \ge& \min\{B_1, B_3B_1\} \int_\Omega \Phi(|\nabla u_n|) - B_3B_1\overline{B_1}|\Omega|. \end{aligned}$$

Consequently,

$$\frac{1}{\|u_n\|} \int_\Omega \frac{\Phi(\sigma_n|\nabla u|)}{\alpha(\sigma_n)} \ge \frac{\min\{B_1, B_3B_1\}}{\|u_n\|} \int_\Omega \Phi(|\nabla u_n|) - \frac{B_3B_1\overline{B_1}|\Omega|}{\|u_n\|}.$$

Since

$$\lim_{\|u\|\to\infty} \frac{1}{\|u\|} \int_\Omega \Phi(|\nabla u|) = \infty$$

(cf. [15]), (44) follows directly from the above estimate. We have proved (H3).

Now, let us prove (H4). Assume $\sigma_n \to 0$ and $u_n \rightharpoonup 0$ in $W_0^1 L_\Phi$ such that $j_{\sigma_n}(u_n) \to 0$, i.e.,

$$\int_\Omega \frac{\Phi(\sigma_n |\nabla u|)}{\alpha(\sigma_n)} \to 0.$$

We show that $u_n \to 0$ in $W_0^1 L_\Phi$. Letting Ω_{1n}, Ω_{2n} be as in (45) and using (46), we get

$$\int_{\Omega_{1n}} \Phi(|\nabla u_n|) \to 0, \qquad \int_{\Omega_{2n}} |\nabla u_n|^p \to 0 \quad \text{as } n \to \infty. \tag{47}$$

Put

$$w_{1n} = \begin{cases} |\nabla u_n| & \text{in } \Omega_{1n}, \\ 0 & \text{in } \Omega_{2n}, \end{cases} \qquad \text{and} \qquad w_{2n} = \begin{cases} |\nabla u_n| & \text{in } \Omega_{2n}, \\ 0 & \text{in } \Omega_{1n}. \end{cases}$$

Hence w_{1n}, w_{2n} are in L_Φ and

$$\int_\Omega \Phi(w_{1n}) \to 0, \qquad \int_\Omega |w_{2n}|^p \to 0 \quad \text{as } n \to \infty.$$

Since Φ and ϕ_p satisfy Δ_2 condition, we have $\|w_{1n}\|_{L_\Phi} \to 0$ and $\|w_{2n}\|_{L^p} \to 0$. Because $\Phi < \phi_p$, $\|w_{2n}\|_{L_\Phi} \to 0$. This implies that $|\nabla u_n| = w_{1n} + w_{2n} \to 0$ in L_Φ, i.e., $u_n \to 0$ in $W_0^1 L_\Phi$. We have checked (H4).

Now, let us consider the homogenization for the lower order term in (26). First, we assume some conditions on $g(x,u,\lambda)$ when u is close to 0. Suppose that for any sequences $\{\sigma_n\}$, $\{u_n\}$, $\{\lambda_n\}$ such that $\sigma_n \to 0^+$, $u_n \to u$, $\lambda_n \to \lambda$ in $\mathbb{R}$, we have

$$\frac{g(x, \sigma_n u_n, \lambda_n)}{\alpha_0(\sigma_n)} \to \lambda |u|^{p-2} u, \tag{48}$$

for a.e. $x \in \Omega$. Moreover, as in (31),

$$\left| \frac{g(x, \sigma u, \lambda)}{\alpha_0(\sigma)} \right| \le C(\lambda)[A(x) + |\Psi'(u)|] \tag{49}$$

for a.e. $x \in \Omega$, all $\sigma \in (0, A_1)$, $u, \lambda \in \mathbb{R}$, where Ψ, A, and C are as in (22) and (23). For $\sigma \in (0, A_1)$, we define $B_\sigma(u,\lambda)$ and $f(u,\lambda)$ by

$$\langle B_\sigma(u,\lambda), v\rangle = \int_\Omega \frac{g(x,\sigma u,\lambda)}{\alpha_0(\sigma)} v, \tag{50}$$

and

$$\langle f(u,\lambda), v\rangle = \int_\Omega \lambda |u|^{p-2} u v, \tag{51}$$

for $u, v \in L_\Psi$. It follows from (48) and (49) that $\lambda |u|^{p-1} \le C(\lambda)[A(x) + \Psi'(u)]$ i.e.,

$$|u|^{p-1} \le C[\Psi'(u) + 1]. \tag{52}$$

Using the arguments in [13], we have from (48) and (49) that the mappings B_σ (for $\sigma > 0$, small) and f are continuous and bounded from $L_\Psi \times \mathbb{R}$ to $L_{\overline{\Psi}}$.

Now, we put $Z = L_\Psi$ with the usual Luxemburg norm $\|\cdot\|_Z = \|\cdot\|_{L_\Psi}$. The dual Z^* of Z is just $L_{\overline{\Psi}}$. Since Ψ and $\overline{\Psi}$ satisfy the Δ_2 condition, Z and Z^* are reflexive. Thus, B_σ and f are continuous and bounded from $Z \times \mathbb{R}$ to Z^*. We check that B_σ and f satisfy conditions (7) and (6). Assume $\{v_n\}$ is a bounded sequence in L_Ψ, $\sigma_n \to 0$, and $\lambda_n \to \lambda$. We show that $\|B_{\sigma_n}(v_n, \lambda_n)\|_{Z^*}$ is bounded. Since

$$\|B_{\sigma_n}(v_n, \lambda_n)\|_{Z^*} = \left\| \frac{g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)} \right\|_{L_{\overline{\Psi}}},$$

we show that $g(\,\cdot\,, \sigma_n v_n, \lambda_n)/\alpha_0(\sigma_n)$ is bounded in $L_{\overline{\Psi}}$. From (49),

$$\left| \frac{g(x, \sigma_n v_n, \lambda)}{\alpha_0(\sigma_n)} \right| \leq C(\lambda_n)[A(x) + |\Psi'(v_n)|]. \tag{53}$$

Since the mapping $u \mapsto \Psi'(u)$ is bounded from L_Ψ to $L_{\overline{\Psi}}$, $A \in L_{\overline{\Psi}}$ and $\{C(\lambda_n)\}$ is bounded, it is clear from from (53) that the sequence $\{g(\,\cdot\,, \sigma_n v_n, \lambda)/\alpha_0(\sigma_n)\}$ is bounded in $L_{\overline{\Psi}}$. Now, assume $\{\sigma_n\}$ and $\{\lambda_n\}$ are as above and that $v_n \to v$ in L_Ψ. We show that

$$\frac{g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)} \to \lambda |v|^{p-2} v \quad \text{in } L_{\overline{\Psi}}. \tag{54}$$

Passing to a subsequence, we can assume that

$$v_n \to v \quad \text{a.e. in } \Omega. \tag{55}$$

Since $\int_\Omega \Psi(v_n - v) \to 0$, $\Psi(v_n - v)$ in $L^1(\Omega)$. Thus, there exists $h \in L^1(\Omega)$ such that

$$\Psi(v_n) \leq h \quad \text{a.e. in } \Omega, \text{ for all } n. \tag{56}$$

From (48) and (55),

$$\frac{g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)} \to \lambda |v|^{p-2} v \quad \text{a.e. in } \Omega. \tag{57}$$

On the other hand, for $x \in \Omega$,

$$\begin{aligned} \overline{\Psi}(|\Psi'(v_n(x))|) = \overline{\Psi}(\Psi'(|v_n(x)|)) &= \int_0^{\Psi'(|v_n(x)|)} (\overline{\Psi})'(t)\,dt \\ &= \int_0^{\Psi'(|v_n(x)|)} (\Psi')^{-1}(t)\,dt \leq \Psi'(|v_n(x)|)|v_n(x)| \\ &\leq \int_0^{2|v_n(x)|} \Psi'(t)\,dt = \Psi(2|v_n(x)|). \end{aligned} \tag{58}$$

Now, since Φ satisfies the Δ_2 condition, so is Φ^*. Hence, Ψ also satisfies Δ_2, i.e.,

$$\Psi(2s) \leq C_0[\Psi(s) + 1] \quad \text{for all } s \in \mathbb{R},$$

for some $C_0 > 0$, fixed. Using this estimate in (58) and taking into account (56), we get

$$\overline{\Psi}(|\Psi'(v_n(x))|) \le \Psi(2|v_n(x)|) \le C_0[\Psi(|v_n(x)|) + 1] \le C_0[h(x) + 1], \tag{59}$$

for a.e. $x \in \Omega$. Now, from (49),

$$\left|\frac{g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)}\right| \le C(\lambda_n)[A + |\Psi'(v_n)|] \le C_1[A + |\Psi'(v_n)|] \tag{60}$$

($C_1 = \text{const}$). Since $\Phi \le \Psi$, we have $\overline{\Psi} \le \overline{\Phi}$ and thus $\overline{\Psi}$ satisfies the Δ_2 condition. In the calculations hereafter, C denotes a generic constant which is independent of $x \in \Omega$ and $n \in \mathbb{N}$. From (52), (59) and (60),

$$\begin{aligned}
&\overline{\Psi}\left[\frac{g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)} - \lambda|v|^{p-2}v\right] \\
&\qquad \le \overline{\Psi}\left[\frac{2g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)}\right] + \overline{\Psi}(2|\lambda||v|^{p-1}) \\
&\qquad \le C\left\{\overline{\Psi}\left[\frac{g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)}\right] + \overline{\Psi}(|v|^{p-1}) + 1\right\} \\
&\qquad \le C\{\overline{\Psi}[A + \Psi'(|v_n|)] + \overline{\Psi}(|v|^{p-1}) + 1\} \\
&\qquad \le C\{\overline{\Psi}(A) + \overline{\Psi}(\Psi'(|v_n|)) + \overline{\Psi}(\Psi'(|v|)) + 1\} \\
&\qquad \le C\{\overline{\Psi}(A) + h + 1\}.
\end{aligned} \tag{61}$$

Since $A \in L_{\overline{\Psi}}$, $\overline{\Psi}(A) \in L^1(\Omega)$. Hence, $\overline{\Psi}(A) + h + 1$ is in $L^1(\Omega)$. From (57) and (61) and the dominated convergence theorem,

$$\int_\Omega \overline{\Psi}\left[\frac{g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)} - \lambda|v|^{p-2}v\right] \to 0,$$

i.e.,

$$\frac{g(\,\cdot\,, \sigma_n v_n, \lambda_n)}{\alpha_0(\sigma_n)} \to \lambda|v|^{p-2}v \quad \text{in } L_{\overline{\Psi}}.$$

We have (54) and thus (6). Note that in applications, g is usually of the form

$$g(x, u, \lambda) = \lambda\psi(u) + h(x, u, \lambda),$$

where

$$\lim_{\sigma\to 0^+} \frac{\psi(\sigma u)}{\alpha_0(\sigma)} = |u|^{p-2}u \quad \text{for all } u \in \mathbb{R},$$

and

$$\lim_{\sigma\to 0^+} \frac{h(x, \sigma u, \lambda)}{\alpha_0(\sigma)} = 0,$$

a.e. uniformly for $x \in \Omega$, uniformly for λ in bounded intervals.

Finally, we note that the embedding $X(= W_0^1 L_\Phi) \hookrightarrow Z(= L_\Psi)$ is compact. We are in the situation to apply the general results in [24]. Using Theorem 1, we obtain the following result about global bifurcation of (1) (or (17)).

THEOREM 2. *Consider the p-Laplacian eigenvalue problem* (*with homogeneous Dirichlet boundary condition*)*:*

$$\begin{cases} p\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v = \lambda\int_\Omega |u|^{p-2}uv \quad \text{for all } v\in W_0^{1,p}(\Omega), \\ u\in W_0^{1,p}(\Omega). \end{cases} \tag{62}$$

(a) *If* $(\lambda,0)$ *is a bifurcation point of* (1) *then* λ *is an eigenvalue of* (62).

(b) *Let*

$$\lambda_0 = \inf\left\{p\frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} : u\in W_0^{1,p}(\Omega)\setminus\{0\}\right\} \tag{63}$$

be the principal eigenvalue of (62). *Then* $(0,\lambda_0)$ *is a bifurcation point of* (1) *and the bifurcation branch* $\mathcal{C}$ *emanating from* $(0,\lambda_0)$ *satisfies the Rabinowitz global alternative* (i)–(ii) *in Theorem* 1.

PROOF. From the above discussion, we see that the homogenized functional j_0 of j is given by

$$j_0(u) = \int_\Omega |\nabla u|^p$$

and the homogenized mapping of the lower order mapping B is defined by

$$\langle f(u,\lambda), v\rangle = \int_\Omega |u|^{p-2}uv.$$

Thus, the homogenized variational inequality is

$$\int_\Omega |\nabla v|^p - \int_\Omega |\nabla u|^p \geq \int_\Omega |u|^{p-2}u(v-u) \quad \text{for all } v\in W_0^{1,\gamma}(\Omega). \tag{64}$$

This variational inequality is equivalent to one with $v\in D(j_0) = W_0^{1,p}(\Omega)$. In $W_0^{1,p}(\Omega)$, the mapping $v\mapsto \int_\Omega |\nabla v|^p$ is of class C^1 with derivative at u given by

$$v\mapsto p\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v.$$

Hence, (64) is equivalent to the p-Laplacian eigenvalue problem (62). The solution operator P_0 associated with (64) is that corresponding to the p-Laplacian operator, i.e.,

$$P_0 : W^{-1,p'}(\Omega)(=[W_0^{1,p}(\Omega)]^*) \to W_0^{1,p}(\Omega),$$
$$u = P_0(f) \Leftrightarrow p\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v = \langle f, v\rangle \quad \text{for all } v\in W_0^{1,p}(\Omega).$$

(a) is a direct consequence of Theorem 1(a). To prove (b), we just observe that λ_0 defined by (63) is a simple, isolated eigenvalue of (62) and the Leray–Schauder degree associated with (62) changes the sign when λ passes through λ_0, i.e.,

$$\mathrm{d}(I - P_0[f(\,\cdot\,,\lambda)], B_r(0), 0) = \begin{cases} 1 & \text{if } \lambda = \lambda_0 - \varepsilon, \\ -1 & \text{if } \lambda = \lambda_0 + \varepsilon, \end{cases}$$

for all $\varepsilon > 0$, sufficiently small (cf. [9]). (b) follows from Theorem 1(b). □

3.3. An example. In this section, we present an example of the above analysis. Let $\Phi(s) = |s|^\gamma \ln(|s| + 1)$, $s \in \mathbb{R}$ $(\gamma > 1)$ and

$$A(s) = \frac{1}{|s|}\frac{d}{ds}\Phi(s) = |s|^{\gamma-2}\bigg[\gamma \ln(|s|+1) + \frac{|s|}{|s|+1}\bigg]. \tag{65}$$

Consider the equation (17) (or (20)), which has the equivalent variational inequality formulation (26), with A and Φ given as above. Theorem 2, applied to this particular case, yields a global bifurcation result for the boundary value problem:

$$\begin{cases} -\mathrm{div}\bigg(|\nabla u|^{\gamma-2}\bigg[\gamma \ln(|\nabla u| + 1) + \dfrac{|\nabla u|}{|\nabla u| + 1}\bigg]\nabla u\bigg) = g(x, u, \lambda) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{66}$$

which has the following variational form

$$\begin{cases} \displaystyle\int_\Omega \bigg[\gamma \ln(|\nabla u| + 1) + \frac{|\nabla u|}{|\nabla u| + 1}\bigg] |\nabla u|^{\gamma-2} \nabla u \cdot \nabla v \\ \qquad\qquad\qquad\qquad \displaystyle = \int_\Omega g(x, u, \lambda) v \quad \text{for all } v \in W_0^1 L_\Phi, \\ u \in W_0^1 L_\Phi. \end{cases} \tag{67}$$

In the sequel, we verify the conditions in Sections 3.1 and 3.2 such that Theorem 2 holds. It is clear that the function $\phi(s) = |s|A(|s|)$ is continuous and strictly increasing on $\mathbb{R}$ and Φ is a Young function of class C^1. Moreover, since

$$|s|^\gamma \ll \Phi(s) \ll |s|^{\gamma+1}$$

(in fact, $\Phi(s) \ll |s|^{\gamma+\varepsilon}$, for all $\varepsilon > 0$), both Φ and $\overline{\Phi}$ satisfy the Δ_2 condition. As a consequence, we have

$$|s|^{\gamma^*} \ll \Phi^*(s) \ll |s|^{(\gamma+1)^*}, \tag{68}$$

where β^* is the Sobolev conjugate of β. We consider equation (20) (or (1)) with A given by (65) and g satisfying (21)–(23). Note that (20) is equivalent to (26). Let us put $\alpha_0(s) = s^\gamma$ and $\alpha(s) = s\alpha_0(s) = s^{\gamma+1}$, $s > 0$. It is clear that if $\sigma_n \to 0^+$ and $s_n \to s$, then

$$\frac{\Phi(\sigma_n s_n)}{\alpha(\sigma_n)} = \frac{(\sigma_n s_n)^\gamma \ln(|\sigma_n s_n| + 1)}{\sigma_n^{\gamma+1}} = |s_n|^{\gamma+1}\frac{\ln(\sigma_n|s_n| + 1)}{\sigma_n|s_n|} \to |s|^{\gamma+1}.$$

Hence, (28) holds with $p = \gamma + 1$. Now, let us check (31), (32) and (33). We show (31) with $B_1 = 1/2$ and $A_1 = s_1 = 1$, i.e.,

$$\frac{(\sigma s)^\gamma \ln(\sigma s + 1)}{\sigma^{\gamma+1}} \geq \frac{1}{2} s^\gamma \ln(s+1). \tag{69}$$

This is equivalent to

$$\ln(\sigma s + 1) \geq \frac{1}{2}\sigma \ln(s+1) \quad \text{for all } \sigma \in (0,1] \text{ and all } s \geq 1. \tag{70}$$

Since $1/(\sigma s + 1) \geq 1/2(s+1)$ for all $\sigma \in (0,1]$, $s \geq 1$, one has

$$\int_1^s \frac{\sigma}{\sigma \xi + 1}\, d\xi \geq \frac{1}{2} \int_1^s \frac{\sigma}{\xi + 1}\, d\xi,$$

for all $s \geq 1$, i.e.,

$$\ln(\sigma \xi + 1) - \ln(\sigma + 1) \geq \frac{\sigma}{2}[\ln(s+1) - \ln 2].$$

Hence,

$$\ln(\sigma \xi + 1) \geq \frac{1}{2}\sigma \ln(s+1) + \left[\ln(\sigma + 1) - \frac{\sigma \ln 2}{2}\right]. \tag{71}$$

Since

$$\ln(\sigma + 1) \geq \sigma/2 \quad \text{for all } \sigma \in (0,1], \tag{72}$$

it follows that

$$\ln(\sigma + 1) - \frac{\sigma}{2}\ln 2 \geq 0 \quad \text{for all } \sigma \in (0,1].$$

This and (71) show (70). Hence, one has (69). Now, let us show that (32) is satisfied with $B_3 = 1/2$, i.e.,

$$\frac{(\sigma s)^\gamma \ln(\sigma s + 1)}{\sigma^{\gamma+1}} \geq \frac{1}{2} s^{\gamma+1} \quad \text{for all } \sigma, s \in (0,1).$$

This is equivalent to

$$\ln(\sigma s + 1) \geq \frac{1}{2}\sigma s \quad \text{for all } \sigma, s \in (0,1),$$

which is a direct consequence of (72). Now, we check (33) with $B_2 = 1$, i.e.,

$$\frac{(\sigma s)^\gamma \ln(\sigma s + 1)}{\sigma^{\gamma+1}} \leq s^{\gamma+1} \quad \text{for all } \sigma \in (0,1) \text{ and all } s \geq 0.$$

This is the same as

$$\frac{\ln(\sigma s + 1)}{\sigma s} \leq 1 + \frac{1}{s^{\gamma+1}},$$

which follows directly from the inequality $\ln(t+1) \leq t$ for all $t \geq 0$.

Now, we assume that g satisfies (48) and (49) (in addition to (22) and (23)). A particular case is

$$f(x, u, \lambda) = \lambda |u|^{\gamma-1} u + h(x, u, \lambda), \tag{73}$$

where

$$\frac{h(x,u,\lambda)}{|u|^\gamma} \to 0 \quad \text{as } u \to 0, \tag{74}$$

a.e. uniformly for $x \in \Omega$, uniformly for λ in bounded intervals. We can choose, in this case, as an example $\Psi(t) = t^{\gamma+1}$, provided $t^{\gamma+1} \ll \Phi^*$, which is satisfied, for example, if $N \le \gamma(\gamma+1)$. Hence, we have (48) (with $p = \gamma+1$). (49) holds if g has the growth condition

$$|g(x,u,\lambda)| \le C(\lambda)[A(x) + |u|^\gamma]$$

for $C(\lambda) \in L^\infty_{\mathrm{loc}}(\mathbb{R}), A \in L^{(\gamma+1)'(\Omega)}$ ($(\gamma+1)'$ is the Hölder conjugate of $\gamma+1$), which, in view of (73) and (74), is equivalent to requiring that h satisfies the same growth condition ($|h(x,u,\lambda)| \le C(\lambda)[A(x)+|u|^\gamma]$). This implies (49) for $\sigma \ge \sigma_0$ (σ_0 is fixed). For $\sigma < \sigma_0$ (small), (49) is a direct consequence of (74). f is in this case given by

$$\langle f(u,\lambda), v\rangle = \lambda \int_\Omega |u|^{\gamma-1} uv.$$

The above arguments, together with Theorem 2, yield the following global bifurcation result for (67).

Theorem 3. *Assume g satisfies* (73) *and* (74).

(I) *If* $(0,\lambda)$ *is a bifurcation point of* (67) *then λ is an eigenvalue of the p-Laplacian equation*

$$\begin{cases} (\gamma+1)\displaystyle\int_\Omega |\nabla u|^{\gamma-1}\nabla u \cdot \nabla v = \lambda \int_\Omega |u|^{\gamma-1}uv \quad \text{for all } v \in W_0^{1,\gamma+1}(\Omega), \\ u \in W_0^{1,\gamma+1}(\Omega). \end{cases} \tag{75}$$

(II) *Let*

$$\lambda_0 = \inf\left\{(\gamma+1)\int_\Omega |\nabla u|^{\gamma+1}\left(\int_\Omega |u|^{\gamma+1}\right)^{-1} : u \in W_0^{1,\gamma+1}(\Omega)\right\}$$

be the principal eigenvalue of (75). *Then* $(0,\lambda_0)$ *is a bifurcation point of* (67) *and the bifurcation branch emanating from* $(0,\lambda_0)$ *satisfies the global alternative in Theorem* 1.

Appendix

In this appendix we state and prove a convergence result which is needed to prove condition (H5) of the functional j in section Section 3.1. It is about a compactness property of quasilinear elliptic operators in Orlicz–Sobolev spaces and extends certain corresponding property (usually referred as property (S) or (α) in the literature, cf. [6], [28]) for operators in Sobolev spaces. The result is presented here because of the somewhat technical nature of its proof.

THEOREM 4. *Assume both Φ and $\overline{\Phi}$ satisfy the Δ_2 condition on Ω. If $\{u_n\}$ is a bounded sequence in $W_0^1 L_\Phi$ and $u \in W_0^1 L_\Phi$ is such that*

$$\int_\Omega [A(|\nabla u_n|)\nabla u_n - A(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) \to 0, \tag{76}$$

then $u_n \to u$ in $W_0^1 L_\Phi$.

PROOF. From our assumption on Φ, there exist $k_0, b_0 > 0$ such that

$$\Phi(2u) \le k_0 \Phi(u) + b_0 \quad \text{for all } u \in \mathbb{R}. \tag{77}$$

Since the strong convergence in L_Φ is equivalent ot the mean convergence, we have

$$u_n \to u \text{ in } W_0^1 L_\Phi \Leftrightarrow \int_\Omega \Phi(|\nabla(u_n - u)|)\,dx \to 0. \tag{78}$$

Note that since $\{u_n\}$ is bounded in $W_0^1 L_\Phi$, $\{\int_\Omega \Phi(|\nabla u_n|)\,dx\}$ is a bounded sequence. On the other hand, because Φ is strictly convex, the function

$$F : \xi \in \mathbb{R}^N \mapsto \Phi(|\xi|)$$

is also strictly convex on $\mathbb{R}^N$. Since the mapping $\xi \mapsto A(|\xi|)\xi$ is continuous in $\mathbb{R}^N$, it is easy to check that F is of class C^1 in $\mathbb{R}^N$ and

$$\langle F'(\xi), \eta\rangle = A(|\xi|)\xi \cdot \eta \quad \text{for all } \xi, \eta \in \mathbb{R}^N.$$

Since F is strictly convex, F' is strictly monotone on $\mathbb{R}^N$. We have

$$[A(|\xi|)\xi - A(|\eta|)\eta] \cdot (\xi - \eta) > 0 \quad \text{for all } \xi, \eta \in \mathbb{R}^N,\ \xi \ne \eta. \tag{79}$$

In particular, the integrand in (76) is nonnegative, i.e.,

$$[A(|\nabla u_n|)\nabla u_n - A(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) \ge 0 \quad \text{a.e. on } \Omega. \tag{80}$$

We show that for each $\varepsilon > 0$ small, there exists $\delta \in (0, \varepsilon)$ such that for all $H \subset \Omega$ measurable with $|H| < \delta$ ($|H|$ denotes the Lebesgue measure of H), we have

$$\int_H \Phi(|\nabla u_n|) \le 6\varepsilon \quad \text{for all } n. \tag{81}$$

In fact, from (22), there exists $n_1 \in \mathbb{N}$ such that

$$\int_\Omega [A(|\nabla u_n|)\nabla u_n - A(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) < \varepsilon, \quad \text{for all } n \ge n_1. \tag{82}$$

From (80), we have

$$\int_H [A(|\nabla u_n|)\nabla u_n - A(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) < \varepsilon, \quad \text{for all } n \ge n_1, \tag{83}$$

and all $H \subset \Omega$ measurable, and thus

$$\begin{aligned}
(84) \qquad \int_H \phi(|\nabla u_n|)|\nabla u_n| &= \int_H A(|\nabla u_n|)|\nabla u_n|^2 \\
&\le \int_H A(|\nabla u_n|)|\nabla u_n|\,|\nabla u| + \int_H A(|\nabla u|)|\nabla u|(|\nabla u_n| + |\nabla u|) + \varepsilon \\
&= \int_H \phi(|\nabla u_n|)|\nabla u| + \int_H \phi(|\nabla u|)|\nabla u_n| + \int_H \phi(|\nabla u|)|\nabla u| + \varepsilon,
\end{aligned}$$

for all $n \ge n_1$, all $H \subset \Omega$ measurable. We have

$$\phi(|\nabla u|)|\nabla u| \le \Phi(2|\nabla u|) \le k_0\Phi(|\nabla u|) + b_0.$$

Hence, $\phi(|\nabla u|)|\nabla u| \in L^1(\Omega)$ and there exists $\delta_1 > 0$ such that

$$(85) \qquad \int_H \phi(|\nabla u|)|\nabla u| < \varepsilon, \quad \text{if } |H| < \delta_1.$$

Using Young's inequality, we have for all $\rho \in (0,1)$,

$$\begin{aligned}
(86) \qquad \phi(|\nabla u_n|)|\nabla u| &\le \overline{\Phi}(\rho\phi(|\nabla u_n|)) + \Phi(\rho^{-1}|\nabla u|) \\
&\le \rho\overline{\Phi}(\phi(|\nabla u_n|)) + \Phi(\rho^{-1}|\nabla u|).
\end{aligned}$$

On the other hand, we have

$$(87) \qquad \overline{\Phi}(\phi(s)) = \int_0^{\phi(s)} \phi^{-1} \le s\phi(s) \le \Phi(2s) \le k_0\Phi(s) + b_0.$$

From the assumption on $\{u_n\}$, we have $M > 0$ such that

$$(88) \qquad \int_\Omega \Phi(|\nabla u_n|) \le M, \quad \text{for all } n.$$

Hence,

$$(89) \qquad \int_H \overline{\Phi}(\phi(|\nabla u_n|)) \le k_0 \int_H \Phi(|\nabla u_n|) + b_0|H| \le k_0 M + b_0|\Omega|.$$

Since $\Phi(\rho^{-1}|\nabla u|) \in L^1(\Omega)$ (Φ satisfies a Δ_2 condition!), there exists $\delta_2 > 0$ such that

$$(90) \qquad \int_H \Phi(\rho^{-1}|\nabla u|) < \varepsilon \quad \text{if } H \subset \Omega,\ |H| < \delta_2.$$

From (86), (89), and (90) with $\rho = \varepsilon(k_0 M + b_0|\Omega|)^{-1}$, we obtain

$$(91) \qquad \int_H \phi(|\nabla u_n|)|\nabla u| \le \rho \int_H \overline{\Phi}(\phi(|\nabla u_n|) + \int_H \Phi(\rho^{-1}(|\nabla u|)) \le 2\varepsilon,$$

if $|H| < \delta_2$. Using again Young's inequality, we have, for $\rho \in (0,1)$,

$$(92) \qquad \phi(|\nabla u|)|\nabla u_n| \le \rho\overline{\Phi}(\rho^{-1}\phi(|\nabla u|) + \rho\Phi(|\nabla u_n|).$$

We now choose $\rho = \varepsilon/M$. Since $\overline{\Phi}$ satisfies a Δ_2 condition, there exist k_3, $b_3 > 0$ (depending on ε) such that

$$\overline{\Phi}(\rho^{-1}s) \leq k_3\overline{\Phi}(s) + b_3 \quad \text{for all } s \in \mathbb{R}.$$

Hence, by (87),

$$\overline{\Phi}(\rho^{-1}\phi(|\nabla u|)) \leq k_3\overline{\Phi}(\phi(|\nabla u|) + b_3 \leq k_3 k_0 \Phi(|\nabla u|) + b_4$$

($b_4 = b_3 + k_3 k_0$). It follows that

$$\int_H \overline{\Phi}(\rho^{-1}\phi(|\nabla u|)) \leq k_3 k_0 \int_H \Phi(|\nabla u|) + b_4|H|.$$

Since $\Phi(|\nabla u|) \in L^1(\Omega)$, there exists $\delta_3 > 0$ such that

$$\int_H \overline{\Phi}(\rho^{-1}\phi(|\nabla u|)) \leq \varepsilon \quad \text{if } H \subset \Omega,\ |H| < \delta_3.$$

For such set H, we have from (92) that

$$\begin{aligned} \int_H \phi(|\nabla u|)|\nabla u_n| &\leq \int_H \overline{\Phi}(\rho^{-1}\phi(|\nabla u|)) + \rho \int_H \overline{\Phi}(|\nabla u_n|) \\ &\leq \varepsilon + M(\varepsilon/M) = 2\varepsilon, \end{aligned} \tag{93}$$

if $H \subset \Omega$, $|H| < \delta_3$. Let $\delta = \min\{\varepsilon, \delta_1, \delta_2, \delta_3\}(> 0)$. It follows from (84), (85), (91), and (93) that

$$\int_H \phi(|\nabla u_n|)|\nabla u_n| \leq 6\varepsilon,$$

for all $n \geq n_1$, all measurable subset H of Ω with $|H| < \delta$. Since $\Phi(s) = \int_0^s \phi \leq s\phi(s)$, we have

$$\int_H \Phi(|\nabla u_n|) \leq \int_H \phi(|\nabla u_n|)|\nabla u_n| \leq 6\varepsilon, \tag{94}$$

if $|H| < \delta$ and $n \geq n_1$. Now, since $\Phi(|\nabla u_n|) \in L^1(\Omega)$ for all n, we have $\delta_4 > 0$ such that

$$\int_H \Phi(|\nabla u_n|) < 6\varepsilon \quad \text{if } n < n_1,\ |H| < \delta_4.$$

Replacing δ by $\min\{\delta, \delta_4\}$ in (94), we have (81).

Now, we prove that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. on } \Omega. \tag{95}$$

From (76) and (79), we have that

$$[A(|\nabla u_n|)\nabla u_n - A(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) \to 0 \quad \text{in } L^1(\Omega).$$

Hence, there exist a subset U of Ω and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $|U| = 0$ and

$$[A(|\nabla u_{n_k}(x)|\nabla u_{n_k}(x) - A(|\nabla u(x)|)\nabla u(x)] \cdot [\nabla u_{n_k}(x) - \nabla u(x)] \to 0, \tag{96}$$

for all $x \in \Omega \setminus U$. Fix $x \in \Omega \setminus U$. The sequence in (96) is bounded and there exists $\overline{\alpha} = \overline{\alpha}(x) > 0$ such that

$$\begin{aligned} A(|\nabla u_{n_k}(x)|)|\nabla u_{n_k}(x)|^2 \le \overline{\alpha} + A(|\nabla u_{n_k}(x)|)|\nabla u_{n_k}(x)||\nabla u(x)| \\ + A(|\nabla u(x)|)|\nabla u(x)||\nabla u_{n_k}(x)| + A(|\nabla u(x)|)|\nabla u(x)|^2. \end{aligned}$$

For $\overline{\alpha}_1 = \overline{\alpha} + A(|\nabla u(x)|)|\nabla u(x)|^2 \in \mathbb{R}$, one has

$$\phi(|\nabla u_{n_k}(x)|)|\nabla u_{n_k}(x)| \le \overline{\alpha}_1 + \phi(|\nabla u_{n_k}(x)|)|\nabla u(x)| + \phi(|\nabla u(x)|)|\nabla u_{n_k}(x)|. \tag{97}$$

Assume $|\nabla u_{n_k}(x)| \to \infty$. From the assumption that $\phi(s) \to \infty$ as $s \to \infty$, we have

$$\phi(s)s > \overline{\alpha}_1 + 1 + 2\phi(s)|\nabla u(x)|, \qquad \phi(s)s > \overline{\alpha}_1 + 1 + 2\phi(|\nabla u(x)|)s,$$

and thus

$$\phi(s)s > \overline{\alpha}_1 + 1 + \phi(s)|\nabla u(x)| + \phi(|\nabla u(x)|)s,$$

for all $s > 0$ sufficiently large. This estimate and (97) show that the sequence $\{|\nabla u_{n_k}(x)|\}$ is bounded. By passing once more to a subsequence, if necessary, we can assume that $\nabla u_{n_k}(x) \to \xi$ in $\mathbb{R}^N$. Letting $k \to \infty$ in (96), one gets

$$[A(|\xi|)\xi - A(|\nabla u(x)|)\nabla u(x)] \cdot [\xi - \nabla u(x)] = 0.$$

From the strict monotonicity property (79), we must have $\xi = \nabla u(x)$. Since this holds for all subsequences of $\{\nabla u_{n_k}(x)\}$, we have (95) for all $x \in \Omega \setminus U$. From (95), we have

$$\Phi(|\nabla u_n(x) - \nabla u(x)|) \to 0 \quad \text{a.e. on } \Omega. \tag{98}$$

Now, we prove that the sequence $\{\Phi(|\nabla u_n - \nabla u|)\}$ is uniformly integrable on Ω. It follows from (77) that

$$\begin{aligned} \Phi(|\nabla u_n - \nabla u|) \le \Phi(|\nabla u_n| + |\nabla u|) \le \frac{1}{2}\Phi(2|\nabla u_n|) + \frac{1}{2}\Phi(2|\nabla u|) \\ \le \frac{1}{2}k_0[\Phi(|\nabla u_n|) + \Phi(|\nabla u|)] + b_0. \end{aligned}$$

Then,

$$\int_\Omega \Phi(|\nabla u_n - \nabla u|) \le \frac{1}{2}k_0 \left[\int_\Omega \Phi(|\nabla u_n|) + \int_\Omega \Phi(|\nabla u|)\right] + b_0|\Omega|, \tag{99}$$

for all n. Now, for $H \subset \Omega$ measurable, $|H| < \delta$, it follows from (81) that

$$\begin{aligned} \int_H \Phi(|\nabla u_n - \nabla u|) \le \frac{1}{2}k_0 \int_H \Phi(|\nabla u_n|) + \frac{1}{2}k_0 \int_H \Phi(|\nabla u|) + b_0|H| \\ \le \frac{1}{2}k_0(6\varepsilon) + \frac{1}{2}k_0 \int_H \Phi(|\nabla u|) + b_0\varepsilon. \end{aligned} \tag{100}$$

Since $\Phi(|\nabla u|) \in L^1(\Omega)$, the right hand side of (100) can be made less than $(4k_0 + b_0)\varepsilon$, say, if $|H| < \delta_5 \le \delta$. Replacing ε by $\varepsilon' = \varepsilon(4k_0 + b_0)^{-1}$ and δ by δ_5, we have

$$\int_H \Phi(|\nabla u_n - \nabla u|) < \varepsilon,$$

for all n, all $H \subset \Omega$ measurable such that $|H| < \delta$. This shows that the sequence $\{\Phi(|\nabla u_n - \nabla u|)\}$ is uniformly integrable on Ω. Together with (98) and Vitali's theorem (cf. e.g. [17]), we have

$$\int_\Omega \Phi(|\nabla u_n - \nabla u|) \to 0.$$

This means that $\nabla u_n \to \nabla u$ in $(L_\Phi)^N$ and therefore $u_n \to u$ in $W_0^1 L_\Phi$. The proof of Theorem 4 is completed. $\square$

The following result is a direct corollary of Theorem 4 about the stability of solutions of the following quasilinear elliptic boundary value problem:

$$\begin{cases} \displaystyle\int_\Omega A(|\nabla u|)\nabla u \cdot \nabla v = \langle f, v\rangle \quad \text{for all } v \in W_0^1 L_\Phi, \\ u \in W_0^1 L_\Phi. \end{cases} \tag{101}$$

COROLLARY 5. *Assume $\{f_n\}$ is a sequence in $(W_0^1 L_\Phi)^*$ such that*

$$f_n \to f \quad \text{in } (W_0^1 L_\Phi)^*.$$

Let u_n be the (*unique*) *solution of* (101) *with f_n instead of f. Then, $u_n \to u$ in $W_0^1 L_\Phi$, where u is the solution of* (101).

Note that for each $f \in (W_0^1 L_\Phi)^*$, (101) has a unique solution u, which is the minimizer of the functional $J(u) = \int_\Omega \Phi(|\nabla u|) - \int_\Omega fu$ on $W_0^1 L_\Phi$.

PROOF OF COROLLARY 5. From (101), we have

$$\int_\Omega A(|\nabla u_n|)\nabla u_n \cdot (\nabla u_n - \nabla u) = \langle f_n, u_n - u\rangle,$$

and

$$\int_\Omega A(|\nabla u|)\nabla u \cdot (\nabla u_n - \nabla u) = \langle f, u_n - u\rangle.$$

Subtracting these equations, we get

$$\begin{aligned} &\int_\Omega [A(|\nabla u_n|)\nabla u_n - A(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) \\ &\qquad = \langle f_n - f, u_n - u\rangle \le \|f_n - f\|_{(W_0^1 L_\Phi)^*} \|u_n - u\|_{W_0^1 L_\Phi}. \end{aligned} \tag{102}$$

Now, from (101) (with f replaced by f_n) with $v = u_n$, we have

$$\int_\Omega A(|\nabla u_n|)|\nabla u_n|^2 = \langle f_n, u_n\rangle.$$

Therefore,

$$\begin{aligned}\int_\Omega \Phi(|\nabla u_n|) &\le \int_\Omega \phi(|\nabla u_n|)|\nabla u_n| = \langle f_n, u_n\rangle \\ &\le \|f_n\|_{(W_0^1 L_\Phi)^*}\|u_n\|_{W_0^1 L_\Phi} \le M_0\|u_n\|_{W_0^1 L_\Phi}\end{aligned}$$

($M_0 = \sup \|f_n\|_{(W_0^1 L_\Phi)^*} < \infty$). Since

$$\lim_{\|u\|_{W_0^1 L_\Phi} \to 0} \frac{1}{\|u\|_{W_0^1 L_\Phi}} \int_\Omega \Phi(|\nabla u|) = \infty$$

(cf. [15] or [13]), this inequality implies that $\{\|u_n\|_{W_0^1 L_\Phi}\}$ is bounded. Hence, the sequence $\{\|u_n - u\|_{W_0^1 L_\Phi}\}$ is bounded. Since $\|f_n - f\|_{(W_0^1 L_\Phi)^*} \to 0$, it follows from (102) that

$$\int_\Omega [A(|\nabla u_n|)\nabla u_n - A(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) \to 0.$$

Now, using Theorem 4, we have $u_n \to u$ in $W_0^1 L_\Phi$. □

Remark 6. Note that Theorem 4 and Corollary 5 imply the Palais–Smale (PS) condition for a class of quasilinear elliptic operators in Orlicz–Sobolev spaces, as considered in [7]. We assume that $g = g(x,u)$ does not depend on λ and ϕ, Φ, g satisfy the conditions in [7] (cf. conditions (H1)–(H5) and Theorem 2.1 of [7]). As in [7], we define

$$G(x,s) = \int_0^s g(x,t)\,dt$$

and

$$I(u) = \int_\Omega \Phi(|\nabla u|)\,dx - \int_\Omega G(x,u)\,dx.$$

We prove that under the conditions in [7] the functional I satisfies the (PS) property. In fact, assume that $\{u_n\}$ is a sequence in $W_0^1 L_\Phi$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in $(W_0^1 L_\Phi)^*$. From the arguments in [7] (Theorem 2.1, Lemma 3.2), we have that $\{u_n\}$ is bounded in $W_0^1 L_\Phi$ and by passing to a subsequence, u_n converges to u weakly in $W_0^1 L_\Phi$ and strongly in L_P. Therefore,

$$\int_\Omega g(x,u_n)\,dx \to 0.$$

Since $I'(u_n) \to 0$, it follows that

$$\int_\Omega A(|\nabla u_n|)\nabla u_n \cdot (\nabla u_n - \nabla u) \to 0$$

as $n \to \infty$. Because $\int_\Omega A(|\nabla u|)\nabla u \cdot (\nabla u_n - \nabla u) \to 0$, we have (76). Thus, by Theorem 4, $u_n \to u$ (strongly) in $W_0^1 L_\Phi$. This shows that T satisfies (PS).

Note that the same observation applies to the operator A defined by

$$\langle A(u), v\rangle = p\int_\Omega \frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}}\nabla u\cdot\nabla v\,dx,$$

as studied in [26]. In this case, by using the $W^{1,p}$-version of Theorem 4 (cf. [6], [28]) and the arguments as above, we can prove that the potential functional

$$I(u) = \int_\Omega (\sqrt{1+|\nabla u|^{2p}}-1)\,dx - \lambda\int_\Omega F(x,u)\,dx$$

also satisfies the (PS) condition. This answers a question left open in [26]. We also note that in [26] and [7], existence of mountain pass type critical points was established without using the (PS) condition but instead by monotonicity arguments. The (PS) property of the potential operators would somewhat simplify the arguments in [7] and [26]. This property would also enable us to employ other variational methods such as the Lusternik–Schnirelman category in the presence of symmetry, saddle point theorem, or linking type theorems to those problems.

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Vy Khoi Le
Department of Mathematics and Statistics
University of Missouri–Rolla
Rolla, MO 65409, USA

E-mail address: vy@umr.edu