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A TOPOLOGICAL APPROACH TO SUPERLINEAR INDEFINITE BOUNDARY VALUE PROBLEMS

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Abstract. We obtain the existence of infinitely many solutions with prescribed nodal properties for some boundary value problems associated to the second order scalar equation $\ddot{x} + q(t)g(x) = 0$, where $g(x)$ has superlinear growth at infinity and $q(t)$ changes sign.

1. Introduction

The study of boundary value problems for the second order ordinary differential equation

$$\ddot{x} + f(t, x) = 0 \tag{1.1}$$

with f superlinear at infinity with respect to x, that is

$$\lim_{s\to\pm\infty} \frac{f(t, s)}{s} = \infty, \tag{1.2}$$

is a topic which has been widely investigated in the literature, using various different approaches.

The case in which the limit in (1.2) holds uniformly with respect to t was studied since the fifties by Morris [37], [38], Ehrmann [23], [24] and Nehari [39]

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for some particular forms of f (see also the references in [17]). The main feature of such kind of problems is the absence of a priori bounds due to the strong oscillatory behaviour of the large solutions [36]. A typical example of equation (1.1) is given by

$$\ddot{x} + q(t)g(x) = 0 \tag{1.3}$$

with g satisfying

$$\lim_{s\to\pm\infty} \frac{g(s)}{s} = \infty. \tag{1.4}$$

In this case, the uniformity with respect to t of the limit in (1.2) corresponds to the condition of definite sign

$$\inf_t q(t) > 0.$$

In this situation, according to Coffman and Ullrich [19], some weak regularity assumptions on the "weight" q (like q continuous and locally of bounded variation) are enough to guarantee the global continuability of the solutions for the associated Cauchy problems and this allows to apply shooting type techniques. A particular type of equation (1.3) is given by

$$\ddot{x} + q(t)x^{2n-1} = 0, \tag{1.5}$$

which has deserved much attention for the rich dynamics exhibited by its solutions. After the work of Laederich and Levi [30] (see also [33], [34], for further progress in that direction) it is known that for $q(\,\cdot\,)$ positive, periodic and sufficiently smooth (e.g., at least of class C^5, according to [33]), all the solutions of (1.5) are bounded, there are infinitely many periodic solutions (harmonics and subharmonics of each order) and most of the solutions with large amplitude are quasiperiodic. The same kind of result holds for suitable perturbations of (1.5), as well as for (1.1) with f smooth enough in both variables and having a polynomial growth in x (see, e.g., [33], [34]). On the other hand, it was proved in [19], that the sole continuity of q is not sufficient even for the global continuability of all the solutions.

It seems that Waltman [47] was the first who considered a changing sign weight for the study of the oscillatory solutions for a superlinear equation of the form (1.5). Observe that here the global continuability of the solutions is no more guaranteed (independently of the degree of regularity of q). In fact, one can see, according to Burton and Grimmer [11], that some solutions will blow up in the intervals of negativity for $q(t)$. Hence, in this situation, the problem of absence of a priori bounds is accompanied by the technical difficulty due to the noncontinuability of some solutions. This makes the phase-plane analysis somehow more delicate. A study of the topological properties of the set of initial points from which depart globally defined solutions of (1.3) was initiated by

Butler in [12], [13]. A condition for the stability of the origin for a perturbed form of (1.5) with q continuous, periodic and changing sign, has been recently obtained by Liu in [35].

In 1976, G. J. Butler [12], proved the existence of infinitely many periodic solutions for equation (1.3) in a situation for which the weight function $q(t)$ may change sign. Butler's argument, which is an ingenious blend of the rapid oscillatory properties of the large solutions when $q > 0$ with the properties of the set of initial points of the continuable solutions when $q < 0$, seems to be "flexible" enough to be adapted to other boundary value problems.

In 1991, L. Lassoued [31] using a variational approach, obtained the existence of one non-constant T-periodic solution for the system of differential equations $\ddot{x} + q(t)G'(x) = 0$, assuming $G : \mathbb{R}^N \to \mathbb{R}$ a superquadratic convex homogeneous function of class C^2 and $q \in L^1([0,T],\mathbb{R})$ a T-periodic function which changes sign. In the case of G even, also the existence of infinitely many solutions was proved. Various investigations along this or related directions were then performed, in particular, with respect to the existence and multiplicity of solutions for Dirichlet, Neumann or mixed boundary value problems associated to elliptic equations [2]–[5], [7]–[10], [29], [32], [42] or to the existence and multiplicity of periodic solutions for Hamiltonian systems [6], [8], [12], [26], [27]. Most of the above quoted results apply to situations (like PDEs or systems) which are widely more general than (1.3) on the other hand, the assumptions involved therein on the nonlinearity require either symmetry conditions or a growth at infinity which is quite close to a power. Moreover, except for the case of the existence of positive solutions, the results obtained up to now, seem to be not very complete with respect to the nodal properties of the solutions.

In a recent article [46], S. Terracini and G. Verzini, dealing with the scalar equation

$$\ddot{x} + q(t)x^3 = 0,$$

obtained a very sharp result for the two-point boundary value problem, proving the existence of solutions with precise nodal properties in the intervals of positivity of $q(t)$ and with exactly one zero in the intervals of negativity of q. Similar conclusions are derived for the periodic problem and for the existence of bounded non-periodic oscillatory solutions defined on the whole real line. The Nehari method employed in [46], works also adding a perturbation of the form $m(t)x + h(t)$ and for more general nonlinearities of power-like growth.

In this paper, following some lines inspired by the Butler approach [12], we prove, like in [46], that the two-point boundary value problem for equation (1.3) has solutions with precise number of zeros in each interval of positivity of q and, moreover, for each interval of negativity, we can fix a priori if the solution will have exactly one zero being also strictly monotone or will have no zeros and exactly one zero of the derivative. As in [12], we combine the oscillatory properties of the solutions with the noncontinuability. Indeed, the condition of superlinear growth at infinity produces a strong oscillatory behaviour of the large solutions in the intervals of positivity for $q(t)$ (see [18], [28], [36]). In the proof, the main problem that we have to face is to "connect" a solution of (1.3) having a certain number n_1 of zeros in an interval $]a,b[$, where $q>0$, to a solution with a possibly different number n_2 of zeros in another interval of positivity $]c,d[$, by passing through the intermediate interval $]b,c[$ where $q<0$. This program is achieved by finding a continuum Γ in the phase-plane such that all the solutions departing from Γ at the time $t=a$ will be continuable till to $t=d$ and have n_1 zeros in $]a,b[$ and n_2 zeros in $]c,d[$. We can also prescribe that either the solution or its derivative, will vanish exactly once in $]b,c[$. A repetition of this argument yields the conclusion.

The class of the functions $g(x)$ to which our result can be applied is of the same kind as that considered by Butler in [12], [13], namely, it contains all the monotone locally Lipschitz functions $g:\mathbb{R}\to\mathbb{R}$ satisfying (1.4) and such that

$$\left|\int^{\pm\infty}\frac{ds}{\sqrt{G(s)}}\right|<\infty, \tag{1.6}$$

where

$$G(x)=\int_0^x g(s)\,ds.$$

Indeed, our class of functions (like Butler's one) is much larger as it contains non-monotone functions, hence, the potential G need not be convex. For example, any locally Lipschitz function $g:\mathbb{R}\to\mathbb{R}$, with $g(s)s>0$ for $s\neq 0$, and such that

$$\exists k>0,\alpha>2:|g(s)|\geq k|s|\log^{\alpha}(1+|s|)\quad\text{for } |s|\gg 1,$$

is suitable for our results. A more precise discussion about the hypotheses that we need to assume on $g(x)$ is given at the end of Section 2 and in the Appendix. As an example of a result that we can obtain, let us consider the following consequence of Theorem 1 in Section 4.

Corollary 1. *Let $g:\mathbb{R}\to\mathbb{R}$ be a locally Lipschitz continuous function with $g(0)=0$ and $g(s)s>0$ for $s\neq 0$. Suppose that g is monotone increasing in a neighbourhood of infinity and let* (1.4) *and* (1.6) *hold. Let $q:[0,\omega]\to\mathbb{R}$ be a continuous and piecewise monotone function with a finite number of zeros such that*

$$[0,\omega]=\left(\bigcup_{i=1}^{k+1}I_i^+\right)\cup\left(\bigcup_{i=1}^{k}I_i^-\right)\cup J,$$

where we assume that $J=[0,\omega]\setminus((\bigcup_{i=1}^{k+1}I_i^+)\cup g(\bigcup_{i=1}^{k}I_i^-))$ *and that*

$$I_1^+,I_1^-,\dots,I_k^+,I_k^-,I_{k+1}^+$$

are $2k+1$ consecutive adjacent nondegenerate closed intervals, with $q \geq 0$ on I_i^+ and $q \leq 0$ on I_i^- and on J ($J = \emptyset$ is not excluded). Then, there are $k+1$ positive integers $n_1^*, \dots, n_{k+1}^*$ such that for each $(k+1)$-uple $\mathbf{n} := (n_1, \dots, n_{k+1})$, with $n_i > n_i^*$, and each k-uple $\boldsymbol{\delta} := (\delta_1, \dots, \delta_k)$, with $\delta_i \in \{0,1\}$, there are at least two solutions $x^+_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ and $x^-_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ of (1.3), *such that $x^\pm_{\mathbf{n},\boldsymbol{\delta}}(0) = x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\omega) = 0$ and*

(1) $\dot{x}^-_{\mathbf{n},\boldsymbol{\delta}}(0) < 0 < \dot{x}^+_{\mathbf{n},\boldsymbol{\delta}}(0)$,
(2) $x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ *has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1-\delta_i$ zeros of the derivative in I_i^-*,
(3) *neither $x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$, nor $\dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$, may vanish in $J \setminus \{0,\omega\}$*,
(4) *for each i, $|x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| + |\dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| \to \infty$, as $n_i \to \infty$, uniformly in $t \in I_i^+$.*

As we have remarked above, more general assumptions on g are allowed. More precisely, we are able to obtain our result by assuming only conditions on the time-mappings associated to the autonomous equations

$$\ddot{x} + g(x) = 0, \quad \ddot{x} - g(x) = 0,$$

which, roughly speaking, describe the dynamic behaviour of equations (1.3) in the intervals where $q(t) > 0$ or $q(t) < 0$, respectively. A precise definition of these conditions, which are called (g_2^+) and (g_2^-) respectively, is given in Section 2; intuitively they mean that the time to run along the orbits tends to zero as we consider orbits which are far and far from the origin.

All the results that we obtain for equation (1.3), could be given for an equation of the form (1.1) under suitable assumptions on f like those in [12]. As an example, we could deal with

$$f(t,x) = q^+(t) f_1(x) + q^-(t) f_2(x),$$

with $f_i(s)s > 0$ for $s \neq 0$ and f_1 and f_2 satisfying (g_2^+) and (g_2^-), respectively. Similar kind of nonlinearities, with $f_1(s) = s^\alpha$ and $f_2(s) = s^\beta$, have been recently considered in [4] for the study of positive solutions of an elliptic problem.

With respect to the weight function $q(t)$, we observe that the assumption of piecewise monotonicity can be removed at some extent. We also point out that, using our approach, it is easy to deal with the case in which there are some intervals where $q \equiv 0$. More details about the meaning of the assumptions for q are given in Section 2 (see also [41], where a similar kind of weights is considered with respect to a Floquet type BVP). Concerning $g(x)$, we note that, by the use of mollifiers and proving the fact that the solutions of (1.3) with fixed nodal properties will be subjected to a priori bounds which are uniform with respect to perturbations of g which are small in the compact-open topology, it is possible to check that the condition of local lipschitzianity for g can be dropped and the continuity of g (paired by an upper bound for $g(x)/x$ in a neighbourhood of zero) is enough to prove all our results. However, in order to make our arguments more transparent, we have preferred not to pursue a detailed investigation in this direction and we have assumed the local Lipschitz condition in order to have the uniqueness of the solutions for the Cauchy problems.

Further results are then given in Section 6 for the Sturm–Liouville boundary conditions. There we also show how it is possible to obtain nodal singular solutions, that is solutions which blow up in some fixed interval of negativity I_j^- and have prescribed oscillatory properties in the I_i^+ for $i \leq j$. Clearly, the same kind of results can be obtained for radially symmetric solutions of the equation $\Delta u + q(|x|)g(u) = 0$ in an annular domain.

In Section 5, we consider the case in which $q : \mathbb{R} \to \mathbb{R}$ and $q(t) \leq 0$ for t in a neighbourhood $]-\infty, a[\cup]b, \infty[$, of infinity and q changes sign on $[a, b]$. Then, combining our approach for the proof of Theorem 1 with a Ważewski type argument considered by C. Conley in [20], we obtain the existence of solutions $x(t)$ which are asymptotic to zero as $t \to \pm\infty$ and have given nodal properties in the interval $[a, b]$ like in the above Corollary 1. Recent results in this direction for Hamiltonian systems having a superlinear growth at infinity and a weight function possibly changing sign, can be found in [14], [22], [25].

As a last consideration, we observe that an iteration of our main argument for the two-point boundary value problem indicates the existence of solutions having a kind of "chaotic" behaviour, when $q(t)$ is defined on $\mathbb{R}^+$ and the intervals of positivity and negativity of q interchange indefinitely. For example, the following result could be obtained by an inductive reasoning along the proof of Theorem 1:

Assume that $g : \mathbb{R} \to \mathbb{R}$ is like in Corollary 1 and let $q : [0, \infty[\to \mathbb{R}$ be a continuous piecewise monotone and ω-periodic function such that $q(0) = q(\sigma) = q(\omega) = 0$ for some $\sigma \in]0, \omega[$, and such that $q > 0$ in $]0, \sigma[$ and $q < 0$ in $]\sigma, \omega[$. Then, there is a positive integer n^*, such that for any integer $N^* \geq n^*$, and having chosen arbitrarily a sequence $(n_j)_j$, and sequence $(\delta_j)_j$, such that

$$n^* \leq n_j \leq N^* \quad \text{and} \quad \delta_j \in \{0, 1\} \quad \text{for all } j = 1, 2, \ldots,$$

there are at least two solutions $u(\,\cdot\,)$ and $v(\,\cdot\,)$ to equation (1.3), such that $u(0) = v(0) = 0$, $\dot{u}(0) > 0 > \dot{v}(0)$ and $u(\,\cdot\,)$, $v(\,\cdot\,)$, have exactly n_j zeros in the intervals $](j-1)\omega, (j-1)\omega + \sigma[$ and exactly δ_j zeros and $1 - \delta_j$ zeros of the derivative in the intervals $[(j-1)\omega + \sigma, j\omega]$.

By construction, the sets $\{\dot{u}(0)\} \subset]0, \infty[$, and $\{\dot{v}(0)\} \subset]-\infty, 0[$, for $u(\,\cdot\,)$ and $v(\,\cdot\,)$ as above, have the power of the continuum and possess a Cantor-like structure. We thank professors Rafael Ortega and Zhong Li, who called our attention on this possible development of our approach. More general results in this direction will be given elsewhere [15].

2. Basic setting

Consider the second order equation

$$\ddot{x} + q(t)g(x) = 0. \tag{2.1}$$

Throughout the paper the following conditions are assumed:

(g_1) $g : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, $g(x)x > 0$ for $x \neq 0$, and such that

$$\lim_{s \to \pm\infty} |g(s)| = \infty.$$

We define

$$G(s) = \int_0^s g(\xi)\, d\xi.$$

Observe that $G : \mathbb{R} \to \mathbb{R}$ is strictly increasing on $[0, \infty[$ and strictly decreasing on $]-\infty, 0]$ and also $G(\pm\infty) = \infty$. Hence there are a right and a left inverse, denoted, respectively, by $G_+^{-1} : [0, \infty[\to [0, \infty[$ and $G_-^{-1} : [0, \infty[\to]-\infty, 0]$.

With respect to the "weight" function $q(t)$, we assume that it is continuously defined on a interval and the following convention is used:

(q_1) If $I \subset \mathrm{dom}(q)$ is any interval such that $q(t) \geq 0$, for all $t \in I$ and $q \not\equiv 0$ on I, then q is locally of bounded variation in I and the set where $q(t) > 0$ is the union of a finite number of open intervals. Moreover, if $[a, b] \subset I$ is such that $q(a) = 0$ (or $q(b) = 0$) and $q(t) > 0$ for all $t \in]a, b[$, then q is monotone in a right neighbourhood of a (or, respectively, in a left neighbourhood of b).

The hypothesis (q_1) is taken from [13], [19] in order to guarantee the continuability of the solutions to the initial value problems for (2.1) in the intervals where q is nonnegative.

Associated to (2.1), the following autonomous equations

$$\ddot{x} + g(x) = 0 \tag{2.2}$$

and

$$\ddot{x} - g(x) = 0 \tag{2.3}$$

are considered. In the corresponding phase-plane $(x, \dot{x})$, equation (2.2) describes a global center around the origin. In fact, the level lines

$$\frac{1}{2}y^2 + G(x) = c > 0,$$

correspond to the (non-trivial) orbits and these orbits are periodic with minimal period

$$\tau^+(c) := \sqrt{2} \int_{G_-^{-1}(c)}^{G_+^{-1}(c)} \frac{1}{\sqrt{c - G(s)}}\, ds.$$

On the other hand, equation (2.3), describes a saddle-like structure in the plane. In fact the union of the origin with its stable and unstable manifolds coincides with the set $y^2 = 2G(x)$, so that the level lines

$$\frac{1}{2}y^2 - G(x) = c \neq 0,$$

correspond to the orbits crossing the positive and the negative axes at the points $(0, \pm\sqrt{2c})$ and $(G_\pm^{-1}(-c), 0)$, according to the fact that $c > 0$ or $c < 0$. The time to run along these orbits is given by the following integrals: the number

$$\frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{c + G(s)}}\, ds \quad \text{if } c > 0,$$

represents the time for each of the two orbits passing through $(0, \pm\sqrt{2c})$ and contained in the upper half-plane $y > 0$ and in the lower half-plane $y < 0$, respectively. The number

$$\sqrt{2} \int_{G_+^{-1}(-c)}^{\infty} \frac{1}{\sqrt{c + G(s)}}\, ds \quad \text{if } c < 0$$

is the time for the orbit passing through $(G_+^{-1}(-c), 0)$ and contained in the right half-plane $x > 0$, while

$$\sqrt{2} \int_{-\infty}^{G_-^{-1}(-c)} \frac{1}{\sqrt{c + G(s)}}\, ds \quad \text{if } c < 0$$

is the time for the orbit passing through $(G_-^{-1}(-c), 0)$ and contained in the left half-plane $x < 0$. If we set

$$\tau^-(c) := \sqrt{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{c + G(s)}}\, ds \quad \text{for } c > 0$$

and

$$\begin{aligned} \tau^-(c) := \sqrt{2} \int_{-\infty}^{G_-^{-1}(-c)} &\frac{1}{\sqrt{c + G(s)}}\, ds \\ &+ \sqrt{2} \int_{G_+^{-1}(-c)}^{\infty} \frac{1}{\sqrt{c + G(s)}}\, ds \quad \text{for } c < 0, \end{aligned}$$

we have that $\tau^\pm(c)$ represents the whole time of moving along the level lines

$$\frac{1}{2}y^2 \pm G(x) = c \neq 0.$$

In the main result of the paper, we shall assume the condition $(g_2) := (g_2^+) \wedge (g_2^-)$, where

(g_2^+) $\lim_{c\to\infty} \tau^+(c) = 0$,
(g_2^-) $\lim_{c\to\pm\infty} \tau^-(c) = 0$.

Hypothesis (g_2) is a weak condition of superlinear growth of g at infinity. For instance, it is satisfied when $\liminf_{|s|\to\infty} g(s)/s\log^\alpha s > 0$, for some $\alpha > 2$. A brief discussion about some (more general) possible assumptions on g and G yielding to (g_2) is given in the Appendix.

3. Technical lemmas

Consider the second order equation

$$\ddot{x} + q(t)g(x) = 0, \tag{3.1}$$

with $g : \mathbb{R} \to \mathbb{R}$ and $q : [0,\omega] \to \mathbb{R}$ continuous and such that (g_1) and (q_1) hold. We are interested in the search of solutions of (3.1) satisfying the two-point boundary condition

$$x(0) = x(\omega) = 0. \tag{3.2}$$

The unique solution $z(t) = (x(t), \dot{x}(t))$ of

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -q(t)g(x), \end{cases} \tag{3.3}$$

with $z(t_0) = p = (p_1, p_2)$ will be denoted by $z(\,\cdot\,; t_0, p)$. For simplicity, we also introduce the following notation: $\mathbb{R}^+$ and $\mathbb{R}^+_0$ are respectively the sets of non-negative and positive real numbers. Moreover, we set $H^+ := \{(x,y) \in \mathbb{R}^2 : x > 0\} \cup \{(0,y) \in \mathbb{R}^2 : y > 0\}$ and $H^- := -H^+$. A path is a continuous one-to-one map (simple curve) from an interval to $\mathbb{R}^2$. When no confusion occurs, we speak at the same time of a path $\gamma : I \to \mathbb{R}^2$ and its image $\Gamma = \gamma(I) \subset \mathbb{R}^2$. The positive open quadrant $(\mathbb{R}^+_0)^2$ is denoted by A_1. The other open quadrants of the plane (axes excluded), counted after A_1 in the counterclockwise sense, are A_2, A_3, A_4, respectively. $B(R)$ and $B[R]$ are, respectively, the open and the closed disk centered at the origin and with radius R, in the plane.

The uniqueness of the solution $z(t) \equiv 0$ implies that the number

$$\mathrm{rot}_{[a,b]}(p) := \int_a^b \frac{q(t)g(x(t;a,p))x(t;a,p) + \dot{x}^2(t;a,p)}{x^2(t;a,p) + \dot{x}^2(t;a,p)}\,dt$$

is well defined for every $p \neq 0$ and represents the angle spanned by the vector $z(t;a,p)$ measured in clockwise sense as t increases from a to b.

Lemma 1. *Let $[b,c] \subset [0,\omega]$ be such that*

$$q \le 0 \text{ and } q \not\equiv 0 \text{ on } [b,c].$$

Consider a half-line

$$\Gamma := \{p \in \mathbb{R}^2 : p = sv,\ s \ge 0\},$$

where v is a fixed unit vector of the plane. Assume that (g_2^-) *is satisfied. Then, there are four positive constants:* $K_b \leq L_b$ *and* $K_c \leq L_c$ *such that the following holds:*

- *for each* $k \in]0, K_b[$*, the solution* $x(\,\cdot\,; b, kv)$ *is defined on* $[b, c]$ *and*

$$\lim_{k\to K_b} |x(c; b, kv)| = \lim_{k\to K_b} |\dot{x}(c; b, kv)| = \infty,$$

- *for each* $k \geq L_b$*, the solution* $x(\,\cdot\,; b, kv)$ *is not defined on* $[b, c]$*,*
- *for each* $k \in]0, K_c[$*, the solution* $x(\,\cdot\,; c, kv)$ *is defined on* $[b, c]$ *and*

$$\lim_{k\to K_c} |x(b; c, kv)| = \lim_{k\to K_c} |\dot{x}(b; c, kv)| = \infty,$$

- *for each* $k \geq L_c$ *, the solution* $x(\,\cdot\,; c, kv)$ *is not defined on* $[b, c]$*.*

In the special situation when Γ is one of the half-axes, more precise information on the solution is available. For example, denoting by $]\alpha_x, \beta_x[$ the maximal interval of existence of the solution $x(\,\cdot\,)$, we have that

- if $v = (1, 0)$ and $k > 0$, then $x(t; b, kv) \geq k$ and $\dot{x}(t; b, kv) > 0$ for $t \in]b, \beta_x[$, while $x(t; c, kv) \geq k$ and $\dot{x}(t; c, kv) < 0$ for $t \in]\alpha_x, c[$,
- if $v = (0, 1)$ and $k > 0$, then $x(t; b, kv) > 0$ and $\dot{x}(t; b, kv) \geq k$ for $t \in]b, \beta_x[$, while $x(t; c, kv) < 0$ and $\dot{x}(t; c, kv) \geq k$ for $t \in]\alpha_x, c[$.

The cases when $v = (-1, 0)$ or $v = (0, -1)$ may be treated in a similar manner.

Proof. At first let us observe that the existence of K_b and K_c will follow from the existence of L_b and L_c, the fact that $x(t) \equiv 0$ is a global solution and the continuous dependence on the initial data.

Secondly, we can assume without loss of generality that actually $q(t) \not\equiv 0$ in any right neighbourhood of b and in any left neighbourhood of c. In fact, if $q(t) \equiv 0$ on $[b, b']$ and on $[c', c]$, the Poincaré maps $p \mapsto z(b'; b, p)$ and $p \mapsto z(c'; c, p)$ homeomorphically transforms our half line Γ in another half line, leaving the origin fixed. This reduction implies that both b and c are accumulation points of the set $\{t \in [b, c] : q(t) < 0\}$.

Let us now consider the existence of L_b: we will show the proof with all details when Γ is given by $\{(u, -hu) : u > 0\}$ for some $h > 0$ since this turns out to be the most difficult case and the other ones can be proved with analogous (or even simpler) calculations. We set $x(t; u) := x(t; b, (u, -hu))$ to simplify the notation. Since $x(\,\cdot\,; u)$ starts positive in b, then it is convex and its graph lies over the line which is tangent in $t = b$, until it remains positive; hence we have that

$$x(t; u) \geq (-hu)(t - b) + u \quad \text{for all } t \in [b, b + 1/h]$$

and

$$x(t; u) \geq u/2 \quad \text{for all } t \in [b, b + 1/2h],$$

whenever $x(t,u)$ is defined. Our assumptions implies that there exist an interval $[b',c'] \subset]b,b+1/2h[$, and a constant $m>0$ such that

$$-q(t) \geq m \quad \text{for all } t \in [b',c'].$$

We will actually show that the right maximal interval of continuability of $x(\,\cdot\,;u)$ is contained in $[b,b']$ when u is large enough: this will obviously implies the existence of L_b.

As a first step, we are going to find that, for all u sufficiently large, $x(\,\cdot\,;u)$ has an internal minimum whose abscissa lies in $]b,b']$. Let us argue by contradiction and suppose that there are a sufficiently small number $\eta>0$ (for instance $\eta < c'-b'$) and two sequences $0<u_n\to\infty$ and $t_n \in [b'+\eta, c']$ such that $\dot{x}_n(t) \leq 0$ for every $t\in[b',t_n]$, where $x_n(t) := x(t;u_n)$. Therefore we have

$$\ddot{x}_n(t)\dot{x}_n(t) = -q(t)g(x_n(t))\dot{x}_n(t) \leq mg(x_n(t))\dot{x}_n(t) \quad \text{for all } t\in[b',t_n].$$

Integrating this inequality between $t\in[b',t_n]$ and t_n we obtain

$$\dot{x}_n^2(t) \geq 2m[G(x_n(t)) - G(x_n(t_n))] + \dot{x}_n^2(t_n) \geq 2m[G(x_n(t)) - G(x_n(t_n))]$$

on $[b',t_n]$. Recalling that $\dot{x}(t)\leq 0$ on $[b',t_n]$, we have

$$-\dot{x}_n(t) \geq \sqrt{2m}\sqrt{[G(x_n(t)) - G(x_n(t_n))]} \quad \text{on } [b',t_n]$$

and then

$$1 \leq -\frac{1}{\sqrt{2m}}\frac{\dot{x}_n(t)}{\sqrt{G(x_n(t)) - G(x_n(t_n))}} \quad \text{on } [b',t_n].$$

A second integration between b' and t_n leads to

$$\begin{aligned}\eta < t_n - b' &\leq \frac{1}{\sqrt{2m}}\int_{x_n(t_n)}^{x_n(b')}\frac{dx}{\sqrt{G(x)-G(x_n(t_n))}}\\ &\leq \frac{1}{\sqrt{2m}}\int_{x_n(t_n)}^{\infty}\frac{dx}{\sqrt{G(x)-G(x_n(t_n))}} \leq \frac{1}{2\sqrt{m}}\tau^-(-G(x_n(t_n))).\end{aligned}$$

Thanks to our choices we know that $x_n(t_n)\geq u_n/2$ by (3.4), so that $x_n(t_n)\to\infty$ as $n\to\infty$. Hence we obtain a contradiction with hypotheses (g_2^-).

Now, let $t_0 = t_0(u) \in]b,c[$, be the minimum point of $x(\,\cdot\,;u)$ for u sufficiently large. Then, for such large u, we have:

(1) $\dot{x}(t_0;u)=0$ and $x(\,\cdot\,;u)$ is a convex function, wherever defined in $[b,c]$,
(2) $t_0(u)\leq b'$,
(3) $x(t;u) \geq x(t_0;u)\to\infty$ as $u\to\infty$, by (3.4).

Now we can show that the right interval of continuability of $x(\,\cdot\,;u)$ is contained in $[b,b']$ for every sufficiently large u. Let us argue again by contradiction: let us suppose that there are a number $0<\eta<c'-b'$ and two sequences $u_n\to\infty$ and $t_n \in]b'+\eta, c'[$, such that $x_n(\,\cdot\,) := x(\,\cdot\,;u_n)$ is continuable up to t_n and $t_0(u_n)\in]b,b']$ for every n (this last condition can be fulfilled thanks to

point 2 above). Then we have that $\dot{x}_n(t) \geq 0$ and $x_n(t) \geq x_n(b') \geq x_n(t_0(u_n))$, as $t \in [b', t_n]$. Estimates, which are completely analogous to those carried on above, lead to

$$\eta < t_n - b' \leq \frac{1}{\sqrt{2m}} \int_{x_n(b')}^{\infty} \frac{dx}{\sqrt{G(x) - G(x_n(b'))}} \leq \frac{1}{2\sqrt{m}} \tau^{-}(-G(x_n(b')))$$

and this again leads to a contradiction with the assumption (g_2^-), since $x_n(b') \to \infty$ as $n \to \infty$.

A completely symmetric argument shows the existence of L_c. □

Lemma 2. *Let $[a,b] \subset [0,\omega]$ be such that $q \geq 0$ and $q \not\equiv 0$ on $[a,b]$. Assume that (g_2^+) is satisfied. Then $|z(b; a, p)| \to \infty$ and $\mathrm{rot}_{[a,b]}(p) \to \infty$ as $|p| \to \infty$. Moreover, suppose that $\Gamma_1 \subset \overline{A}$ and $\Gamma_2 \subset \overline{B}$ are two unbounded continua, where $\overline{A}$ and $\overline{B}$ are two (not necessarily distinct) closed quadrants of the plane; let also $R > 0$ be such that*

$$\Gamma_i \cap B[R] \neq \emptyset \quad \text{for } i = 1, 2.$$

Then there is $n^* = n^*_R$ such that the following holds:

- *if Γ_1 and Γ_2 are both in H^+ or both in H^-, then, for every $n \geq n^*$ and n even, there is at least one solution $z(\,\cdot\,)$ of (3.3) such that*

$$z(a) \in \Gamma_1, \quad z(b) \in \Gamma_2, \quad |z(t)| \geq R \quad \text{for all } t \in [a,b]$$

  and $x(\,\cdot\,)$ has exactly n zeros on $]a,b]$,
- *if $\Gamma_1 \subset H^+$ and $\Gamma_2 \subset H^-$ (or vice versa) then, for every $n \geq n^*$ and n odd, there is at least one solution $z(\,\cdot\,)$ of (3.3) such that*

$$z(a) \in \Gamma_1, \quad z(b) \in \Gamma_2, \quad |z(t)| \geq R \quad \text{for all } t \in [a,b]$$

  and $x(\,\cdot\,)$ has exactly n zeros on $]a,b]$.

This result is essentially contained in [45] for the case in which $g(s)/s \to \infty$ as $s \to \pm\infty$. The treatment in the situation when the weaker condition (g_2^+) holds follows from the argument developed in [21].

Lemma 3. *Assume (g_2). Let $a < b < c$, with $[a,c] \subset [0,\omega]$ be such that*

$$q \geq 0 \text{ and } q \not\equiv 0 \text{ on } [a,b], \quad q \leq 0 \text{ and } q \not\equiv 0 \text{ on } [b,c].$$

Then there is a constant R^* (depending only on g and $q|_{[b,c]}$) such that the following holds: for each $R > 0$, there is $n^* = n^*_R > 0$ such that, for each $n > n^*$, and for each path $\gamma : [\alpha, \beta[\to \overline{A_1}$ (respectively, $\gamma : [\alpha, \beta[\to \overline{A_3}$), with

$$|\gamma(\alpha)| \leq R \quad \text{and} \quad |\gamma(s)| \to \infty \text{ as } s \to \beta^-,$$

there is an interval $]\alpha_n, \beta_n] \subset]\alpha, \beta[$, such that for each $s \in]\alpha_n, \beta_n]$ we have:

- *$z(t; a, \gamma(s))$ is defined for all $t \in [a,c]$,*

- *$x(\,\cdot\,;a,\gamma(s))$ has exactly n zeros in $]a,b[$, no zeros in $[b,c]$ and exactly one change of sign of the derivative in $]b,c[$.*

Moreover, setting $\gamma_n(s) := z(c;a,\gamma(s))$ for all $s \in\,]\alpha_n,\beta_n]$, we have that

$$|\gamma_n(\beta_n)| \le R^* \quad \textit{and} \quad |\gamma_n(s)| \to \infty \quad \textit{as } s \to \alpha_n^+$$

and γ_n lies in $\overline{A_1}$ or in $\overline{A_3}$ according to the fact that n is even or odd (respectively, γ_n lies in $\overline{A_3}$ or in $\overline{A_1}$ according to the fact that n is even or odd).

Proof. Applying Lemma 1 on $[b,c]$ with $v = (0,\pm 1)$ and $v = (\pm 1, 0)$, we find that there is $R^* > 0$ such that for every $p \in (\{0\}\times\mathbb{R}) \cup (\mathbb{R}\times\{0\})$, with $|p| \ge R^*$, the solutions $z(\,\cdot\,;b,p)$ and $z(\,\cdot\,;c,p)$ are not continuable on the whole $[b,c]$. Moreover, let

$$R' := \max\{|\xi| : |z(b;a,\xi)| \le R^*\},$$

which is finite since $q \ge 0$ on $[a,b]$ and then we have the global continuability of initial value problems for our equation. For every $R > 0$ we set

$$n^* = n_R^* := \left\lceil \frac{\sup\{\mathrm{rot}_{[a,b]}(p) : 0 < |p| \le \max\{R,R'\}\}}{\pi} \right\rceil.$$

Note that $g(s)/s$ is bounded in a neighbourhood of zero and hence n_R^* is well defined.

Let us consider a curve $\gamma : [\alpha,\beta[\, \to \overline{A_1}$ such that

$$|\gamma(\alpha)| \le R \quad \text{and} \quad \lim_{s\to\beta^-} |\gamma(s)| = \infty.$$

Without loss of generality we may assume also that $\gamma(s) \ne (0,0)$ for every $s \in [\alpha,\beta[$. By Lemma 2 we can find two continuous functions $\rho : [\alpha,\beta[\, \to \mathbb{R}_0^+$ and $\theta : [\alpha,\beta[\, \to \mathbb{R}$ such that

(1) $z(b;a,\gamma(s)) = (\rho(s)\cos\theta(s), \rho(s)\sin\theta(s))$ for all $s \in [\alpha,\beta[$,
(2) $\lim_{s\to\beta^-} \rho(s) = \infty$ and $\lim_{s\to\beta^-} \theta(s) = -\infty$,
(3) $\theta(s) + \mathrm{rot}_{[a,b]}(\gamma(s)) \in [0,\pi/2]$ for all $s \in [\alpha,\beta[$.

(Condition (3) can be achieved since θ is uniquely determined up to multiples of 2π.) Then, for every $n > n^*$, let $\alpha',\beta' \in [\alpha,\beta[$, be such that $\theta(\alpha') = -(n-1/2)\pi$, $\theta(\beta') = -(n+1/2)\pi$ and $-(n+1/2)\pi < \theta(s) < -(n-1/2)\pi$ for every $s \in\,]\alpha',\beta'[$. We have that $x(\,\cdot\,;a,\gamma(s))$ has exactly n zeros in $]a,b[$, for all $s \in\,]\alpha',\beta']$.

We suppose for definiteness that n is even: then $z(b;a,\gamma(s)) \in \mathbb{R}_0^+ \times \mathbb{R}$ as s ranges in $]\alpha',\beta'[$; the other case can be treated in a completely symmetric way. We remark that $z(\,\cdot\,;a,\gamma(\alpha'))$, $z(\,\cdot\,;a,\gamma(\beta'))$ are not continuable on the whole $[a,c]$ since $z(b;a,\gamma(\alpha')), z(b;a,\gamma(\beta')) \in \{0\}\times\mathbb{R}$, $|z(b;a,\gamma(\alpha'))| \ge R^*$ and

$|z(b;a,\gamma(\beta'))| \geq R^*$, by the definition of n^*; therefore, it can be deduced, by a simple analysis of equation (3.1), that there are $t_1, t_2 \in \,]b,c]$ such that

$$\lim_{t\to t_1^-} x(t;a,\gamma(\alpha')) = \lim_{t\to t_1^-} \dot{x}(t;a,\gamma(\alpha')) = \infty$$

and

$$\lim_{t\to t_2^-} x(t;a,\gamma(\beta')) = \lim_{t\to t_2^-} \dot{x}(t;a,\gamma(\beta')) = -\infty.$$

Let

$$\beta'' := \inf\{s \in \,]\alpha',\beta'[\, : x(t;a,\gamma(s)) < 0 \text{ for some } t \in \,]b,c]\}.$$

In the definition of β'' we do not care if $x(\,\cdot\,;a,\gamma(s))$ is not continuable up to c: we simply ask that it is negative somewhere in the part of its interval of continuability which lies to the right of b. Of course we have $\alpha' < \beta'' < \beta'$.

Now we show that $x(\,\cdot\,;a,\gamma(\beta''))$ is actually continuable up to c and is nonnegative and decreasing on $[b,c]$, with $x(c;a,\gamma(\beta'')) = 0$ and $\dot{x}(c;a,\gamma(\beta'')) < 0$. From the continuous dependence on the initial data we deduce that $x(t;a,\gamma(\beta'')) \geq 0$ for every $t \in [b,c]$, wherever it is defined. Moreover, if $x(t;a,\gamma(s)) < 0$ for some $t \in \,]b,c]$ and $s \in \,]\alpha',\beta'[$, then $x(\,\cdot\,;a,\gamma(s))$ has a zero in $]b,c[$, since $x(b;a,\gamma(s)) > 0$ for every $s \in \,]\alpha',\beta'[$. On the other hand, every solution of (3.1) on $[b,c]$ is convex, wherever it is positive, and concave elsewhere; this implies that every solution x of (3.1) may have at most one zero and, if this is the case, x is strictly monotone decreasing. Thus $x(\,\cdot\,;a,\gamma(\beta''))$ is decreasing since it is limit of decreasing functions by the continuous dependence on the initial data. Together with the nonnegativity, the decreasing monotonicity implies that $x(\,\cdot\,;a,\gamma(\beta''))$ is continuable on $[a,c]$. Finally, if $x(c;a,\gamma(\beta''))$ were positive, we should have, again by the continuous dependence on the initial data, that $x(t;a,\gamma(s)) > 0$ for every $t \in [b,c]$ and every s in a neighbourhood of β'', and the definition of β'' should be violated. Moreover, we obtain also that $\dot{x}(c;a,\gamma(\beta'')) < 0$ by the uniqueness of the trivial solution. In particular we can deduce that all the solutions $x(\,\cdot\,;a,\gamma(s))$ are positive (and then convex) on $[b,c\,[$ (where defined) for all $s \in [\alpha',\beta'']$.

Now let

$$\alpha_n := \inf\{\sigma \in \,]\alpha',\beta''] : z(\,\cdot\,;a,\gamma(s)) \text{ is continuable on } [a,c], \text{ for all } s \in [\sigma,\beta'']\},$$

thus $\alpha' \leq \alpha_n < \beta''$,

$$\lim_{s\to\alpha_n^+} x(c;a,\gamma(s)) = \lim_{s\to\alpha_n^+} \dot{x}(c;a,\gamma(s)) = \infty$$

and $x(\,\cdot\,;a,\gamma(s))$ is continuable on $[a,c]$ for all $s \in \,]\alpha_n,\beta'']$. Since we have $\dot{x}(c;a,\gamma(\beta'')) < 0$, there is $\beta_n \in \,]\alpha_n,\beta''[$, such that $\dot{x}(c;a,\gamma(\beta_n)) = 0$ and $\dot{x}(c;a,\gamma(s)) > 0$ for all $s \in \,]\alpha_n,\beta_n[$. As a consequence of this construction,

$\dot{x}(\,\cdot\,; a, \gamma(s))$ must change sign exactly once in $]b, c[$ and, by the convexity, the zero set of $\dot{x}(\,\cdot\,)$ is either a point or a closed subinterval of $]b, c[$.

If we set $\gamma_n(s) := z(c; a, \gamma(s))$ as s ranges in $]\alpha_n, \beta_n]$, then $|\gamma_n(s)| \to \infty$ as $s \to \alpha_n^+$ and $|\gamma_n(\beta_n)| < R^*$, since, by construction, $\gamma_n(\beta_n) \in \mathbb{R} \times \{0\}$ and $z(\,\cdot\,; c, \gamma_n(\beta_n))$ is continuable on $[b, c]$.

An analogous argument can be used in the case $\gamma : [\alpha, \beta[\to \overline{A_3}$. $\square$

It is clear that $\dot{x}$ has exactly one zero in $]b, c[$, if $q(t)$ does not vanish identically on any subinterval of $]b, c[$.

A completely symmetric result is the following.

Lemma 4. *Assume* (g$_2$). *Let $a < b < c$, with $[a, c] \subset [0, \omega]$ be such that*

$$q \geq 0 \text{ and } q \not\equiv 0 \text{ on } [a, b], \quad q \leq 0 \text{ and } q \not\equiv 0 \text{ on } [b, c].$$

Then there is a constant R^* (depending only on g and $q|_{[b,c]}$) such that the following holds: for each $R > 0$, there is $n^* = n^*_R > 0$ such that, for each $n > n^*$, and for each path $\gamma : [\alpha, \beta[\to \overline{A_1}$ (respectively, $\gamma : [\alpha, \beta[\to \overline{A_3}$), with

$$|\gamma(\alpha)| \leq R \quad \text{and} \quad |\gamma(s)| \to \infty \text{ as } s \to \beta^-,$$

there is an interval $[\alpha_n, \beta_n[\subset]\alpha, \beta[$, such that for each $s \in [\alpha_n, \beta_n[$, we have:

- *$z(t; a, \gamma(s))$ is defined for all $t \in [a, c]$,*
- *$x(\,\cdot\,; a, \gamma(s))$ has exactly n zeros in $]a, b[$, exactly one zero in $]b, c[$ and no zeros of the derivative in $[b, c]$.*

Moreover, setting $\gamma_n(s) := z(c; a, \gamma(s))$ for all $s \in [\alpha_n, \beta_n[$, we have that

$$|\gamma_n(\alpha_n)| \leq R^* \quad \text{and} \quad |\gamma_n(s)| \to \infty \quad \text{as } s \to \beta_n^-$$

and γ_n lies in $\overline{A_3}$ or in $\overline{A_1}$ according to the fact that n is even or odd (respectively, γ_n lies in $\overline{A_1}$ or in $\overline{A_3}$ according to the fact that n is even or odd).

Proof. We describe only the main changes to be performed on the proof of Lemma 3 in order to obtain Lemma 4. The first part of the proof can be left unchanged until it is remarked that there are $t_1, t_2 \in]b, c]$ such that

$$\lim_{t \to t_1^-} x(t; a, \gamma(\alpha')) = \lim_{t \to t_1^-} \dot{x}(t; a, \gamma(\alpha')) = \infty$$

and

$$\lim_{t \to t_2^-} x(t; a, \gamma(\beta')) = \lim_{t \to t_2^-} \dot{x}(t; a, \gamma(\beta')) = -\infty.$$

At this point we set

$$\alpha_n := \sup\{s \in]\alpha', \beta'[: x(t; a, \gamma(s)) > 0 \text{ for all } t \in]b, c] \text{ where it is defined}\},$$

hence $\alpha' < \alpha_n < \beta'$.

Now it is possible to show by the same kind of argument that $x(\,\cdot\,;a,\gamma(\alpha_n))$ is actually continuable up to c and on $[b,c]$ is nonnegative and decreasing, with $x(c;a,\gamma(\alpha_n))=0$ and $\dot x(c;a,\gamma(\alpha_n))<0$. In particular we deduce that all the solutions $x(\,\cdot\,;a,\gamma(s))$ are strictly decreasing (hence their derivatives have no zeroes) and with exactly one zeroes on $]b,c]$ for all $s\in[\alpha_n,\beta']$. Now let

$$\beta_n := \sup\{\sigma\in\,]\alpha_n,\beta'] : z(\,\cdot\,;a,\gamma(s)) \text{ is continuable on } [a,c] \text{ for all } s\in[\alpha_n,\sigma]\},$$

thus $\alpha_n<\beta_n\le\beta'$,

$$\lim_{s\to\beta_n^-} x(c;a,\gamma(s)) = \lim_{s\to\beta_n^-} \dot x(c;a,\gamma(s)) = -\infty$$

and $x(\,\cdot\,;a,\gamma(s))$ is continuable on $[a,c]$ for all $s\in[\alpha_n,\beta_n[$.

If we set $\gamma_n(s):=z(c;a,\gamma(s))$ as s ranges in $[\alpha_n,\beta_n[$, then $|\gamma_n(s)|\to\infty$ as $s\to\beta_n^-$ and $|\gamma_n(\alpha_n)|<R^*$, since, by construction, $\gamma_n(\alpha_n)\in\{0\}\times\mathbb{R}$ and $z(\,\cdot\,;c,\gamma_n(\alpha_n))$ is continuable on $[b,c]$.

An analogous argument can be used in the case $\gamma:[\alpha,\beta[\,\to\overline{A_3}$. □

4. Main result

After these preliminary lemmas we are now in position to prove our result for the two-point boundary value problem.

Theorem 1. *Assume* (q_1), (g_1) *and* (g_2). *Let $k\ge 0$, be an integer. Suppose that there are $2k+1$ consecutive adjacent nondegenerate closed intervals*

$$I_1^+, I_1^-, \ldots, I_k^+, I_k^-, I_{k+1}^+,$$

such that $q\ge 0$, $q\not\equiv 0$ on I_i^+ and $q\le 0$, $q\not\equiv 0$ on I_i^-. We assume also that $q\le 0$, $q\not\equiv 0$ on

$$J := [0,\omega]\setminus\bigg(\bigg(\bigcup_{i=1}^{k+1} I_i^+\bigg)\cup\bigg(\bigcup_{i=1}^{k} I_i^-\bigg)\bigg),$$

if $J\ne\emptyset$. Then, there are $k+1$ positive integers $n_1^*,\ldots,n_{k+1}^*$ such that for each $(k+1)$-uple $\mathbf{n}:=(n_1,\ldots,n_{k+1})$, with $n_i>n_i^*$, and each k-uple $\boldsymbol{\delta}:=(\delta_1,\ldots,\delta_k)$, with $\delta_i\in\{0,1\}$, there are at least two solutions $x^+_{\mathbf{n},\boldsymbol{\delta}}(\cdot)$ and $x^-_{\mathbf{n},\boldsymbol{\delta}}(\cdot)$ of (3.1)–(3.2)*, such that:*

1. $\dot x^-_{\mathbf{n},\boldsymbol{\delta}}(0)<0<\dot x^+_{\mathbf{n},\boldsymbol{\delta}}(0)$,
2. $x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\cdot)$ *has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1-\delta_i$ changes of sign of the derivative in I_i^-,*
3. *neither $x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\cdot)$ nor $\dot x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\cdot)$, may vanish in $J\setminus\{0,\omega\}$,*
4. *for each i, $|x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)|+|\dot x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)|\to\infty$, as $n_i\to\infty$, uniformly in $t\in I_i^+$.*

Note that $\dot{x}^{\pm}_{\mathbf{n},\boldsymbol{\delta}}(\omega) \times \dot{x}^{\pm}_{\mathbf{n},\boldsymbol{\delta}}(0)$ is positive or negative according to the fact that the total number of zeros is even or odd. This clearly allows to determine the sign of the slopes of the solutions at $t = \omega$.

Proof. First of all, just to fix the starting point, let us assume that

$$J = \emptyset, \tag{4.1}$$

so that $q \geq 0$ and $q \not\equiv 0$ in a right neighbourhood of 0 and in a left neighbourhood of ω. At the end of the proof, we briefly describe how to deal with the other cases which occur when (4.1) is not satisfied. We also suppose that $k \geq 2$. The situations when $k = 0$ or $k = 1$ are just sub-cases of our proof.

According to the above positions, we can split $[0, \omega]$ as

$$[0,\omega] = [a_1, b_1] \cup [b_1, c_1] \cup [a_2, b_2] \cup [b_2, c_2] \cup \ldots \cup [a_k, b_k] \cup [b_k, c_k] \cup [a_{k+1}, b_{k+1}],$$

with $a_1 = 0$, $b_{k+1} = \omega$ and $I_i^+ = [a_i, b_i]$, $I_i^- = [b_i, c_i]$.

Step 1. Suppose that $\delta_1 = 0$ in $\boldsymbol{\delta}$. In this case we use Lemma 3 which ensures the existence of a constant $R_1^* > 0$ and a positive integer n_1^*, such that the following holds with respect to the curve

$$\gamma : [0, \infty[\; \to \overline{A_1} \quad \text{with } \gamma(s) = (0, s)$$

(here, the choice of R is arbitrary). Fix any $n_1 > n_1^*$. Then, there is an interval $\left]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}\right] \subset \,]0, \infty[$, such that for each $s \in \left]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}\right]$, we have:

- $z(t; a_1, \gamma(s))$ is defined for all $t \in [a_1, c_1]$,
- $x(\,\cdot\,; a_1, \gamma(s))$ has exactly n_1 zeros in $]a_1, b_1[$, no zeros in $[b_1, c_1]$ and exactly one change of sign of the derivative in $]b_1, c_1[$.

Moreover, setting

$$\gamma_{(n_1,\delta_1)}(s) := z(c_1; a_1, \gamma(s)) \quad \text{for all } s \in \left]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}\right],$$

we have that

$$|\gamma_{(n_1,\delta_1)}(\beta_{(n_1,\delta_1)})| \leq R_1^*, \quad \text{and} \quad |\gamma_{(n_1,\delta_1)}(s)| \to \infty, \text{ as } s \to \alpha^+_{(n_1,\delta_1)}$$

and $\gamma_{(n_1,\delta_1)}$ lies in $\overline{A_1}$ or in $\overline{A_3}$ according to the fact that n_1 is even or odd.

On the other hand, if $\delta_1 = 1$ in $\boldsymbol{\delta}$, we use Lemma 4 and obtain a completely similar result, but with

- $x(\,\cdot\,; a_1, \gamma(s))$ has exactly n_1 zeros in $]a_1, b_1[$, exactly one zero in $]b_1, c_1[$, and no zeros of the derivative in $[b_1, c_1]$

and $\gamma_{(n_1,\delta_1)}$ lies in $\overline{A_3}$ or in $\overline{A_1}$ according to the fact that n_1 is even or odd.

Step 2. We repeat now inductively this kind of argument. Without loss of generality, we suppose that γ_{n_1} lies in $\overline{A_1}$. The proof is completely symmetric if γ_{n_1} lies in $\overline{A_3}$.

Suppose again that $\delta_2 = 0$ in $\boldsymbol{\delta}$. We use again Lemma 3 which ensures the existence of a constant $R_2^* > 0$ and a positive integer n_2^*, depending on R_1^*, but not depending on n_1, such that the following holds with respect to the curve $\gamma_{n_1} :]\alpha_{n_1}, \beta_{n_1}] \to \overline{A_1}$.

Fix an arbitrary $n_2 > n_2^*$. Then, there is an interval

$$\big[\alpha_{((n_1,n_2),(\delta_1,\delta_2))}, \beta_{((n_1,n_2),(\delta_1,\delta_2))}\big[\subset \big]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}\big],$$

such that for each $s \in \big[\alpha_{((n_1,n_2),(\delta_1,\delta_2))}, \beta_{((n_1,n_2),(\delta_1,\delta_2))}\big[$ we have:

- $z(t; a_1, \gamma(s))$ is defined for all $t \in [a_1, c_2]$,
- $x(\,\cdot\,; a_1, \gamma(s))$ has exactly n_2 zeros in $]a_2, b_2[$, no zeros in $[b_2, c_2]$ and exactly one change of sign of the derivative in $]b_2, c_2[$.

Moreover, setting

$$\gamma_{((n_1,n_2),(\delta_1,\delta_2))}(s) := z(c_2; a_1, \gamma(s))$$

for all $s \in \big[\alpha_{((n_1,n_2),(\delta_1,\delta_2))}, \beta_{((n_1,n_2),(\delta_1,\delta_2))}\big[$, we have that

$$|\gamma_{((n_1,n_2),(\delta_1,\delta_2))}(\alpha_{((n_1,n_2),(\delta_1,\delta_2))})| \le R_2^*,$$

and

$$|\gamma_{((n_1,n_2),(\delta_1,\delta_2))}(s)| \to \infty \quad \text{as } s \to \beta^-_{((n_1,n_2),(\delta_1,\delta_2))}$$

and $\gamma_{((n_1,n_2),(\delta_1,\delta_2))}$ lies in $\overline{A_1}$ or in $\overline{A_3}$ according to the fact that n_2 is even or odd.

On the other hand, if $\delta_2 = 1$ in $\boldsymbol{\delta}$, we use Lemma 4 and obtain a completely similar result, but with

- $x(\,\cdot\,; a_1, \gamma(s))$ has exactly n_2 zeros in $]a_2, b_2[$, exactly one zero in $]b_2, c_2[$, and no zeros of the derivative in $[b_2, c_2]$

and $\gamma_{((n_1,n_2),(\delta_1,\delta_2))}$ lies in $\overline{A_3}$ or in $\overline{A_1}$ according to the fact that n_2 is even or odd.

As remarked above, it is clear that the same argument works if $\gamma_{(n_1,\delta_1)}$ lies in $\overline{A_3}$. Actually, we could now summarize all the different possibilities as follows: $\gamma_{((n_1,n_2),(\delta_1,\delta_2))}$ lies in $\overline{A_1}$ or in $\overline{A_3}$ according to the fact that $n_1 + n_2 + \delta_1 + \delta_2$, that is, the total number of zeros in $]a_1, c_2]$ is even or odd.

Step 3. Repeating this argument k-times, we find an unbounded continuum Γ^* which is the image of a continuous curve

$$h := \gamma_{((n_1,\dots,n_k),\boldsymbol{\delta})}$$

defined on a suitable half-open bounded interval $I \subset \big]\alpha_{(n_1,\delta_1)}, \beta_{(n_1,\delta_1)}\big]$ such that $\Gamma^* \cap B[R_k^*] \ne \emptyset$ and

$$\Gamma^* \subset \overline{A_1} \quad \text{or} \quad \Gamma^* \subset \overline{A_3}$$

according to the fact that $\sum_{i=1}^k (n_i + \delta_i)$ is even or odd.

We have that for each $s \in I$ the solution $x(\,\cdot\,; a_1, \gamma(s))$ is defined on $[a_1, c_k]$ and, for each $i = 1, \dots, k$, it possesses exactly n_i zeros on $]a_i, b_i[$, exactly δ_i zeros in $[b_i, c_i]$ and exactly $1 - \delta_i$ changes of sign of the derivative in $[b_i, c_i]$. By construction, we notice that the curve $h : I \to \Gamma^*$ is a homeomorphism and the numbers n_i (for $i = 1, \dots, k$) may be chosen independently one from each other, depending only on the choice of n_i^* which, in turn, depend on R_{i-1}^* and R_i^*.

Step 4. At this point, we can use Lemma 2 with $\Gamma_1 = \Gamma^*$, while Γ_2 will be the positive or the negative $\dot{x}$-axis in the phase-plane, according that we look for an even or an odd number of zeros in $]a_{k+1}, b_{k+1}]$. Lemma 2 guarantees that there is n_{k+1}^* depending only on R_k^* and the behaviour of g and $q|_{[a_{k+1}, b_{k+1}]}$ such that for each $n_{k+1} > n_{k+1}^*$ there is at least one point $p \in \Gamma^*$ such that the solution $z(\,\cdot\,; a_{k+1}, p)$ of (3.3) is defined on $[a_{k+1}, b_{k+1}]$, satisfies

$$x(b_{k+1}; a_{k+1}, p) = 0$$

and $x(\,\cdot\,; a_{k+1}, p)$ has exactly n_{k+1} zeros in $]a_{k+1}, b_{k+1}[$. Moreover, we have that $z(b_{k+1}; a_{k+1}, p)$ and $p \in \Gamma^*$ belong to the same quadrant or to opposite quadrants, according to the fact that n_{k+1} is even or is odd. More precisely, this means that if $\Gamma^* \subset \overline{A_1}$ (respectively, if $\Gamma^* \subset \overline{A_3}$) then $\dot{x}(b_{k+1}; a_{k+1}, p) > 0$ if and only if n_{k+1} is odd (respectively, if and only if n_{k+1} is even).

In conclusion, we can take

$$s^* = h^{-1}(p)$$

and have that the solution $x(\,\cdot\,; a_1, (0, s^*))$ is defined on $[a_1, b_{k+1}] = [0, \omega]$ it has precisely n_i zeros in the interior of the I_i^+ intervals, for each $i = 1, \dots, k+1$ and also it has exactly δ_i zeros and $1 - \delta_i$ changes of sign of the derivative in the interior of the I_i^- intervals. By definition, $\dot{x}(0; a_1, (0, s^*)) = s^* > 0$, so that we see that the solution starts with positive slope. Clearly, a completely symmetric argument yields a second solution with the same nodal properties and starting with negative slope. Thus, the main part of the proof is concluded.

Step 5. We briefly discuss now how to proceed in the case that $k = 0$ or $k = 1$.

In the former situation we have that $q \geq 0$ and $q \not\equiv 0$, for each $t \in [0, \omega]$. This is a slight variant of the "classical" case in which $q(\,\cdot\,)$ does not change sign (see, e.g. [45]). To prove the result, (which is essentially already contained in Lemma 2) it will be sufficient to argue like in Step 4 and obtain a solution having a "large" number of zeros and connecting the positive (respectively, the negative) $\dot{x}$-axis to the $\dot{x}$-axis.

If $k = 1$, it will be sufficient to jump directly from the Step 1 to the Step 4, avoiding the intermediate steps.

Step 6. It remains to discuss the case in which $J \neq \emptyset$.

In this situation, let us consider at first the possibility that $J = [0, a_1[$. If this happens, using Lemma 1 with $v = (0,1)$, we find an interval $[0, K[$ (for $K = K_0$) and a continuous one-to-one map $\sigma : [0, K[\to \overline{A_1}$ such that $\sigma(0) = (0,0)$ and $|\sigma(s)| \to \infty$ as $s \to K^-$. Such a map is defined by $\sigma(s) = z(a_1; 0, (0, s))$. By definition, $\sigma(s)$ lies in the interior of the first quadrant for $s > 0$. Now, we can repeat exactly the same argument in Step 1, replacing γ with σ. At the end of Step 4, we'll find a point

$$s^* = h^{-1}(p) \in]0, K[$$

and have that the solution $x(\,\cdot\,; 0, (0, s^*))$ is defined on $[0, b_{k+1}] = [0, \omega]$ it has precisely n_i zeros in the interior of the I_i^+ intervals, for each $i = 1, \ldots, k+1$ and also it has exactly δ_i zeros and $1 - \delta_i$ changes of sign of the derivative in the interior of the I_i^- intervals and there are no zeros of x or $\dot{x}$ in $]0, a_1]$.

If $J \supset]b_{k+1}, \omega]$, we proceed without changes from Steps 1 to 3 (taking also into account the first part of Step 6, if J contains also points in a right neighbourhood of 0). Then, we use Lemma 1 and find two intervals $[0, K'[$ and $]-K'', 0]$ (K' and K'' are given by K_b when we apply the lemma to $[b, c] = [b_{k+1}, \omega]$ and with $v = (0,1)$ and $v = (0,-1)$, respectively) such that their image under the map $(0, s) \mapsto z(b_{k+1}; \omega, (0, s))$ is the union of two unbounded paths S^+ and S^- passing through $(0,0)$ and contained in the second and in the fourth quadrant, respectively. At this moment, we can repeat the same argument of Step 4, but replacing the $\dot{x}$-axis, with $S^+ \cup S^-$. Hence, we find a point $s^* = h^{-1}(p)$ and have that the solution $x(\,\cdot\,; 0, (0, s^*))$ is defined on $[0, \omega]$ it has precisely n_i zeros in the interior of the I_i^+ intervals, for each $i = 1, \ldots, k+1$ and also it has exactly δ_i zeros and $1 - \delta_i$ changes of sign of the derivative in the interior of the I_i^- intervals and there are no zeros of x or $\dot{x}$ in $[b_{k+1}, \omega[$. In this manner, we have also checked the claim about the absence of zeros for x and $\dot{x}$ in $J \setminus \{0, \omega\}$ and therefore the proof is complete. □

5. Asymptotic solutions

In this section, we discuss the case when the function q is defined on the whole real line and we look for conditions ensuring that there are solutions decaying at infinity after an arbitrarily large number of oscillations. Problems of this kind raised the interest of several authors in the recent years. Accordingly, from now on, we assume that $q, g : \mathbb{R} \to \mathbb{R}$ are continuous functions satisfying (q_1) and (g_1). It is also convenient to set

$$Q(t) := \int_0^t q(s)\, ds.$$

Lemma 5. *Let us assume there is $b \in \mathbb{R}$ such that $Q(\,\cdot\,)$ is strictly decreasing on $[b,\infty[$, with $Q(\infty) = -\infty$. Then there are two unbounded continua $\Gamma^+ \subset \overline{A_4}$ and $\Gamma^- \subset \overline{A_2}$, such that*

(i) $(0,0) \in \Gamma^+ \cap \Gamma^-$,

(ii) *for all $p \in \Gamma^+ \cup \Gamma^-$, $x(\,\cdot\,; b, p)$ is continuable on $[b,\infty[$ and*

$$\lim_{t\to\infty} x(t;b,p) = \lim_{t\to\infty} \dot{x}(t;b,p) = 0,$$

(iii) *$x(t;b,p) > 0$ for every $t \in [b,\infty[$ and $p \in \Gamma^+$, while $x(t;b,p) < 0$ for every $t \in [b,\infty[$ and $p \in \Gamma^-$.*

We will use the following lemma, which is proved in [43].

Lemma 6. *Let $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1, -1 < x < 1\}$, $P_0 = (0,-1)$, $P_1 = (0,1)$, $Q_0 = (-1,0)$ and $Q_1 = (1,0)$ and assume that there is a set $C \subset D^\circ = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ such that $\overline{C} \cap D = C$ and every continuous curve in D from P_0 to P_1 meets C. Then there is a connected set $\Gamma \subseteq C$ such that $\overline{\Gamma} \cap D = \Gamma$ and $Q_0, Q_1 \in \overline{\Gamma}$.*

Proof of Lemma 5. We will show only the existence of Γ^+, since Γ^- can be found in a completely symmetric way.

We claim at first that if $\gamma : [0,1] \to \overline{A_4}$ is any continuous curve with $\gamma(0) \in \mathbb{R}^+_0 \times \{0\}$ and $\gamma(1) \in \{0\} \times \mathbb{R}^-_0$ then there is $\overline{s} \in]0,1[$, such that

(a) $x(\,\cdot\,; b, \gamma(\overline{s}))$ is continuable on $[b,\infty[$,

(b) $x(t; b, \gamma(\overline{s})) > 0$ for all $t \in [b,\infty[$,

(c) $\lim_{t\to\infty} x(t; b, \gamma(\overline{s})) = \lim_{t\to\infty} \dot{x}(t; b, \gamma(\overline{s})) =0$.

To this aim let $W := \{(x,y,t) : x \ge 0,\ y \le 0 \text{ and } t \ge b\}$ as subset of the extended phase space and let us remark that every solution x, whose trajectory $t \mapsto (x(t), \dot{x}(t), t)$ remains in W for $t \ge b$ (as long as it is defined), must satisfy (a)–(c). In fact, such an x must be nonnegative and decreasing and, therefore, it satisfies (a) and (b) and

$$\lim_{t\to\infty} x(t) = \ell \ge 0 \quad \text{and} \quad \lim_{t\to\infty} \dot{x}(t) = 0.$$

If $\ell > 0$, there is a constant $m > 0$ such that $g(x(t)) \ge m$ for all $t \in [b,\infty[$. Then we can evaluate

$$-\dot{x}(b) = \int_b^\infty \ddot{x}(t)\,dt = \int_b^\infty [-q(t)]g(x(t))\,dt \ge m \int_b^\infty [-q(t)]\,dt,$$

that is q is integrable on $[b,\infty[$, which is a contradiction. Hence x satisfies condition (c), too, and the problem is now finding $\overline{s} \in]0,1[$, such that the trajectory of $x(\,\cdot\,; b, \gamma(\overline{s}))$ remains in W.

Consider now the semiflow $\pi : (w,s) \mapsto w \cdot s$ induced on the phase space $X := \{(x,y,t) : t \ge b\}$ by the system $\dot{x} = y$, $\dot{y} = -q(t)g(x)$, $\dot{t} = 1$ and define the sets $U := \{(x,0,t) : x > 0 \text{ and } t \geq b\}$ and $V := \{(0,y,t) : y < 0 \text{ and } t \geq b\}$ which are contained in the boundary ∂W of W, relatively to X.

If $w_0 = (x_0, 0, t_0) \in U$, we have that $w_0 \cdot s \notin W$ for $0 < s \leq \varepsilon$. In fact, if $\varepsilon > 0$ is small enough, we have that $g(x(t_0+s)) \geq \eta > 0$ for $0 < s \leq \varepsilon$ (for a suitable η). Hence, $y(t_0+s) - y(t_0) = -\int_0^s q(t_0+\xi) g(x(t_0+\xi)\, d\xi \geq \eta(Q(t_0) - Q(t_0+s)) > 0$, as $Q(\,\cdot\,)$ is strictly decreasing. Similarly, one can check that $w_0 \cdot s \notin W$ for $0 < s \leq \varepsilon$, when $w_0 \in V$. On the other hand, if $w_0 \in \partial W \setminus (U \cup V)$, then $w_0 \cdot s \in W$ for all s, since $x(t_0+s) = y(t_0+s) = 0$ for all $s \geq b - t_0$ and therefore, for each point w_0 of W, either $w_0 \cdot s \in W$, for all $s \geq 0$, or there is a first s, such that $w_0 \cdot s \in U \cup V$. In this manner, we have proved that $U \cup V$ is the set of exit points for W.

The Ważewski lemma (see [20] for a similar argument) now implies that the map ϕ, which sends a point $(x_0, y_0, t_0) \in W$ to the first exit point from W of the positive semitrajectory starting from (x_0, y_0, t_0), is continuous whenever defined.

We come back now to our curve γ, recalling that $\gamma(0) = (x_0, 0)$, with $x_0 > 0$ and $\gamma(1) = (0, y_0)$, with $y_0 < 0$, and we evaluate ϕ along $(\gamma(s), b)$. Clearly, $\phi(\gamma(0), b) = (\gamma(0), b) \in U$ and $\phi(\gamma(1), b) = (\gamma(1), b) \in V$. Assume, by contradiction, that we cannot find our $\overline{s} \in \,]0,1[$: then $(\gamma(s), b)$ belongs to the domain of ϕ for every $s \in [0,1]$ and therefore $\phi(\gamma(s), b) \in U \cup V$, for all $s \in [0,1]$. Observe also that $\{\phi(\gamma(s), b) : s \in [0,1]\}$ is a connected set, since ϕ and γ are continuous. But this is not possible since such connected set should be contained in $U \cup V$ and intersect both U and V which are disjoint and open relatively to ∂W. Thus, our claim is proved.

Let now $\Omega^+ := \{p \in \overline{A_4} : x(\,\cdot\,; b, p) \text{ satisfies (a)–(c)}\}$.

We wish to find a connected and unbounded component of Ω^+, which contains the origin, by Lemma 6. To this aim let us define

$$\psi(x,y) := \left(2\,\frac{x+y}{1+(x-y)^2},\, 1 - \frac{2}{1+(x-y)^2} \right).$$

It is possible to check that ψ is a homeomorphism of $\overline{A_4}$ onto $\overline{D} \setminus \{Q_1\}$, such that

(1) $\psi((0,0)) = Q_0$,
(2) $\psi(\mathbb{R}_0^+ \times \{0\}) = \{(x,y) : x^2 + y^2 = 1 \text{ and } x > 0\}$ and $\psi(\{0\} \times \mathbb{R}_0^-) = \{(x,y) : x^2 + y^2 = 1 \text{ and } x < 0\}$,
(3) if Z is any neighbourhood of Q_1 then $\psi^{-1}(Z \cap D)$ is unbounded.

What we have proved above ensures that the set $C := \psi(\Omega^+)$ satisfies the hypotheses of Lemma 6 and, hence, there is a connected $\Gamma \subseteq C$ such that $Q_0, Q_1 \in \overline{\Gamma}$. Our search is concluded letting $\Gamma^+ := \psi^{-1}(\Gamma)$; in fact, with our choices, Γ^+ is connected, contains the origin by the connectedness and the fact that $(0,0) \in \overline{\Gamma} \cap C$ and it is unbounded by property 3 of ψ. $\square$

A dual result is the following.

Lemma 7. *Let us assume there is $a \in \mathbb{R}$ such that $Q(\,\cdot\,)$ is strictly decreasing on $]-\infty, a]$, with $Q(-\infty) = \infty$. Then there are two unbounded continua $\Gamma^+ \subset \overline{A_1}$ and $\Gamma^- \subset \overline{A_3}$, such that*

(i) $(0,0) \in \Gamma^+ \cap \Gamma^-$,
(ii) *for all $p \in \Gamma^+ \cup \Gamma^-$, $x(\,\cdot\,; a, p)$ is continuable on $]-\infty, a]$ and*

$$\lim_{t\to-\infty} x(t; b, p) = \lim_{t\to-\infty} \dot{x}(t; b, p) = 0,$$

(iii) *$x(t; b, p) > 0$ for every $t \in \,]-\infty, a]$, and $p \in \Gamma^+$, while $x(t; b, p) < 0$ for every $t \in \,]-\infty, a]$, and $p \in \Gamma^-$.*

At this moment, we can obtain a result of existence of solutions decaying to zero at infinity and having sharp nodal properties in the intervals of positivity and negativity of $q(t)$.

Theorem 2. *Assume* (q_1), (g_1) *and* (g_2). *Let us suppose that there are $a, b \in \mathbb{R}$ with $a < b$ such that $Q(\,\cdot\,)$ is strictly decreasing on $]-\infty, a] \cup [b, \infty[$, with $Q(\mp\infty) = \pm\infty$. Let $k \geq 0$, be an integer. Suppose that there are $2k+1$ consecutive adjacent nondegenerate closed intervals*

$$I_1^+, I_1^-, \ldots, I_k^+, I_k^-, I_{k+1}^+,$$

with $q \geq 0$, $q \not\equiv 0$ on I_i^+ and $q \leq 0$, $q \not\equiv 0$ on I_i^-, such that

$$[a, b] = \left(\left(\bigcup_{i=1}^{k+1} I_i^+ \right) \cup \left(\bigcup_{i=1}^{k} I_i^- \right) \right).$$

Then, there are $k+1$ positive integers $n_1^*, \ldots, n_{k+1}^*$ such that for each $(k+1)$-uple $\mathbf{n} := (n_1, \ldots, n_{k+1})$, with $n_i > n_i^*$, and each k-uple $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_k)$, with $\delta_i \in \{0, 1\}$, there are at least two solutions $x^+_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ and $x^-_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ of (3.1) *such that:*

(1) *$x^-_{\mathbf{n},\boldsymbol{\delta}}(t) < 0$ and $\dot{x}^-_{\mathbf{n},\boldsymbol{\delta}}(t) < 0$ for all $t \in \,]-\infty, a]$, while $x^+_{\mathbf{n},\boldsymbol{\delta}}(t) > 0$ and $\dot{x}^+_{\mathbf{n},\boldsymbol{\delta}}(t) > 0$ for all $t \in \,]-\infty, a]$,*
(2) *$x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1 - \delta_i$ changes of sign of the derivative in I_i^-,*
(3) *$x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t) \times \dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(t) \neq 0$ for all $t \in [b, \infty[$,*
(4) *for each i, $|x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| + |\dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| \to \infty$, as $n_i \to \infty$, uniformly in $t \in I_i^+$,*
(5) *$|x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| + |\dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| \to 0$ as $t \to \pm\infty$.*

To prove this, we follow the argument of Theorem 1, but, instead of proving that the solutions meet the y-axis at $t = 0$ and $t = \omega$, we require now that the solutions meet at the time $t = a$ the continua of Lemma 5, say $\Gamma_a^\pm$, and at the time $t = b$ the continua of Lemma 7, say $\Gamma_b^{\pm}$. The only difference is that now these continua need not to be the images of any continuous curve, that is we could be no more able to parametrize them by continuous functions of a real variable. However this technical difficulty can be easily overcome by a standard approximation procedure (see [12, Lemma 1] for instance). In fact, at first it can be shown that we have to consider only a bounded portion of each continua $\Gamma_a^{\pm}$ and $\Gamma_b^{\pm}$, since we are interested in finding solutions with a fixed number of zeros in I_1^+ and I_{k+1}^+ (it is a consequence of Lemma 2). Secondly these portions of continua can be uniformly approximated by images of continuous curves $\Gamma_{a,\varepsilon}^{\pm}$ and $\Gamma_{b,\varepsilon}^{\pm}$, in the sense that $\Gamma_{a,\varepsilon}^{\pm}$ and $\Gamma_{b,\varepsilon}^{\pm}$ lies inside an ε-neighbourhood of the bounded portions of $\Gamma_a^{\pm}$ and $\Gamma_b^{\pm}$, respectively. Hence, following the arguments of the proof of Theorem 1, we can find at least two solutions $x_{\mathbf{n},\boldsymbol{\delta},\varepsilon}^{\pm}$ such that they have the prescribed nodal behaviour in $[a,b]$ and moreover satisfy $(x_{\mathbf{n},\boldsymbol{\delta},\varepsilon}^{\pm}(a), \dot{x}_{\mathbf{n},\boldsymbol{\delta},\varepsilon}^{\pm}(a)) \in \Gamma_{a,\varepsilon}^{\pm}$ and $(x_{\mathbf{n},\boldsymbol{\delta},\varepsilon}^{\pm}(b), \dot{x}_{\mathbf{n},\boldsymbol{\delta},\varepsilon}^{\pm}(b)) \in \Gamma_{b,\varepsilon}^{+} \cup \Gamma_{b,\varepsilon}^{-}$. The required solutions $x_{\mathbf{n},\boldsymbol{\delta}}^{\pm}$ can be found by letting $\varepsilon \to 0$.

Note that from the number of zeros of the solutions it is possible to know the sign of $x_{\mathbf{n},\boldsymbol{\delta}}^{\pm}(t)$ and $\dot{x}_{\mathbf{n},\boldsymbol{\delta}}^{\pm}(t)$ for $t \geq b$.

It is possible to obtain a result for the existence of solutions decaying at infinity and with a singularity at a point $t^* \in I_j^-$ by combining Theorem 4 with Theorem 2.

6. Related results

In this section, we present some variants of Theorem 1. In particular we consider general Sturm–Liouville boundary conditions and the existence of solutions with a point of singularity in a prescribed interval.

Let us consider at first the boundary condition

$$
\begin{cases} \alpha x(0) + \beta \dot{x}(0) = 0, \\ \gamma x(\omega) + \delta \dot{x}(\omega) = 0, \end{cases} \tag{6.1}
$$

where $|\alpha| + |\beta| > 0$ and $|\gamma| + |\delta| > 0$. With the same technique of Theorem 1, the following result can be proved.

Theorem 3. *Under the same assumptions of Theorem* 1*, there are* $k+1$ *positive integers* $n_1^*, \ldots, n_{k+1}^*$ *such that for each* $(k+1)$*-uple* $\mathbf{n} := (n_1, \ldots, n_{k+1})$*, with* $n_i > n_i^*$*, and each* k*-uple* $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_k)$*, with* $\delta_i \in \{0,1\}$*, there are at least two solutions* $x_{\mathbf{n},\boldsymbol{\delta}}^{+}(\cdot)$ *and* $x_{\mathbf{n},\boldsymbol{\delta}}^{-}(\cdot)$ *of* (3.1)–(6.1)*, such that:*

(1) $\alpha \dot{x}_{\mathbf{n},\boldsymbol{\delta}}^{-}(0) - \beta x_{\mathbf{n},\boldsymbol{\delta}}^{-}(0) < 0 < \alpha \dot{x}_{\mathbf{n},\boldsymbol{\delta}}^{+}(0) - \beta x_{\mathbf{n},\boldsymbol{\delta}}^{+}(0)$,

(2) $x_{\mathbf{n},\boldsymbol{\delta}}^{\pm}(\cdot)$ *has exactly* n_i *zeros in* I_i^+*, exactly* δ_i *zeros in* I_i^- *and exactly* $1 - \delta_i$ *changes of sign of the derivative in* I_i^-*,*

(3) *for each* i*,* $|x_{\mathbf{n},\boldsymbol{\delta}}^{\pm}(t)| + |\dot{x}_{\mathbf{n},\boldsymbol{\delta}}^{\pm}(t)| \to \infty$*, as* $n_i \to \infty$*, uniformly in* $t \in I_i^+$.

Depending of the position of the lines $\alpha x+\beta y=0$ and $\gamma x+\delta y=0$, one could easily discuss the different possibilities that $x^{\pm}_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$, or $\dot x^{\pm}_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$, may not vanish in $J\setminus\{0,\omega\}$, or they have exactly one or two changes of sign. We also remark that with small modifications in the proof, it is possible to find solutions as above and satisfying the non-homogeneous Sturm–Liouville boundary conditions

$$\begin{cases} \alpha x(0)+\beta\dot x(0)=r_1,\\ \gamma x(\omega)+\delta\dot x(\omega)=r_2. \end{cases}$$

We refer to [16], [45] and the references therein for preceding results in this direction for the case of positive definite weight.

We briefly discuss now the case of solutions which blow up in a fixed interval of negativity, but maintaining their nodal properties in the intervals when $q\ge 0$. First we need two preparatory lemmas.

Lemma 8. *Let $[b,c]\subset[0,\omega]$ be such that*

$$q\le 0 \quad \textit{and} \quad q\not\equiv 0 \quad \textit{on } [b,c]$$

and assume that (g_2^-) is satisfied. Then there are two unbounded continua $\Gamma_1\subset\overline{A_4}$ and $\Gamma_2\subset\overline{A_2}$ such that, for every $p\in\Gamma_1\cup\Gamma_2$, $x(\,\cdot\,;b,p)$ is defined on $[b,\beta_p[\,\subset[b,c[$, $x(t;b,p)\ne 0$ for all $t\in[b,\beta_p[$, and $\dot x(\,\cdot\,;b,p)$ has exactly one change of sign in $[b,\beta_p[$. Moreover, we have that

$$\lim_{t\to\beta_p} x(t;b,p)=\lim_{t\to\beta_p}\dot x(t;b,p)=\infty \quad \textit{for all } p\in\Gamma_1$$

and

$$\lim_{t\to\beta_p} x(t;b,p)=\lim_{t\to\beta_p}\dot x(t;b,p)=-\infty \quad \textit{for all } p\in\Gamma_2.$$

Lemma 9. *Let $[b,c]\subset[0,\omega]$ be such that*

$$q\le 0 \quad \textit{and} \quad q\not\equiv 0 \quad \textit{on } [b,c]$$

and assume that (g_2^-) is satisfied. Then there are two unbounded continua $\Gamma_1\subset\overline{A_2}$ and $\Gamma_2\subset\overline{A_4}$ such that for every $p\in\Gamma_1\cup\Gamma_2$, $x(\,\cdot\,;b,p)$ is defined on $[b,\beta_p[\,\subset[b,c[$, $\dot x(t;b,p)\ne 0$ for all $t\in[b,\beta_p[$, and $x(\,\cdot\,;b,p)$ has exactly one zero in $[b,\beta_p[$. Moreover, we have that

$$\lim_{t\to\beta_p} x(t;b,p)=\lim_{t\to\beta_p}\dot x(t;b,p)=\infty \quad \textit{for all } p\in\Gamma_1,$$

and

$$\lim_{t\to\beta_p} x(t;b,p)=\lim_{t\to\beta_p}\dot x(t;b,p)=-\infty \quad \textit{for all } p\in\Gamma_2.$$

The proofs of these two results are omitted, as they follow quite closely those of Lemmas 3 and 4. The fact that Γ_1 and Γ_2 can be chosen as connected sets, can be proved by Lemma 6, using the same argument contained in Lemma 5. At this moment, repeating the scheme in the proof of Theorem 1 (with the modifications introduced for Theorem 2, since Γ_1 and Γ_2 need not to be images of continuous curves), we can obtain:

Theorem 4. *Let us suppose $k \geq 1$. Under the same assumptions of Theorem 1, let us fix $j \in \{1, \dots, k\}$. Then, there are j positive integers $n_1^*, \dots, n_j^*$ such that for each j-uple $\mathbf{n} := (n_1, \dots, n_j)$, with $n_i > n_i^*$, and each j-uple $\boldsymbol{\delta} := (\delta_1, \dots, \delta_j)$, with $\delta_i \in \{0, 1\}$, there are at least two solutions $x^+_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ and $x^-_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ of* (3.1) *which are defined on $[0, \beta_+[$ and $[0, \beta_-[$, respectively, with $\beta_\pm \in I_j$, and such that:*

(1) *$x^\pm_{\mathbf{n},\boldsymbol{\delta}}(0) = 0$ and $\dot{x}^-_{\mathbf{n},\boldsymbol{\delta}}(0) < 0 < \dot{x}^+_{\mathbf{n},\boldsymbol{\delta}}(0)$,*
(2) *$x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$ has exactly n_i zeros in I_i^+, exactly δ_i zeros in I_i^- and exactly $1 - \delta_i$ changes of sign of the derivative in I_i^-,*
(3) *$|x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| \to \infty$ and $|\dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| \to \infty$ as $t \to \beta_\pm$,*
(4) *neither $x^\pm_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$, nor $\dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(\,\cdot\,)$, may vanish in $J \cap\,]0, \beta_\pm[$,*
(5) *for each i, $|x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| + |\dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)| \to \infty$, as $n_i \to \infty$, uniformly in $t \in I_i^+$.*

Note that one can decide if $x^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)$ and $\dot{x}^\pm_{\mathbf{n},\boldsymbol{\delta}}(t)$ are positive or negative in a left neighbourhood of $\beta_\pm$ by counting their number of zeros.

A completely symmetric result holds if we look for solutions defined on intervals of the form $]\alpha_\pm, \omega]$, which vanish in ω and blow up at $\alpha_\pm \in I_j$. Of course, a version of Theorem 4 can be stated with respect to the Sturm–Liouville boundary conditions or for the case of unbounded intervals on the line of Theorem 2.

It is possible to obtain extensions of Theorems 1, 3, 4 for the equation $\ddot{x} + c\dot{x} + q(t)g(x) = 0$, as well (see also [41]).

7. Appendix: Estimates on the time-maps

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous map such that

$$\lim_{s \to \pm\infty} g(s)\operatorname{sgn}(s) = \infty.$$

Clearly, the primitive G of g is strictly monotone in a neighbourhood of $\pm\infty$ and thus inverses $G_\pm^{-1}$ are defined. We are looking for some conditions ensuring the validity of (g_2^+) and (g_2^-).

With respect to the first condition, which means that the period of the closed orbits of $\dot{x} = y$, $\dot{y} = -g(x)$ tends to zero as the energy of the orbits tends to infinity, we can use various results in the literature, starting with the classical work of Opial [40] (see also [44]). A typical assumption which implies (g_2^+) is the condition of superlinear growth at infinity

$$(\mathrm{g}_3) \qquad \lim_{s \to \pm\infty} \frac{g(s)}{s} = \infty.$$

Other possibilities for (g_2^+) when (g_3) does not hold, are discussed in [21]. For instance, we have that (g_2^+) follows from

$$\exists A > 0 : \lim_{s\to\pm\infty} \frac{G(s+A)-G(s)}{s^2} = \infty.$$

With respect to (g_2^-), which is a condition on the time along the orbits of $\dot{x} = y$, $\dot{y} = g(x)$, we observe the following kind of integrals have to be evaluated:

$$f_1(x) := \int_{-\infty}^{\infty} \frac{1}{\sqrt{x + G(s)}}\, ds, \tag{7.1}$$

and

$$f_2(x) := \int_{G_+^{-1}(x)}^{\infty} \frac{1}{\sqrt{G(s) - x}}\, ds \tag{7.2}$$

(there is a similar integral depending on $G_-^{-1}(x)$ which can be handled in the same way like f_2).

Our goal is to study under which conditions $f_1(x), f_2(x) \to 0$ as $x \to \infty$.

For the first question we easily see that $f_1(x) \to 0$ as $x \to \infty$ if and only if

$$\left| \int^{\pm\infty} \frac{1}{\sqrt{G(s)}}\, ds \right| < \infty. \tag{g_4}$$

For the second condition, we have the following lemma.

LEMMA 10. *Assume* (g_3), (g_4) *and suppose that there is a constant* $k > 1$ *such that*

$$\liminf_{s\to\pm\infty} \frac{G(ks)}{G(s)} > 1. \tag{g_5}$$

Then (g_2^-) *holds.*

PROOF. By the above remark we can confine ourselves to the proof of $f_2(x) \to 0$ as $x \to \infty$. With a simple change of variables, we consider the function

$$f_3(x) := \int_x^{\infty} \frac{1}{\sqrt{G(s) - G(x)}}\, ds.$$

Clearly, $f_2(x) \to 0$ if and only if $f_3(x) \to 0$. Now we split the integral as

$$\int_x^{kx} \frac{1}{\sqrt{G(s) - G(x)}}\, ds + \int_{kx}^{\infty} \frac{1}{\sqrt{G(s) - G(x)}}\, ds.$$

For the first one, we have that

$$\begin{aligned} \int_x^{kx} \frac{1}{\sqrt{G(s) - G(x)}}\, ds &= \int_x^{kx} \frac{1}{\sqrt{\int_x^s g(\xi)d\xi}}\, ds \le \int_x^{kx} \frac{1}{\sqrt{s-x}\sqrt{g_{\min}(x)}}\, ds \\ &= \frac{2\sqrt{(k-1)x}}{\sqrt{g_{\min}(x)}} = 2\sqrt{(k-1)}\sqrt{\frac{x}{\xi_x}}\sqrt{\frac{\xi_x}{g(\xi_x)}} \end{aligned}$$

where we have set $g_{\min}(x) := \min_{[x,kx]} g(\xi)$ and $\xi_x \in [x,kx]$ is such that $g(\xi_x) = g_{\min}(x)$. From (g_3) it easily follows that the first integral tends to zero as $x \to \infty$.

By (g_5) there exist $c > 1$ and $\overline{s} > 0$ such that

$$G(ks) \geq cG(s) \quad \text{for all } s \geq \overline{s}.$$

Moreover, we can assume without loss of generality that G is monotone increasing on $[\overline{s}, \infty[$. Hence, for the second integral we can estimate:

$$\begin{aligned}\int_{kx}^{\infty} \frac{1}{\sqrt{G(s)-G(x)}}\,ds &= \int_{kx}^{\infty} \frac{1}{\sqrt{G(s)}\sqrt{1-G(x)/G(s)}}\,ds \\ &\leq \int_{kx}^{\infty} \frac{1}{\sqrt{G(s)}\sqrt{1-G(x)/G(kx)}}\,ds \\ &\leq \sqrt{\frac{c}{c-1}}\int_{kx}^{\infty} \frac{1}{\sqrt{G(s)}}\,ds\end{aligned}$$

for all $x \geq \overline{s}$. Thus also the second integral tends to zero as $x \to \infty$ since (g_4) holds. □

We remark that condition (g_5) is related to a well-known assumption which appears in the theory of Orlicz–Sobolev spaces. Indeed, it is easy to prove that (g_5) holds when G_+^{-1} and $|G_-^{-1}|$ satisfy a Δ_2-*condition at infinity* (see [1, p. 232]). We notice that a sufficient condition for the validity of (g_5) is that g is *monotone increasing* in a neighbourhood of infinity. In fact, if $x > 0$ is large enough, by the monotonicity, we have that $G(2x) = \int_0^{2x} g(s)\,ds \geq 2\int_0^x g(s)\,ds = 2G(x)$ (the case for $x << 0$ is treated in the same manner) and therefore (g_5) is proved. In this case, however, we can obtain much more, namely we have:

Lemma 11. *Assume* (g_4) *with g a monotone function in a neighbourhood of infinity. Then* (g_2^-) *holds.*

Proof. Via the change of variables $\xi = G(s) - G(x)$ and $s = G_+^{-1}(\xi + G(x))$, the function $f_3(x)$ defined in the proof of Lemma 10 can be written in the following way:

$$f_3(x) = \int_0^{\infty} \frac{d\xi}{g(G_+^{-1}(\xi + G(x)))\sqrt{\xi}}.$$

By definition, the function $\phi_x : \xi \mapsto 1/g(G_+^{-1}(\xi + G(x)))\sqrt{\xi}$ is positive and summable on $]0, \infty[$ for large $x > 0$. Moreover, by the monotonicity of g, we see that $\phi_x(\xi)$ is monotonically pointwise decreasing to zero as $x \to \infty$. As a consequence, the monotone convergence theorem implies that $f_3(x) \to 0$ as $x \to \infty$. □

Remark 1. Suppose that $h : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$\lim_{s \to \pm\infty} h(s)\operatorname{sgn}(s) = \infty$$

and assume also that

$$|g(x)| \geq |h(x)| \quad \text{for } |x| \gg 1.$$

Then, it is straightforward to check that the validity of the condition (g_2) with respect to the function h implies the same for the function g. In particular, if, for $|x|$ large we have that g "dominates" a monotone increasing function h at infinity, with $h(x)/x \to \infty$ as $x \to \pm\infty$ and $|\int^{\pm\infty} ds/\sqrt{H(s)}| < \infty$, where $H' = h$, then (g_2) holds for g.

From this remark, we see that a (non-necessarily monotone) function g which is larger than $kx \log^{\alpha}(|x| + 1)$ for $|x|$ large and for some $k > 0$ and $\alpha > 2$, satisfies (g_2).

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