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ON A CONTROLLABILITY PROBLEM FOR SYSTEMS GOVERNED BY SEMILINEAR FUNCTIONAL DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Dedicated to the memory od Juliusz P. Schauder

Abstract. For a Banach space E, a given pair $(\overline{p},\overline{x}) \in [0,a] \times E$, and control system governed by a semilinear functional differential includion of the form

$$x'(t) \in Ax(t) + F(t, x(t), Tx)$$

the existence of a mild trajectory of $x(t)$ satisfying the condition $x(\overline{p}) = \overline{x}$ is considered. Using topological methods we develop an unified approach to the cases when a multivalued nonlinearity F is Carathéodory upper semicontinuous or almost lower semicontinuous and an abstract extension operator T allows to deal with variable and infinite delay. For the Carathéodory case, the compactness of the solutions set and, as a corollary, an optimization result are obtained.

1. Introduction

In the present paper we study a problem of attaining a given point at a prescribed time for a system governed by a semilinear functional differential inclusion in a Banach space. Notice that inclusions of that type appear in the

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description of processes of controlled heat transfer (see, e.g. [13]), in hybrid systems with dry friction, in transmission line process and other problems (see [11] and references therein).

Several authors have introduced hereditary structures for functional differential equations with bounded or unbounded delay. Recently, has been showed that these structures are equivalent (see [12]).

The frame of our functional argument has been introduced in [14] for the study of terminal value problems for functional differential equations with unbounded delay.

In this argument, the continuous extension operator T generalizes the known structures and allows us to deal with variable and infinite delay (see, e.g. [3], [9], [15]).

We use the technique of condensing multivalued maps to develop an unified approach to the cases when multivalued nonlinearity is Carathéodory upper semicontinuous or almost lower semicontinuous. We obtain the existence result (Theorem 4) basing, in the first case, on the Bohnenblust–Karlin fixed point theorem, and, in the second case, on the Fryszkowski continuous selection result ([8]) and the Schauder fixed point theorem. For the case of Carathéodory nonlinearity, we also prove the compactness of the solutions set (Theorem 5) and, as a corollary, we obtain an optimization result.

2. Preliminaries

Let X and Y be topological spaces, $\mathcal{P}(Y)$ denote the collection of all nonempty subsets of Y. A multivalued map (multimap) $\mathcal{F} : X \to \mathcal{P}(Y)$ is said to be

(i) *upper semicontinuous* (u.s.c.) if $\mathcal{F}^{-1}(V) = \{x \in X : \mathcal{F}(x) \subset V\}$ is an open subset of X for every open $V \subset Y$,

(ii) *lower semicontinuous* (l.s.c.) if $\mathcal{F}^{-1}(W)$ is a closed subset of X for every closed $W \subset Y$ (for futher details, see, eg. [2], [10], [11]).

Recall also the following notions (see, e.g. [1], [11]). Let E be a Banach space; 2^E denotes the collection of all subsets of E. A function $\beta : 2^E \to \mathbb{R} \cup \{+\infty\}$ is called a (real) *measure of noncompactness* (MNC) in E if

$$\beta(\overline{\mathrm{co}}\,\Omega) = \beta(\Omega)$$

for every $\Omega \in 2^E$ and $\beta(\Omega) < +\infty$ for every bounded Ω.

A MNC is called

(i) *monotone* if $\Omega_0, \Omega_1 \in 2^E$, $\Omega_0 \subset \Omega_1$ implies $\beta(\Omega_0) \le \beta(\Omega_1)$,

(ii) *nonsingular* if $\beta(\{a\} \cup \Omega) = \beta(\Omega)$ for every $a \in E, \Omega \in 2^E$,

(iii) *regular* if $\beta(\Omega) = 0$ is equivalent to the relative compactness of Ω.

As the example of the MNC possessing all these properties, we may consider the Hausdorff MNC

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net }\}.$$

In the sequel, we will need the following its property that can be easily verified: if $L : E \to E$ is a bounded linear operator, then, for every bounded set $\Omega \subset E$, we have

$$\chi(L\Omega) \leq \|L\|\chi(\Omega).$$

Everywhere in the following E will denote a separable Banach space.

We consider some properties of the space of all Bochner summable functions $L^1([a,b];E)$.

A multifunction $\mathcal{G} : [a,b] \to \mathcal{P}(E)$ is said to be

(i) *integrable* provided it has a summable selection $g \in L^1([a,b];E)$, i.e. $g(t) \in \mathcal{G}(t)$ for a.e. $t \in [a,b]$,

(ii) *integrably bounded* if there exists a summable function $\mu \in L^1_+([a,b])$ such that

$$\|\mathcal{G}(t)\| := \sup\{\|g\| : g \in \mathcal{G}(t)\} \leq \mu(t) \text{ for a.e. } t \in [a,b].$$

The set of all summable selections of the multifunction $\mathcal{G}$ will be denoted by $S^1_{\mathcal{G}}$. If the multifunction $\mathcal{G} : [a,b] \to \mathcal{P}(E)$ is integrable, then the integral of $\mathcal{G}$ on every measurable subset $I \subset [a,b]$ may be defined as

$$\int_I \mathcal{G}(s)\,ds := \left\{ \int_I g(s)\,ds : g \in S^1_{\mathcal{G}} \right\}.$$

We recall that the multifunction $\mathcal{G} : [a,b] \to \mathcal{P}(E)$ is said to be *measurable* if $\mathcal{G}^{-1}(V)$ is Lebesgue measurable for every open set $V \subset E$ (for equivalent definitions and details, see, e.g. [2], [4], [10], [11]).

We denote by χ_E the Hausdorff MNC in the space E.

Lemma 1 ([13], see also [11]). *Let a multifunction $\mathcal{G} : [a,b] \to \mathcal{P}(E)$ be integrable, integrably bounded and $\chi_E(\mathcal{G}(t)) \leq q(t)$ for a.e. $t \in [a,b]$, where $q \in L^1_+[a,b]$. Then*

$$\chi\bigg(\int_a^b \mathcal{G}(s)\,ds\bigg) \leq \int_a^b q(s)\,ds.$$

In the following we will suppose that

(A) $A : \operatorname{dom}(A) \subset E \to E$ is a densely defined linear operator generating a C_0-group e^{At} (see, e.g. [7], [16]).

We will say that the map $S : L^1([a,b];E) \to C([a,b];E)$ defined as

$$S(f)(t) = \int_t^b e^{A(t-s)} f(s)\,ds$$

is the Cauchy operator. As in [5] (see also [11]), the following its property may be verified.

Lemma 2. *If the sequence $\{f_n\} \subset L^1([a,b];E)$ is integrably bounded and the set $\{f_n(t)\}$ is relatively compact in E for a.e. $t \in [a,b]$, then the sequence $\{Sf_n\} \subset C([a,b];E)$ is relatively compact.*

Further, consider the following notion. A nonempty set $M \subset L^1([a,b];E)$ is said to be *decomposable* provided for every $f, g \in M$ and each Lebesgue measurable subset $m \in [a,b]$,

$$f \cdot \mathbf{1}_m + g \cdot \mathbf{1}_{[a,b]\setminus m} \in M$$

where $\mathbf{1}_m$ is the characteristic function of the set m.

Lemma 3 ([8]). *Let X be a compact metric space. Then every l.s.c. multimap $\mathcal{T} : [a,b] \to \mathcal{P}(L^1([a,b];E))$ with closed decomposable values has a continuous selection, i.e. there exists a continuous map $f : X \to L^1([a,b];E)$ such that $f(x) \in \mathcal{T}(x)$ for all $x \in X$.*

For $a > 0$ fixed, we put $I = (-\infty, a]$. We denote by $C = C_{\mathrm{loc}}(I;E)$ the space of all continuous functions endowed with the topology of uniform convergence on every compact subset of I. It is known that this topology can be induced by a countable family of seminorms

$$p_n(x) = \sup\{\|x(t)\| : t \in [-n,n] \cap I\}, \quad n \in \mathbb{N}.$$

Therefore, the topology of C is compatible with the metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{2^{-n} p_n(x-y)}{1 + p_n(x-y)}.$$

Since E is a Banach space, C is a linear metric space.

3. Controllability problem

Let us give a pair $(\overline{p}, \overline{x}) \in [0,a] \times E$ and an extension operator

$$T : C([0,a];E) \to C, \quad Tx_{|[0,a]} = x \quad \text{for } x \in C([0,a];E).$$

We suppose that TD is bounded for every bounded $D \subset C([0,a];E)$. We consider the following controllability problem for systems governed by a semilinear functional differential inclusion of the form

$$P(\overline{p}, \overline{x}) = \begin{cases} x'(t) \in Ax(t) + F(t, x(t), Tx) & \text{a.e. } t \in [0, \overline{p}], \\ x(\overline{p}) = \overline{x}. \end{cases}$$

Let us denote by $K(E)[Kv(E)]$ the collection of all nonempty compact [convex] subsets of E. For the multivalued nonlinearity $F : [0,a] \times E \times C \to K(E)$, we will suppose two alternative groups of hypotheses.

The first group includes the following assumptions.

- ($\mathrm{F1}_U$) F has nonempty compact convex values, i.e. it acts into the collection $Kv(E)$,
- ($\mathrm{F2}_U$) for every fixed $\xi \in E$ and $c \in C$ the multifunction $F(\,\cdot\,,\xi,c) : [0,a] \to Kv(E)$ admits a measurable selection,
- ($\mathrm{F3}_U$) for a.e. $t \in [0,a]$ the multimap $F(t,\,\cdot\,,\,\cdot\,) : E \times C \to Kv(E)$ is u.s.c.

Notice that condition ($\mathrm{F2}_U$) is fulfilled if the multifunction $F(\,\cdot\,,\xi,c)$ is measurable for every $(\xi,c) \in E \times C$.

The alternative supposition is the following.

- (F_L) $F : [0,a] \times E \times C \to K(E)$ is almost lower semicontinuous (a.l.s.c.) in the sense that there exists a sequence of disjoint compact sets $\{I_j\}$, $I_j \subset [0,a]$, with $\mathrm{meas}([0,a]\setminus\bigcup_j I_j) = 0$, such that the restriction of F on each set $I_j \times E \times C$ is l.s.c. (cf. [6]).

In both cases, the remaining hypotheses are the same.

- (H1) (Volterra property) for a.e. $t \in [0,a]$ and for every $\xi \in E$ and $c_1, c_2 \in C$ with $c_{1|]-\infty,t]} = c_{2|]-\infty,t]}$, we have

$$F(t,\xi,c_1) = F(t,\xi,c_2),$$

- (H2) for every bounded $\Omega \subset C([0,a];E)$ there exists a function $\mu_\Omega \in L^1_+([0,a])$ such that for all $x \in \Omega$ we have that

$$\|F(t,x(t),Tx)\| \le \mu_\Omega(t) \quad \text{for a.e. } t \in [0,a].$$

To describe the next assumption, let us denote by the symbol χ' the Hausdorff MNC in the space $C((-\infty,t];E)$, $t \in [0,a]$ endowed with the metric similar to that in C.

- (H3) there exists a function $k \in L^1_+([0,a])$ such that, for every bounded set $D \subset C([0,a];E)$, we have that, for a.e. $t \in [0,a]$,

$$\chi_E(F(t,D(t),TD)) \le k(t)\max\{\chi_E(D(t)), \chi'(TD_{|]-\infty,t]})\}.$$

On the extension operator $T : C([0,a];E) \to C$ we take the following assumptions.

- (T1) T is continuous,
- (T2) (Volterra-type property) $Tx_{|]-\infty,0]} = Ty_{|]-\infty,0]}$ if $x(0) = y(0)$.

To give the next property of T, let us use, for every $b \in (0,a]$, the following MNC in the space $C([0,b];E)$:

$$\phi(\Omega) = \sup_{t\in[0,b]} \chi_E(\Omega(t))$$

for any bounded $\Omega \subset C([0,b];E)$ (notice that ϕ coincides with the Hausdorff MNC on equicontinuous sets [1]).

We suppose that

(T3) there exists a real number $m \geq 1$ such that, for every bounded set $D \subset C([0,a];E)$ and for every $b \in (0,a]$, we have

$$\chi'\left(TD_{|(-\infty,b]}\right) \leq m\phi(D_{|[0,b]}).$$

Note that operator T allows us to deal with variable and infinite delay. For example, T may be a translation operator defined, for a given $x^* \in C$, by

$$Tx(t) = \begin{cases} x(t) & t \in [0,a], \\ x^*(t) - x^*(0) + x(0) & t < 0. \end{cases}$$

Remark 1. For $x \in C([0,a];E)$, the images $F(t,x(t),Tx)$ are defined only by values of x on $[0,t]$, $t \in (0,a]$. In fact, if we consider any extension $\widetilde{x}$ of $x_{|[0,t]}$ to the whole interval $[0,a]$, then, by means of conditions (H1) and (T2), the value of $F(t,x(t),T\widetilde{x})$ does not depend on the way we extend x. Thus, we still denote by x the extension $\widetilde{x}$.

For any $p \in (0,a]$ and $x \in C([0,p];E)$, the multifunction $F(t,x(t),Tx)$ admits a measurable selection and, moreover, (H2) yields that it is integrable. In fact, in the case when F satisfies $(F1_U)$–$(F3_U)$, it follows from the Castaing's theorem (see, e.g. Proposition 3.5(a) in [6] or Theorem 1.3.5 in [11]). In the case (F_L), the general properties of multimaps (see, e.g. [2], [10], [11]) imply that the multifunction $F(t,x(t),Tx)$ is l.s.c. on $(\bigcup_j I_j) \cap [0,p]$ and hence it is measurable on $[0,p]$.

So, in both cases, we may define the superposition multioperator $\mathcal{P}_F : C([0,p];E) \to \mathcal{P}(L^1([0,p];E))$ as

$$\mathcal{P}_F(x) = S^1_{F(\,\cdot\,,x(\,\cdot\,),Tx)}.$$

We will say that a function $x \in C([0,\overline{p}];E)$ is a *mild solution* of problem $P(\overline{p},\overline{x})$ provided it has the following representation

$$x(t) = e^{A(t-\overline{p})}\overline{x} - \int_t^{\overline{p}} e^{A(t-s)} f(s)\,ds \quad \text{for } f \in \mathcal{P}_F(x).$$

Theorem 4. *Under assumptions* (A) *and*

(i) $(F1_U)$–$(F3_U)$ *or*

(ii) (F_L)

and (H1)–(H3), (T1)–(T3) *for every* $\overline{x} \in E$ *there exists* $h \in (0,a]$ *such that the problem* $P(\overline{p},\overline{x})$ *has a mild solution for every* $\overline{p} \in (0,h]$.

Proof. We will divide the proof into several steps. The first two of them are the same in both cases (i) and (ii).

Step 1. Let us take an arbitrary $\delta > 0$. Since e^{At} is the C_0-group, there exists a real positive number h_1 such that

$$\|e^{A\theta}\overline{x} - \overline{x}\| < \delta/2 \quad \text{for all } \theta,\ \theta \le h_1.$$

Further, in the space $C([0,a];E)$ we consider a ball $\mathcal{B} = \overline{B}_\delta(\overline{x}(\,\cdot\,))$, where $\overline{x}(\,\cdot\,)$ is the function identically equal to $\overline{x}$.

Now, let $h_2 = \delta/(2r\|\mu_{\mathcal{B}}\|_1)$, where $r = \sup_{\theta \le a} \|e^{A\theta}\|$, $\mu_{\mathcal{B}}$ be the function from condition (H2) and $\|\mu_{\mathcal{B}}\|_1$ its norm in the space $L^1([0,a])$.

We put $h := \min\{h_1, h_2\}$ and we take $\overline{p} \in (0,h]$.

We consider the integral multioperator associated to $P(\overline{p},\overline{x})$, $\Gamma : C([0,\overline{p}];E) \to \mathcal{P}(C([0,\overline{p}];E))$ defined by

$$\Gamma(x) = \left\{ y : y(t) = e^{A(t-\overline{p})}\overline{x} - \int_t^{\overline{p}} e^{A(t-s)} f(s)\,ds,\ f \in \mathcal{P}_F(x) \right\}.$$

It is clear that every mild solution $x(\,\cdot\,)$ of $P(\overline{p},\overline{x})$ is a fixed point of Γ: $x \in \Gamma(x)$.

Let us verify that the multioperator Γ transforms the ball $\mathcal{B}' = \overline{B}_\delta(\overline{x}(\,\cdot\,)) \subset C([0,\overline{p}];E)$ into itself. In fact, if $x \in \mathcal{B}'$ and $y \in \Gamma(x)$, then $y(t) = e^{A(t-\overline{p})}\overline{x} - \int_t^{\overline{p}} e^{A(t-s)} f(s)\,ds$, $f \in \mathcal{P}_F(x)$ and we have the following estimation for any $t \in [0,\overline{p}]$:

$$\begin{aligned}\|y(t) - \overline{x}(t)\| &\le \|e^{A(t-\overline{p})}\overline{x} - \overline{x}\| + \left\| \int_t^{\overline{p}} e^{A(t-s)} f(s)\,ds \right\| \\ &\le \delta/2 + \int_t^{\overline{p}} \|e^{A(t-s)}\|\|f(s)\|\,ds \le \delta/2 + hr\|\mu_{\mathcal{B}}\|_1 \le \delta.\end{aligned}$$

Step 2. We construct a nonempty compact convex subset of $\mathcal{B}'$ which is invariant with respect to the action of Γ. To do so, we consider in $C([0,\overline{p}];E)$ the MNC

$$\psi(\Omega) = \sup_{t \in [0,\overline{p}]} e^{-Rm\int_0^t k(\theta)\,d\theta} \chi_E(\Omega(t))$$

for any bounded $\Omega \subset C([0,\overline{p}];E)$, where $R > r$, $k(\,\cdot\,)$ is the function from condition (H3) and $m \ge 1$ is the real number of condition (T3). It is easy to see that the MNC ψ is monotone and nonsingular, but not regular in general.

Let us demonstrate that the multioperator Γ is $(r/R,\psi)$ – condensing, i.e. for any $\Omega \subset \mathcal{B}'$ we will have that

$$\psi(\Gamma(\Omega)) \le \frac{r}{R}\psi(\Omega).$$

Without loss of generality, we will assume that $h > 0$ taken at Step 1 is small enough to provide

$$e^{Rm\int_0^h k(\theta)\,d\theta} \le 2.$$

By means of the properties of the Hausdorff MNC and applying conditions (H3) and (T3), for $0 \le t \le s \le \overline{p}$ we have

$$\begin{aligned}\chi_E(e^{A(t-s)}F(s,\Omega(s),T\Omega)) &\le \|e^{A(t-s)}\|\chi_E(F(s,\Omega(s),T\Omega)) \\ &\le rk(s)\max\{\chi_E(\Omega(s)),\chi'(T\Omega_{|]-\infty,s]})\} \\ &\le rk(s)\max\{\chi_E(\Omega(s)), m\phi(\Omega_{|[0,s]})\}.\end{aligned}$$

Since $m \ge 1$, we obtain

$$\begin{aligned}\chi_E(e^{A(t-s)}F(s,\Omega(s),T\Omega)) &\le rmk(s)\sup_{\theta\in[0,s]}\chi_E(\Omega(\theta)) \\ &\le rm\psi(\Omega)k(s)\sup_{\theta\in[0,s]} e^{Rm\int_0^\theta k(\rho)d\rho} \\ &= rm\psi(\Omega)k(s)e^{Rm\int_0^s k(\theta)\,d\theta}.\end{aligned}$$

Applying the properties of the Hausdorff MNC and Lemma 1, for every $t \in [0,\overline{p}]$, we have

$$\begin{aligned}\chi_E(\Gamma(\Omega)(t)) &\le \chi_E\bigg(e^{A(t-\overline{p})}\overline{x} - \int_t^{\overline{p}} e^{A(t-s)}F(s,\Omega(s),T\Omega)\,ds\bigg) \\ &= \chi_E\bigg(\int_t^{\overline{p}} e^{A(t-s)}F(s,\Omega(s),T\Omega)\,ds\bigg) \\ &\le \int_t^{\overline{p}} rmk(s)\psi(\Omega)e^{Rm\int_0^s k(\theta)\,d\theta}\,ds \\ &= \frac{r}{R}\psi(\Omega)\int_t^{\overline{p}} Rmk(s)e^{Rm\int_0^s k(\theta)d\theta}\,ds \\ &= \frac{r}{R}\psi(\Omega)[e^{Rm\int_0^{\overline{p}} k(\theta)\,d\theta} - e^{Rm\int_0^t k(\theta)\,d\theta}] \\ &= \frac{r}{R}\psi(\Omega)e^{Rm\int_0^t k(\theta)\,d\theta}[e^{Rm\int_t^{\overline{p}} k(\theta)\,d\theta} - 1] \le \frac{r}{R}\psi(\Omega)e^{Rm\int_0^t k(\theta)\,d\theta},\end{aligned}$$

and hence

$$e^{-Rm\int_0^t k(\theta)\,d\theta}\chi_E(\Gamma(\Omega)(t)) \le \frac{r}{R}\psi(\Omega)$$

yielding

$$\psi(\Gamma(\Omega)) \le \frac{r}{R}\psi(\Omega).$$

Now, we consider the collection $\{N_\nu\}$ of all convex closed subsets of $\mathcal{B}'$ containing origin and Γ-invariant, i.e. $\Gamma(N_\nu) \subset N_\nu$ for all ν. Notice that this collection is nonempty, since it includes $\mathcal{B}'$. We set

$$N = \bigcap_\nu N_\nu.$$

From the minimality of N, it follows that $N = \overline{\mathrm{co}}\,(\Gamma(N) \cup \{0\})$ and hence

$$\psi(N) = \psi(\Gamma(N)) = 0$$

therefore, we have that $\chi_E(N(t)) = 0$ for all $t \in [0,\overline{p}]$.

Conditions (H3) and (T3) yield that $\chi_E(F(t,N(t),TN)) = 0$ for a.e. $t \in [0,\overline{p}]$ and by using the property of the Cauchy operator expressed in Lemma 2, we obtain that $\Gamma(N)$ is relatively compact and hence

$$X = \overline{\mathrm{co}}\,(\Gamma(N))$$

is the desiderable compact convex subset of $\mathcal{B}'$ invariant with respect to the action of Γ.

Step 3. Case (i). Following the lines of Theorem 5.1.2 of [11], we may verify that the integral multioperator Γ is u.s.c. on X and has convex compact values. So, we may apply the Bohnenblust–Karlin fixed point theorem (see, e.g. Theorem 3.1.5 of [11]) to deduce that the fixed point set $\mathrm{Fix}\,\Gamma$ is nonempty.

Step 4. Case (ii). It is clear that the superposition multioperator $\mathcal{P}_F$ has closed and decomposable values. Following the lines of [6, Section 9], and [11, Section 5.5], we may verify that $\mathcal{P}_F$ is l.s.c.

Applying Lemma 3 to the restriction of $\mathcal{P}_F$ on X, we obtain that there exists its continuos selection

$$q : X \to L^1([0,\overline{p}];E).$$

We consider a map $\gamma : X \to X$ defined as

$$\gamma(x)(t) = e^{A(t-\overline{p})}\overline{x} - \int_t^{\overline{p}} e^{A(t-s)} q(x)(s)\,ds.$$

Since the Cauchy operator is continuous, the map γ is also continuous, therefore, it is a continuous selection of the integral multioperator Γ.

From the Schauder fixed point Theorem, it follows that $\mathrm{Fix}\gamma \neq \emptyset$ and hence $\mathrm{Fix}\,\Gamma \neq \emptyset$. □

4. Compactness of the solutions set

In this Section, in the framework of case (i) and under a more severe assumption on the boundedness of the nonlinearity F, we prove the compactness of the solutions set $S(\overline{p},\overline{x})$ of the controllability problem $P(\overline{p},\overline{x})$ for all $\overline{p}$ small enough.

As an immediate consequence, we obtain the existence of an optimal trajectory for $P(\overline{p},\overline{x})$.

Theorem 5. *Under assumptions* (A), (F1$_U$)–(F3$_U$)*,* (H1), (H3) *and*

(H2') *there exists a function* $\alpha \in L^1_+[0,a]$ *such that for every* $x \in C([0,a];E)$ *we have that*

$$\|F(t,x(t),Tx)\| \leq \alpha(t)(1+\|x(t)\|) \quad \textit{for a.e. } t \in [0,a],$$

for every $\overline{x} \in E$ there exists $h \in (0,a]$ such that the set $S(\overline{p},\overline{x})$ of all mild solutions of the problem $P(\overline{p},\overline{x})$ is a (nonempty) compact for every $\overline{p} \in (0,h]$.

Proof. Let us demonstrate that the set $S(\overline{p},\overline{x})$, for any $\overline{p} \in (0,h]$, $\overline{x} \in E$, is apriori bounded. In fact, if $x(\,\cdot\,) \in S(\overline{p},\overline{x})$, then for $t \in [0,\overline{p}]$ we have that

$$x(t) = e^{A(t-s)}\overline{x} - \int_t^{\overline{p}} e^{A(t-s)} f(s)\,ds$$

where $f \in \mathcal{P}_F(x)$.

By the condition (H2'), we obtain that

$$\begin{aligned}\|x(t)\| &\le \|e^{A(t-\overline{p})}\|\|\overline{x}\| + \int_t^{\overline{p}} \|e^{A(t-s)}\|\|f(s)\|\,ds \\ &\le r\bigg(\|\overline{x}\| + \int_t^{\overline{p}} \alpha(s)(1+\|x(s)\|)\,ds\bigg) \\ &\le r\bigg(\|\overline{x}\| + \|\alpha\|_{L^1} + \int_t^{\overline{p}} \alpha(s)\|x(s)\|\,ds\bigg).\end{aligned}$$

By a Gronwall type inequality, we have that

$$\|x(t)\| \le r(\|\overline{x}\| + \|\alpha\|_{L^1})e^{r\|\alpha\|_{L^1}} = M.$$

Now, repeating the steps of the proof of Theorem 4, we choose $\delta > 0$ large enough to provide that the ball $\overline{B}_\delta(\overline{x}(\,\cdot\,))$ will contain the ball $\overline{B}_M(0)$.

Further, by considering the collection of Γ-invariant sets $\{N_\nu\}$ at Step 2, we will additionally assume that every such set apriori contains $S(\overline{p},\overline{x}) = \operatorname{Fix}\Gamma$. Then, the set N and, hence, X also contain $S(\overline{p},\overline{x})$.

The proof ends by the observation that the fixed points set $\operatorname{Fix}\Gamma$ of the u.s.c. multioperator Γ is a closed, and hence compact, subset of X. □

Corollary 6. *Under assumptions of Theorem 5, there exists a mild solution x^* of the problem $P(\overline{p},\overline{x})$ for $\overline{p} \in (0,h]$ such that*

$$j(x^*) = \max_{x(\,\cdot\,)\in S(\overline{p},\overline{x})} j(x(\,\cdot\,))$$

where $j : C([0,\overline{p}];E) \to \mathbb{R}$ is a given continuous cost functional.

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