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ATTRACTOR AND DIMENSION FOR DISCRETIZATION OF A DAMPED WAVE EQUATION WITH PERIODIC NONLINEARITY

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Abstract. The existence and Hausdorff dimension of the global attractor for discretization of a damped wave equation with the periodic nonlinearity under the periodic boundary conditions are studied for any space dimension. The obtained Hausdorff dimension is independent of the mesh sizes and the space dimension and remains small for large damping, which conforms to the physics.

1. Introduction

Consider the damped wave equation with periodic nonlinearity

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} - \triangle u + g(u) = f, \quad x \in \Omega,\ t \geq 0 \tag{1}$$

with the periodic boundary conditions

$$\begin{cases} u(x,t)|_{x\in\Gamma_j} = u(x,t)|_{x\in\Gamma_{j+n}}, \\ \Big(-\dfrac{\partial u}{\partial \nu}(x,t)\Big|_{x\in\Gamma_j} = \Big) \dfrac{\partial u}{\partial x_j}(x,t)|_{x\in\Gamma_j} \\ \qquad\qquad = \dfrac{\partial u}{\partial x_j}(x,t)|_{x\in\Gamma_{j+n}} \Big(= \dfrac{\partial u}{\partial \nu}(x,t)|_{x\in\Gamma_{j+n}}\Big), \\ j = 1,\ldots,n,\ t > 0, \end{cases} \tag{2}$$

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and the initial value conditions

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \Omega,$$

where $u = u(x,t)$ is a real-valued function on $\Omega \times [0,\infty)$, $f = f(x) \in L^2(\Omega)$, $\alpha > 0$, $D(-\triangle) = H^2_{\text{per}}(\Omega)$, the space of H^2 functions which are spatially periodic, $\Omega = \prod_{j=1}^{n}(0,1) \subset \mathbb{R}^n$, $n \in \mathbb{N}$,

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}, \quad \Gamma_{j+n} = \partial\Omega \cap \{x_j = 1\}$$

are the faces of the boundary $\partial\Omega$ on Ω, $j = 1, \ldots, n$, and $g(u) \in C^2(\mathbb{R};\mathbb{R})$ satisfies:

$$(3) \qquad |g(u)| \le c, \quad g(u+T) = g(u), \quad T > 0, \quad |g'(u)| \le C \text{ (constant)}.$$

We consider the spatially finite difference discretized version of problem (1)–(2).

Let $m \in \mathbb{N}$, $h = 1/m$. We approximate a function $u(x) : \Omega \to \mathbb{R}$ by $u = u_k$:

$$u_k = u(k_1 h, \ldots, k_n h) = u\left(\frac{k_1}{m}, \ldots, \frac{k_n}{m}\right),$$

where $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \cap \{1 \le k_1, \ldots, k_n \le m\}$.

We can think of u_k as a vector in $\mathbb{R}^{m^n}$. For convenience, we reorder the subscripts of components of any $v \in \mathbb{R}^{m^n}$ as follows:

$$(4) \quad v = (v_{1\ldots111}, v_{1\ldots112}, \ldots, v_{1\ldots11m}, \ldots, v_{1\ldots1m1}, v_{1\ldots1m2}, \ldots, v_{1\ldots1mm}, \\ \ldots, v_{mm\ldots m1}, v_{mm\ldots m2}, \ldots, v_{mm\ldots mm})^T \in \mathbb{R}^{m^n},$$

where "T" is the transpose operation for matrixes. Let

$$M = \{v \in \mathbb{R}^{m^n} \mid \text{subscripts of components of } v \text{ are ordered as in (4)}\}.$$

Since we consider the periodic boundary conditions, we extend the indexes of any $v \in M$ by periodicity:

$$(5) \qquad v_k = v_{(k_1 \bmod (m)), \ldots, (k_n \bmod (m))}, \quad \text{for all } k = (k_1, \ldots, k_n) \in \mathbb{Z}^n,$$

where

$$p \bmod (m) = \begin{cases} m & p \text{ is a multiple of } m, \\ p \bmod m & \text{otherwise.} \end{cases}$$

Let $D_1, \ldots, D_n, D, A$ denote the finite difference discretizations of the linear operators $\partial/\partial x_1, \ldots, \partial/\partial x_n, \nabla, -\Delta$ of the continuous version, respectively. For

$v \in M$, the $(k_1, \dots, k_n)$-th component of v is denoted by $v_{(k_1,\dots,k_n)}$, we define the linear operators $D_1, \dots, D_n, A : M \to M$ by:

$$
\begin{aligned}
(D_1 v)_{(k_1,\dots,k_n)} &= m(v_{(k_1,\dots,k_n)} - v_{(k_1-1,k_2,\dots,k_n)}),\\
(D_2 v)_{(k_1,\dots,k_n)} &= m(v_{(k_1,\dots,k_n)} - v_{(k_1,k_2-1,\dots,k_n)}),\\
&\dots\dots\dots\dots\dots\\
(D_n v)_{(k_1,\dots,k_n)} &= m(v_{(k_1,\dots,k_n)} - v_{(k_1,\dots,k_n-1)}),\\
(Av)_{(k_1,\dots,k_n)} &= m^2(2nv_{(k_1,\dots,k_n)} - v_{(k_1+1,k_2,\dots,k_n)} - v_{(k_1-1,k_2,\dots,k_n)}\\
&\quad - \dots - v_{(k_1,\dots,k_n+1)} - v_{(k_1,\dots,k_n-1)}),
\end{aligned} \tag{6}
$$

and $D : M \to M \times \dots \times M$ as

$$
Dv = \begin{pmatrix} D_{1v} \\ \vdots \\ D_{nv} \end{pmatrix}.
$$

where $(k_1, \dots, k_n) \in \mathbb{Z}^n \cap \{1 \le k_j \le m,\ j = 1, \dots, n\}$.

The spacially finite difference discretized version of the systems (1)–(2) can be written as

$$
\frac{d^2u}{dt^2} + \alpha\frac{du}{dt} + Au + G_0(u) = \Gamma \tag{7}
$$

and the initial value conditions as

$$
u(0) = u^{(0)}, \quad \frac{du}{dt}(0) = u^{(1)}, \tag{8}
$$

where

$$
\begin{aligned}
u = &(u_{1\dots111}, \dots, u_{1\dots11m}, \dots, u_{1\dots1m1}, \dots, u_{1\dots1mm},\\
&\dots, u_{mm\dots m1}, \dots, u_{mm\dots mm})^T \in M,\\
u^{(i)} = &(u^{(i)}_{11\dots11}, u^{(i)}_{11\dots12}, \dots, u^{(i)}_{mm\dots mm})^T \in M, \quad (i = 0, 1),
\end{aligned}
$$

and

$$
\Gamma = (\Gamma_{11\dots11}, \dots, \Gamma_{mm\dots mm})^T \in M,
$$

the sampling of f with $(1/m^n)\sum_{k_1,\dots,k_n=1}^{m} \Gamma^2_{k_1,\dots,k_n}$ uniformly bounded with respect to m, and

$$
G_0(u) = (g(u_{11\dots11}), g(u_{11\dots12}), \dots, g(u_{mm\dots mm}))^T \in M,
$$

the sampling of $g(u)$.

For system (7)–(8) where the nonlinearity $g(u) = \sin u$ in one space dimension $n = 1$, Yin Yan in [1] proved the existence of the global attractor and gave an upper bound of Hausdorff dimension of the attractor for $\alpha > 0$. But this upper bound is directly proportional to the coefficient α of damping when $\alpha \ge \sqrt{6}$, and tends to infinity as $\alpha \to \infty$, which are not precise in the physical sense. S. Zhou in [2] improved the estimate in [1] and obtained a more strict upper bound of the dimension for the global attractor by carefully estimating and splitting the positivity of the linear operator in the corresponding evolution equation of the first order in time. The obtained Hausdorff dimension of the global attractor is independent of the mesh sizes and space dimension remains small for large damping.

In this paper, by using similar technique in [2], we generalize the estimate of [2] to any space dimension $n \in \mathbb{N}$ and obtain an upper bound of the Hausdorff dimension of the global attractor for system (7)–(8). The result is the following theorem.

Theorem 1. *The semigroup determined by* (7)–(8) *possesses a global attractor in M and the Hausdorff dimension d_H of the global attractor satisfies:*

$$d_H \le 2 + \min\left\{\ell \,\middle|\, \ell \in \mathbb{N},\ \frac{1}{\ell}\sum_{j=1}^{[\ell/2]+1}\frac{1}{\widetilde{\lambda}_j} \le \frac{\lambda_1\alpha^2}{4C^2\sqrt{\alpha^2+4\lambda_1}(\alpha+\sqrt{\alpha^2+4\lambda_1})}\right\}, \tag{9}$$

$$\le 2 + \min\left\{\ell \,\middle|\, \ell \in \mathbb{N},\ \frac{1}{\ell}\sum_{j=1}^{[\ell/2]+1}\frac{1}{\widetilde{\lambda}_j} \le \frac{4\alpha^2}{C^2\sqrt{\alpha^2+16\pi^2}(\alpha+\sqrt{\alpha^2+16\pi^2})}\right\}. \tag{10}$$

Where $\lambda_1 = 4m^2\sin^2\pi/m$ and $16 \le \lambda_1 = \widetilde{\lambda}_1 \le \widetilde{\lambda}_2 \le \ldots \le \widetilde{\lambda}_\ell \le \ldots \le \widetilde{\lambda}_{[m/2]^n+1}$ are the ordering, from small to large, of set

$$\{16(l_1^2+\ldots+l_n^2) \mid 0 \le l_1,\ldots,l_n \le [m/2] \quad \text{but } l_1+\ldots+l_n \ge 1\}. \tag{11}$$

Particularly, if

$$\lambda_1^2\alpha^2 > 4C^2\sqrt{\alpha^2+4\lambda_1}(\alpha+\sqrt{\alpha^2+4\lambda_1}),$$

then $d_H \le 2$.

It is easy to see from (9) that d_H is uniformly bounded for sufficiently large α because

$$\frac{\lambda_1^2\alpha^2}{4C^2\sqrt{\alpha^2+4\lambda_1}(\alpha+\sqrt{\alpha^2+4\lambda_1})} \to \frac{\lambda_1^2}{8C^2} \tag{12}$$

as $\alpha \to \infty$.

2. Preliminaries

At first, we consider the properties of operator A. Obviously, the linear operator $A: M \to M$ defined by (6) is symmetric, so it can be diagonalized. For the eigenvalues of A, we have the following information.

Let

$$e(l,i)=\begin{cases}1 & \text{if } l=0,\\ \sin\left(\dfrac{2l\pi}{m}i\right) & \text{if } 1\le l\le\left[\dfrac{m}{2}\right],\\ \cos\left(\dfrac{2(m-l)\pi}{m}i\right) & \text{if } \left[\dfrac{m}{2}\right]+1\le l\le m-1,\end{cases}\tag{13}$$

where $[m/2]$ is the largest integer not greater than $m/2$, and for any $0 \le l_1,\dots,l_n \le m-1$ and $1 \le k_1,\dots,k_n \le m$, define $e(l_1,\dots,l_n) \in M$ by

$$e(l_1,\dots,l_n)_{(k_1,\dots,k_n)} = e(l_1,k_1)\cdot\ldots\cdot e(l_n,k_n).\tag{14}$$

Lemma 1. *The eigenvalues of A are as follows:*

$$\lambda_{(l_1,\dots,l_n)} = 4m^2\left(\sin^2\frac{l_1\pi}{m}+\ldots+\sin^2\frac{l_n\pi}{m}\right)\tag{15}$$

and the corresponding eigenvectors are $e(l_1,\dots,l_n)$, i.e.,

$$Ae(l_1,\dots,l_n) = 4m^2\left(\sin^2\frac{l_1\pi}{m}+\ldots+\sin^2\frac{l_n\pi}{m}\right)e(l_1,\dots,l_n),\tag{16}$$

for any $l_1,\dots,l_n = 0,\dots,m-1$. Particularly, 0 is a simple eigenvalue of A with the corresponding eigenvector

$$e=(e_{(k_1,\dots,k_n)})\in M,\quad \text{where } e_{(k_1,\dots,k_n)}=1\ (1\le k_1,\dots,k_n\le m).\tag{17}$$

Proof. It is easy to see from (13)–(14) that for any $0 \le l_1,\dots,l_n \le m-1$, $k_1,\dots,k_n \in \mathbb{Z}$,

$$e(l_1,\dots,l_n)_{(k_1,\dots,k_n)} = e(l_1,\dots,l_n)_{(k_1 \operatorname{mod}(m),\dots,k_n \operatorname{mod}(m))}.$$

Here we consider the case $0 \le l_1,\dots,l_n \le [m/2]$ only. In other cases, we can prove the lemma similarly.

Write $\beta_i = 2l_i\pi/m$, $i=1,\dots,n$. By (6), (13) and (14), it is easy to check that for any $0 \le l_1,\dots,l_n \le m-1$, $k_1,\dots,k_n \in \mathbb{Z}$,

$$\frac{1}{m^2}Ae(l_1,\dots,l_n)_{(k_1,\dots,k_n)} = 4\left(\sin^2\frac{l_1\pi}{m}+\ldots+\sin^2\frac{l_n\pi}{m}\right)e(l_1,\dots,l_n)_{(k_1,\dots,k_n)}.$$

The proof is completed. □

Let q_1 and q_2 are two different arrangements of $l_1,\dots,l_n$, $0 \le l_1,\dots,l_n \le m-1$, then by (15), we have $\lambda_{q_1}=\lambda_{q_2}$. Since $\sin x \ge 2x/\pi$ for $x\in[0,\pi/2]$,

$$\begin{aligned}16 &\le \lambda_{(1,0,\dots,0)} = \lambda_{(0,1,\dots,0)} = \ldots = \lambda_{(0,\dots,0,1)} = \lambda_{(m-1,0,\dots,0)}\\ &= \lambda_{(0,m-1,\dots,0)} = \ldots = \lambda_{(0,\dots,0,m-1)} = 4m^2\sin^2\frac{\pi}{m}\\ &\le \lambda_{(l_1,\dots,l_n)} \le 4m^2.\end{aligned}\tag{18}$$

and it is easy to see that if one/ones of $l_1, \dots, l_n$ is/are replaced by $m - l_1$ or $m - l_2$ or $\dots$ or $m - l_n$, respectively, then the corresponding eigenvalue of A remains invariant for any $0 \le l_1, \dots, l_n \le [m/2]$ and $l_1 + \dots + l_n \ge 1$. For example, we have

$$\lambda_{(l_1,\dots,l_n)} = \lambda_{(m-l_1,\dots,m-l_n)}, \quad \text{for all } 0 \le l_1, \dots, l_n \le [m/2] \text{ but } l_1 + \dots + l_n \ge 1.$$

So, we need to consider the case of $0 \le l_1, \dots, l_n \le [m/2]$ only.

Let $z, z^{(1)}, z^{(2)} \in M$ with their components $z_{k_1\dots k_n}$, $z^{(1)}_{k_1\dots k_n}$, $z^{(2)}_{k_1\dots k_n}$ for $1 \le k_1, \dots, k_n \le m$. We define the weighted inner products and norms as

$$(19) \qquad \begin{aligned} (z^{(1)}, z^{(2)}) &= \frac{1}{m^n} \sum_{k_1,\dots,k_n=1}^{m} z^{(1)}_{k_1\dots k_n} z^{(2)}_{k_1\dots k_n}, \\ |z| = (z,z)^{1/2} &= \left(\frac{1}{m^n} \sum_{k_1\dots k_n=1}^{m} z^2_{k_1\dots k_n} \right)^{1/2}, \\ \|z\| = (Az,z)^{1/2} &= (Dz, Dz)^{1/2}. \end{aligned}$$

Write $E = \{e\}^{\perp M}$, the orthogonal complement of span$\{e\}$ in M, which is an invariant subspace of the linear operator A. It is easy to see that $|\cdot|$ is a norm in M, $\|\cdot\|$ is only a semi-norm in M, but it is a norm in E. We also have the following inequality:

$$(20) \qquad \|z\|^2 \ge \lambda_{(1,0,\dots,0)} |z|^2 \ge 16|z|^2, \quad \text{for all } z \in E,$$

which corresponds to the Poincáre inequality.

Let

$$E_0 = (E, |\cdot|), \quad E_1 = (E, \|\cdot\|),$$

and

$$V_0 = (E_1 \times S^1) \times (E_0 \times R), \quad V_1 = E_1 \times E_0,$$

where $S^1 = R^1/TZ$ is the one-dimensional torus. Introduce a orthogonal projector

$$P : M \mapsto \{e\}^{\perp M} = E,$$

which induces a projector from V_0 to V_1(also denoted by P). Write $\overline{u} = Pu$, $\overline{\Gamma} = P\Gamma$, then

$$\begin{aligned} \overline{u} &= u - \left(\frac{1}{m^n} \sum_{k_1,\dots,k_n=1}^{m} u_{k_1\dots k_n} \right) e, \\ \overline{\Gamma} &= \Gamma - \left(\frac{1}{m^n} \sum_{k_1,\dots,k_n=1}^{m} \Gamma_{k_1\dots k_n} \right) e, \end{aligned}$$

and the projection of system (7) to E is

$$\frac{d^2\overline{u}}{dt^2} + \alpha\frac{d\overline{u}}{dt} + A\overline{u} + G_0(u) - \left(\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} g(u_{k_1\dots k_n})\right)e = \overline{\Gamma}, \tag{21}$$

and the initial value conditions (8) is

$$\overline{u}(0) = \overline{u^{(0)}}, \quad \frac{d\overline{u}}{dt}(0) = \overline{u^{(1)}}, \tag{22}$$

where

$$\overline{u^{(i)}} = u^{(i)} - \left(\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} u^{(i)}_{k_1\dots k_n}\right)e, \quad (i = 0, 1).$$

Since $G_0(u)$ in (7) is globally Lipschitzian continuous with respect to u in M and equation (7) can be solved backwards in time t, globally existence and uniqueness of solutions of (7) are evident for any $t \in R$. If $u(t) \in M$ is a solution of (7), then $u(t)$ can be decomposed into

$$u(t) = \overline{u}(t) + m(t)e, \tag{23}$$

where

$$m(t) = \frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} u_{k_1\dots k_n}. \tag{24}$$

Since (7) is invariant if we add an amount lTe ($l \in \mathbb{Z}$) to u for any integer l, the solution $u(t)$ of (7) induces a nonlinear flow

$$S(t) : (u^{(0)}, u^{(1)}) \in V_0 \to \left(u(t), \frac{du}{dt}(t)\right) \in V_0, \quad t \geq 0.$$

3. Global attractor

Firstly, we consider the absorbing properties of flow $S(t)|_{V_1}$, $t \geq 0$, in V_1. Let $\varphi = (\overline{u}, \overline{v})^T$, $\overline{v} = d\overline{u}/dt + \varepsilon\overline{u}$, where ε is chosen as

$$\varepsilon = \frac{\lambda_1\alpha}{\alpha^2 + 4\lambda_1}, \tag{25}$$

where $\lambda_1 = \lambda_{(1,0,\dots,0)} = 4m^2\sin^2\pi/m$, then system (21) can be written as

$$\varphi_t + \Lambda\varphi + G(\varphi) = H, \tag{26}$$

where

$$H = (0, \overline{\Gamma})^T, \quad G(\varphi) = \left(0, G_0(u) - \left(\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} g(u_{k_1\dots k_n})\right)e\right)^T,$$

$$\Lambda = \begin{pmatrix} \varepsilon I & -I \\ A - \varepsilon(\alpha - \varepsilon)I & (\alpha - \varepsilon)I \end{pmatrix}. \tag{27}$$

By (19) and (20), we can define the inner product and norm in V_1 as

$$(\varphi,\psi)_{V_1} = (A|_E\overline{u_1},\overline{u_2}) + (\overline{v_1},\overline{v_2}), \quad |\varphi|_{V_1} = (\varphi,\varphi)_{V_1}^{1/2} \tag{28}$$

for $\varphi = (\overline{u_1},\overline{v_1})^T$, $\psi = (\overline{u_2},\overline{v_2})^T \in V_1$.

Lemma 2. *For any* $\varphi = (\overline{u},\overline{v})^T \in V_1$,

$$(\Lambda\varphi,\varphi)_{V_1} \geq \sigma|\varphi|_{V_1}^2 + \frac{\alpha}{2}|\overline{v}|^2, \tag{29}$$

where

$$\sigma = \frac{\lambda_1\alpha}{\sqrt{\alpha^2+4\lambda_1}(\alpha+\sqrt{\alpha^2+4\lambda_1})}. \tag{30}$$

Proof. From (27) and (28), for any $\varphi = (\overline{u},\overline{v})^T \in V_1$, we have

$$\begin{aligned}
&(\Lambda\varphi,\varphi)_{V_1} - \sigma|\varphi|_{V_1}^2 - \frac{\alpha}{2}|\overline{v}|^2 \\
&\quad = (\varepsilon-\sigma)\|\overline{u}\|^2 + \left(\frac{\alpha}{2}-\varepsilon-\sigma\right)|\overline{v}|^2 - \varepsilon(\alpha-\varepsilon)(\overline{u},\overline{v}) \qquad \text{by (20)} \\
&\quad \geq (\varepsilon-\sigma)\|\overline{u}\|^2 + \left(\frac{\alpha}{2}-\varepsilon-\sigma\right)|\overline{v}|^2 - \frac{\varepsilon(\alpha-\varepsilon)}{\sqrt{\lambda_1}}\|\overline{u}\|\cdot|\overline{v}| \\
&\quad \geq (\varepsilon-\sigma)\|\overline{u}\|^2 + \left(\frac{\alpha}{2}-\varepsilon-\sigma\right)|\overline{v}|^2 - \frac{\varepsilon\alpha}{\sqrt{\lambda_1}}\|\overline{u}\|\cdot|\overline{v}|.
\end{aligned}$$

A simple computation by (25) and (30) shows

$$4(\varepsilon-\sigma)\left(\frac{\alpha}{2}-\varepsilon-\sigma\right) = \frac{\varepsilon^2\alpha^2}{\lambda_1}.$$

Thus, the proof is completed. □

Let $\varphi = (\overline{u},\overline{v})^T \in V_1$ be the solution of (26). Taking the inner product $(\,\cdot\,,\,\cdot\,)_{V_1}$ of (26) with $\varphi = (\overline{u},\overline{v})^T \in V_1$ in which $\overline{v} = d\overline{u}/dt + \varepsilon\overline{u}$, we have

$$\frac{1}{2}\frac{d}{dt}|\varphi|_{V_1}^2 = -(\Lambda\varphi,\varphi)_{V_1} - (G(\varphi),\varphi)_{V_1} + (H,\varphi)_{V_1}. \tag{31}$$

By (28) and (29),

$$-2(\Lambda\varphi,\varphi)_{V_1} \leq -2\sigma|\varphi|_{V_1}^2 - \alpha|\overline{v}|^2, \tag{32}$$

$$
\begin{aligned}
(33)\quad & -2(G(\varphi),\varphi)_{V_1}+2(H,\varphi)_{V_1}\\
&=-2\bigg(G_0(u)-\bigg(\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} g(u_{k_1\dots k_n})\bigg)e,\overline{v}\bigg)+2(\overline{\Gamma},\overline{v})\\
&=2(\overline{\Gamma},\overline{v})-2\frac{1}{m^n}\sum_{l_1,\dots,l_n=1}^{m}\bigg(g(u_{l_1\dots l_n})-\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} g(u_{k_1\dots k_n})\bigg)\overline{v}_{l_1\dots l_n}\\
&\le 2|\overline{\Gamma}||\overline{v}|\\
&\quad +2|\overline{v}|\bigg(\frac{1}{m^n}\sum_{l_1,\dots,l_n=1}^{m}\bigg(g(u_{l_1\dots l_n})-\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} g(u_{k_1\dots k_n})\bigg)^2\bigg)^{1/2}\\
&\le 2(|\overline{\Gamma}|+2c)\cdot|\overline{v}|\le \alpha|\overline{v}|^2+\frac{(|\overline{\Gamma}|+2c)^2}{\alpha},
\end{aligned}
$$

where c is defined by (3), thus by (31), (32) and (33), we have

$$
(34)\qquad \frac{d}{dt}|\varphi|^2_{V_1}\le -2\sigma|\varphi|^2_{V_1}+\frac{(|\overline{\Gamma}|+2c)^2}{\alpha}.
$$

Applying the Gronwall inequality, we obtain the following absorbing inequality in the space V_1:

$$
(35)\quad |\varphi(t)|^2_{V_1}\le (||\overline{u^{(0)}}||^2+|\overline{u^{(1)}}+\varepsilon\overline{u^{(0)}}|^2)\exp(-2\sigma t)+\frac{(|\overline{\Gamma}|+2c)^2}{2\sigma\alpha}[1-\exp(-2\sigma t)],
$$

or

$$
\limsup_{t\to+\infty}|\varphi(t)|^2_{V_1}\le\frac{(|\overline{\Gamma}|+2c)^2}{2\sigma\alpha}.
$$

Now, we consider the existence of the global attractor of $S(t)$, $t\ge 0$ in V_1.

If $u=u(t)$ is the solution of (7), then $\overline{u}=Pu$, the orthogonal projection of $u\in M$ into $\overline{u}\in E$, satisfies (35). Thus we have

$$
u(t)=\overline{u}(t)+m(t)e,
$$

and

$$
(36)\qquad \frac{du}{dt}(t)=\frac{d\overline{u}}{dt}(t)+\frac{dm}{dt}(t)e,
$$

where

$$
(37)\qquad m(t)=\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} u_{k_1\dots k_n}.
$$

By (7) and (37),

$$
\frac{d^2m}{dt^2}(t)+\alpha\frac{dm}{dt}(t)+\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m} g(u_{k_1\dots k_n})=\frac{1}{m^n}\sum_{k_1,\dots,k_n=1}^{m}\Gamma_{k_1\dots k_n}.
$$

Integrating this equality,

$$\begin{aligned}(38)\quad &\left|\frac{dm}{dt}(t)\right| \\ &= \left|\frac{dm}{dt}(0)e^{-\alpha t} + \frac{1}{m^n}\int_0^t \sum_{k_1,\dots,k_n=1}^{m} (\Gamma_{k_1\dots k_n} - g(u_{k_1\dots k_n}(\tau)))e^{-\alpha(t-\tau)}\,d\tau\right| \\ &\le \left|\frac{dm}{dt}(0)\right|e^{-\alpha t} + \frac{1}{\alpha}(|\Gamma| + c)(1 - e^{-\alpha t}).\end{aligned}$$

By the definition of V_0,

$$(39)\qquad \left|\left(u(t), \frac{du}{dt}(t)\right)\right|_{V_0}^2 = ||\overline{u}(t)||^2 + |m(t)|^2 + \left|\frac{d\overline{u}}{dt}(t)\right|^2 + \left|\frac{dm}{dt}(t)\right|^2.$$

By the fact that $m(t) \in S^1 = R/TZ$,

$$(40)\qquad |m(t)|^2 \le T^2,$$

and, by (38),

$$(41)\qquad \left|\frac{dm}{dt}(t)\right| \le |u^{(1)}|e^{-\alpha t} + \frac{1}{\alpha}(|\Gamma| + c)(1 - e^{-\alpha t}).$$

By (20),

$$\begin{aligned}(42)\quad ||\overline{u}(t)||^2 + \left|\frac{d\overline{u}}{dt}(t)\right|^2 &\le ||\overline{u}(t)||^2 + (|\overline{v}| + \varepsilon|\overline{u}|)^2 \\ &\le \left(1 + \frac{1}{8}\varepsilon^2\right)||\overline{u}(t)||^2 + 2|\overline{v}|^2 \le \mu(||\overline{u}(t)||^2 + |\overline{v}|^2),\end{aligned}$$

where $\mu = \max\{1 + \varepsilon^2/8, 2\}$, by (35) and (42),

$$\begin{aligned}(43)\quad &||\overline{u}(t)||^2 + \left|\frac{d\overline{u}}{dt}(t)\right|^2 \\ &\le \mu\left\{\left(\left(1 + \frac{1}{8}\varepsilon^2\right)||\overline{u^{(0)}}||^2 + 2|\overline{u^{(1)}}|^2\right)e^{-2\sigma t} + \frac{(|\overline{\Gamma}| + 2c)^2}{2\sigma\alpha}\left(1 - e^{-2\sigma t}\right)\right\} \\ &\le \mu^2(||\overline{u^{(0)}}||^2 + |\overline{u^{(1)}}|^2)e^{-2\sigma t} + \frac{\mu(|\overline{\Gamma}| + 2c)^2}{2\sigma\alpha}\left(1 - e^{-2\sigma t}\right),\end{aligned}$$

then together with (40), (41) and (43), (39) yields

$$\begin{aligned}(44)\quad \left|\left(u(t), \frac{du}{dt}(t)\right)\right|_{V_0}^2 &\le \mu\left(\left(1 + \frac{1}{8}\varepsilon^2\right)||\overline{u^{(0)}}||^2 + 2|\overline{u^{(1)}}|^2\right)e^{-2\sigma t} \\ &\quad + \frac{(|\overline{\Gamma}| + 2c)^2}{2\sigma\alpha}(1 - e^{-2\sigma t}) + T^2 \\ &\quad + (|u^{(1)}|e^{-\alpha t} + \frac{1}{\alpha}(|\Gamma| + c)(1 - e^{-\alpha t}))^2.\end{aligned}$$

Therefore, we have the following lemma.

Lemma 3. *There exists a constant*

$$\rho_0 = \left(T^2 + \frac{\mu(|\overline{\Gamma}| + 2c)^2}{2\sigma\alpha} + \frac{2}{\alpha^2}(|\Gamma| + c)^2\right)^{1/2}$$

such that for any $\rho_1 > \rho_0$ *and any* $R_0 > 0$, *if the initial value* $(u^{(0)}, u^{(1)})^T$ *satisfies*

$$||u^{(0)}||^2 + |u^{(1)}|^2 \le R_0^2,$$

then the solution $u(t)$ *of* (7) *satisfies*

$$\left|\left(u(t), \frac{du}{dt}(t)\right)\right|_{V_0} \le \rho_1$$

for any

$$t \ge T_0 = \frac{1}{2\sigma} \log \frac{(\mu^2 + 2)R_0^2 + \frac{2}{\alpha^2}(|\Gamma| + c)^2}{\rho_1^2 - \rho_0^2}.$$

As a direct consequence of Lemma 3, we have the existence of the global attractor.

Theorem 2. *The nonlinear semi-flow of* (7) *possesses a global attractor* ß *in* V_0.

4. Hausdorff dimension of the global attractor

We note that the projection $P : M \to E$ induces a projection on V_0, denoted by P again.

Lemma 4. *Let* $S(t)$ *be the semi-flow of* (7) *and* $\omega(PS(t))$ *be the* ω*-limit set of the restricted semi-flow of* (26), *we have*

$$P\text{ß} = \omega(PS(t)). \tag{45}$$

Proof. For any $(w, z)^T \in$ ß, assume that there exist initial conditions $u^{(0)}$, $u^{(1)}$ such that the solution $u(t)$ of (7) with $u(0) = u^{(0)}$, $\frac{du}{dt}(0) = u^{(1)} \in M$, satisfies $u(t_j) \to w$ in $E_1 \times S^1$ and $\frac{du}{dt}(t_j) \to z$ in $E_0 \times R$ for some sequence $t_j \to \infty$. Then $\overline{u}(t_j) = Pu(t_j) \to \overline{w} = Pw$ in E_1 and $\frac{d\overline{u}}{dt}(t_j) = P\frac{du}{dt}(t_j) \to \overline{z} = Pz$ in E_0, i.e., Pß $\subset \omega(PS(t))$.

Conversely, assume for some initial conditions $u^{(0)}$, $u^{(1)}$ and a sequence $t_j \to \infty$ such that $Pu(t_j) \to \overline{w}$ in E_1 and $P\frac{du}{dt}(t_j) \to \overline{z}$ in E_0. Since $m(t)$ and $\frac{dm}{dt}(t)$ defined by (37) and (36) are both bounded, then there exists a subsequence of $\{t_j\}$, denoted by $\{t_{j_i}\}$, such that

$$\left(u(t_{j_i}), \frac{du}{dt}(t_{j_i})\right) \to (w, z) \quad \text{in } V_0,$$

where $Pw = \overline{w}$, $Pz = \overline{z}$. This implies $\omega(PS(t)) \subset P$ß. □

Lemma 5. *The Hausdorff dimension $d_H(ß)$ of the global attractor $ß \subset V_0$ satisfies*

$$d_H(ß) \leq d_H(Pß) + 2. \tag{46}$$

Proof. Since $ß \subset Pß \times S^1 \times R^1$, then

$$d_H(ß) \leq d_H(Pß \times S^1 \times R^1) \leq d_H(Pß) + 2. \qquad \square$$

According to the above lemmas, we only need to consider the Hausdorff dimension of $Pß$, the global attractor of system (21).

To estimate the dimension of the attractor for system (21), we consider the first variation equation of (26)

$$\Psi' = \overline{F'(\varphi)(U,V)^T} = -\Lambda\Psi + G'(\varphi)(U,V)^T, \Psi(0) = (\overline{\xi},\overline{\eta})^T \in V_1, \tag{47}$$

where $\Psi = (\overline{U},\overline{V})^T$, $\varphi = (\overline{u},\overline{v})^T$ is a solution of (26)–(22),

$$G'(\varphi)(U,V)^T = \left(0, \begin{pmatrix} g'(u_{11\ldots11})U_{11\ldots11} \\ g'(u_{11\ldots12})U_{11\ldots12} \\ \vdots \\ g'(u_{mm\ldots mm})U_{mm\ldots mm}\end{pmatrix} - \left(\frac{1}{m^n}\sum_{k_1,\ldots,k_n=1}^{m} g'(u_{k_1\ldots k_n})U_{k_1\ldots k_n}\right)e\right)^T, \tag{48}$$

and

$$u = (u_{11\ldots11}, u_{11\ldots12}, \ldots, u_{mm\ldots mm})^T \in M$$

is a solution of (7), (8), and

$$U = (U_{11\ldots11}, U_{11\ldots12}, \ldots, U_{mm\ldots mm})^T, \quad V = \frac{dU}{dt} + \varepsilon U$$

is a solution of the variation equation of (7), (8) with initial value conditions

$$\begin{aligned}
U(0) &= \xi = (\xi_{11\ldots11}, \xi_{11\ldots12}, \ldots, \xi_{mm\ldots mm})^T, \\
V(0) &= \frac{dU}{dt}(0) + \varepsilon U(0) = \eta = (\eta_{11\ldots11}, \eta_{11\ldots12}, \ldots, \eta_{mm\ldots mm})^T, \\
\overline{U} &= U - \frac{1}{m^n}\sum_{k_1,\ldots,k_n=1}^{m} U_{k_1\ldots k_n}, \\
\overline{\xi} &= \xi - \frac{1}{m^n}\sum_{k_1,\ldots,k_n=1}^{m} \xi_{k_1\ldots k_n}, \\
\overline{\eta} &= \eta - \frac{1}{m^n}\sum_{k_1,\ldots,k_n=1}^{m} \eta_{k_1\ldots k_n}.
\end{aligned}$$

LEMMA 6. *For any orthonormal family of elements of* V_1, $\{\xi_j, \eta_j\}_{j=1}^{\ell}$,

$$\sum_{j=1}^{\ell} |\xi_j|^2 \le \sum_{j=1}^{\ell} \frac{1}{\lambda_j}, \tag{49}$$

where $0 < \lambda_1 \le \ldots \le \lambda_\ell \le \ldots \le \lambda_{m^n-1}$ *are eigenvalues of operator* $A|_E$.

PROOF. See lemma VI. 6.3. in [3]. □

LEMMA 7. *Consider the system* (26). *Let* Φ *denote a set of* ℓ *vectors*

$$\{\Phi_1, \ldots, \Phi_\ell\}$$

which are orthonormal in V_1. *If*

$$\sup_{\Phi \subset V_1} \sup_{\varphi \in P\text{ß}} \sum_{j=1}^{\ell} ((-\Lambda\Phi_j, \Phi_j)_{V_1} + (G'(\varphi)\Phi_j, \Phi_j)_{V_1}) < 0, \tag{50}$$

then the Hausdorff dimension of the global attractor Pß *of* (26) *is less than or equals to* ℓ, *i.e.,*

$$d_H(P\text{ß}) \le \ell.$$

PROOF. This is a direct consequence of theorem V. 3.3, equations (V.3.47)–(V.3.49) and identity (VI.6.24) of [3]. □

LEMMA 8. *The Hausdorff dimension* $d_H(P\text{ß})$ *of the global attractor for system* (26) *satisfies*

$$d_H \le \min\left\{\ell \,\middle|\, \ell \in \mathbb{N},\ \frac{1}{\ell}\sum_{j=1}^{[\ell/2]+1} \frac{1}{\widetilde{\lambda}_j} < \frac{\lambda_1\alpha^2}{4C^2\sqrt{\alpha^2+4\lambda_1}(\alpha+\sqrt{\alpha^2+4\lambda_1})}\right\} \tag{51}$$

where C *is defined by* (3), $\lambda_1 = 4m^2\sin^2\pi/m$ *and* $16 \le \lambda_1 = \widetilde{\lambda}_1 \le \ldots \le \widetilde{\lambda}_\ell \le \ldots \le \widetilde{\lambda}_{[m/2]^n+1}$ *are the ordering, from small to large, of set*

$$\{16(l_1^2 + \ldots + l_n^2) \mid 0 \le l_1, \ldots, l_n \le [m/2], \quad \text{but } l_1 + \ldots + l_n \ge 1\}.$$

PROOF. Let $\ell \in \mathbb{N}$ be fixed. Consider ℓ solutions $\Psi_1, \ldots, \Psi_\ell$ of (47). At a given time τ, let $Q_\ell(\tau)$ be the orthogonal projector in V_1 onto the space spanned by $\Psi_1, \ldots, \Psi_\ell$. Let $\Phi^j(\tau) = (\overline{\xi^j}(\tau), \overline{\eta^j}(\tau))^T \in V_1$, $j = 1, \ldots, \ell$, denote an orthonormal basis of $Q_\ell(\tau)V_1 = \text{span}\{\Psi_1(\tau), (\tau), \ldots, \Psi_\ell(\tau)\}$. Consider

$$\begin{aligned} Tr\overline{F'(\varphi(\tau))} \circ Q_\ell(\tau) &= \sum_{j=1}^{\ell} (\overline{F'(\varphi(\tau))\Phi^j(\tau)}, \Phi^j(\tau))_{V_1} \\ &= -\sum_{j=1}^{\ell} [(\Lambda\Phi^j, \Phi^j)_{V_1} - (G'(\varphi)\Phi^j, \Phi^j)_{V_1}]. \end{aligned}$$

By (29) and $|\Phi^j|_{V_1} = 1$,

$$-(\Lambda\Phi^j, \Phi^j)_{V_0} \leq -\sigma - \frac{\alpha}{2}|\overline{\eta^j}|^2$$

By (28) and (48),

$$\begin{aligned}
&|(G^{'}(\varphi)\Phi^j, \Phi^j)_{V_1}| \\
&= \left|\left(\begin{pmatrix} g'(u_{11\ldots11})\overline{\xi^j_{11\ldots11}} \\ g'(u_{11\ldots12})\overline{\xi^j_{11\ldots12}} \\ \vdots \\ g'(u_{mm\ldots mm})\overline{\xi^j_{mm\ldots mm}} \end{pmatrix} - \left(\frac{1}{m^n}\sum_{k_1,\ldots,k_n=1}^{m} g'(u_{k_1\ldots k_n})\overline{\xi^j_{k_1\ldots k_n}}\right)e, \overline{\eta^j}\right)\right| \\
&\leq \left|\frac{1}{m^n}\sum_{k_1,\ldots,k_n=1}^{m} g'(u_{k_1\ldots k_n}) \cdot \overline{\xi^j_{k_1\ldots k_n}} \cdot \overline{\eta^j_{k_1\ldots k_n}}\right| \\
&\quad + \left|\left(\frac{1}{m^n}\sum_{k_1,\ldots,k_n=1}^{m} g'(u_{k_1\ldots k_n})\overline{\xi^j_i}\right) \cdot \left(\frac{1}{m^n}\sum_{k_1,\ldots,k_n=1}^{m} \overline{\eta^j_{k_1\ldots k_n}}\right)\right| \leq 2C|\overline{\xi^j}| \cdot |\overline{\eta^j}|.
\end{aligned}$$

Hence,

$$\begin{aligned}
(52) \qquad Tr\overline{F'(\varphi(\tau))} \circ Q_\ell(\tau) &\leq -\ell\sigma - \frac{\alpha}{2}\sum_{j=1}^{\ell}|\overline{\eta^j}|^2 + \sum_{j=1}^{\ell} 2C|\overline{\xi^j}| \cdot |\overline{\eta^j}| \\
&\leq -\ell\sigma + \frac{2C^2}{\alpha}\sum_{j=1}^{\ell}|\overline{\xi^j}|^2 \quad (\text{by}(49)) \\
&\leq -\ell\sigma + \frac{2C^2}{\alpha}\sum_{j=1}^{\ell}\frac{1}{\lambda_j}.
\end{aligned}$$

Since $\sin x \geq 2x/\pi$ for $x \in [0, 1/2]$, the eigenvalues of the operator $A|_E$ as follows:

$$\lambda_{(l_1,\ldots,l_n)} = 4m^2\left(\sin^2\frac{l_1\pi}{m} + \ldots + \sin^2\frac{l_n\pi}{m}\right) \geq 16(l_1^2 + \ldots + l_n^2),$$

for any $0 \leq l_1, \ldots, l_n \leq [m/2]$.

Let $0 < \widetilde{\lambda}_1 \leq \ldots \leq \widetilde{\lambda}_\ell \leq \ldots \leq \widetilde{\lambda}_{[m/2]^n+1}$ be the ordering, from small to large, of set

$$\{16(l_1^2 \ldots + l_n^2) | 0 \leq l_1, \ldots, l_n \leq [m/2] \quad \text{but } l_1 + \ldots + l_n \geq 1\}.$$

Thus, by (52),

$$(53) \qquad Tr\overline{F'(\varphi(\tau))} \circ Q_\ell(\tau) \leq -\ell\sigma + \frac{4C^2}{\alpha}\sum_{j=1}^{[\ell/2]+1}\frac{1}{\widetilde{\lambda}_j}.$$

If

$$\frac{1}{\ell}\sum_{j=1}^{[\ell/2]+1}\frac{1}{\widetilde{\lambda}_j} < \frac{\alpha\sigma}{4C^2} = \frac{\lambda_1\alpha^2}{4C^2\sqrt{\alpha^2 + 4\lambda_1}(\alpha + \sqrt{\alpha^2 + 4\lambda_1})}$$

then, by (53),

$$Tr\overline{F'(\varphi(\tau))} \circ Q_\ell(\tau) < 0.$$

By Lemma 7, (51) is true. The proof is completed. □

COROLLARY 1. *If*

$$\lambda_1^2\alpha^2 > 4C^2\sqrt{\alpha^2 + 4\lambda_1}(\alpha + \sqrt{\alpha^2 + 4\lambda_1}),$$

then $d_H(Pß) = 0$.

PROOF. In this case, $\ell = 1$ in (52) and $(\overline{F'(\varphi(\tau))\Phi(\tau)}, \Phi(\tau))_{V_1} < 0$ for any unit element $\Phi = (\overline{\xi}, \overline{\eta})^T \in V_1$. So, the largest Lyapunov exponent of $Pß$: $\mu_1 < 0$, hence, $d_H(Pß) = 0$.

Combining with Theorem 2, Lemma 5, Lemma 8, and Corollary 1, we complete the proof of Theorem 1. □

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