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DISCONTINUOUS MAYER CONTROL PROBLEM UNDER STATE-CONSTRAINTS

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Dedicated to the memory of Juliusz P. Schauder

Abstract. This paper deals with Mayer's problem for control systems with state constraints and, possibly, discontinuous terminal cost. The main result of this paper consists in the characterization of the value function as the unique solution to an Hamilton–Jacobi equation. The above characterization extends results already obtained in the case of regular cost functions and under some controlability assumptions on the boundary of the set of constraints.

1. Introduction

We investigate the Mayer Problem

$$\text{minimize } g(x(T)) \tag{1}$$

over solutions of the differential inclusion

$$\begin{cases} \text{(i)} & x'(t) \in F(x(t)) \text{ for almost every } t \in [t_0, T], \\ \text{(ii)} & x(t_0) = x_0, \end{cases} \tag{2}$$

satisfying state constraints

$$x(t) \in K \quad \text{for all } t \in [t_0, T] \tag{3}$$

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where $g : \mathbb{R}^n \mapsto \mathbb{R}_+$, $T \geq t_0 \geq 0$, K is a nonempty closed subset of $\mathbb{R}^n$ and $F : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a set valued map.

The value function corresponding to the optimal control problem (1)–(3) is given by

$$V_g^K(t_0, x_0) = \inf_{\begin{cases} x'(t) \in F(x(t)), \\ x(t_0) = x_0, \\ x(t) \in K, & \text{for all } t \in [t_0, T], \end{cases}} g(x(T)) \tag{4}$$

If the value function is a differentiable function then it is the classical solution to the corresponding Hamilton–Jacobi–Bellman equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + H\left(x, \dfrac{\partial u}{\partial x}\right) = 0 & \text{for } (t, x) \in [0, T[\times \mathbb{R}^n, \\ u(T, x) = g(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{5}$$

where

$$H(x, p) := \min_{z \in F(x)} \langle z, p \rangle. \tag{6}$$

Since pioneering works of [8] the constant effort have been made by several authors to generalize the notion of solution to Hamilton–Jacobi equations in such a way that the value function V_g^K would be the unique solution to the corresponding equation (5) for a wide class of optimal control problems. Continuous viscosity solutions are widely described in is [9]. The constrained optimal control problem was studied for the first time in [16], where the value function of an infinite horizon control problem with space constraints was characterized as a continuous solution to a corresponding Hamilton–Jacobi–Bellman equation. To ensure the continuity of the value function the dynamics of the control system have to satisfy some controllability condition at the boundary of the set of space constraints.

Semicontinuous solutions and semicontinuous value functions was introduced by Barron–Jensen [5] and Frankowska [11]. Control systems with state constraints and various controllability conditions at the boundary which guarantee semicontinuity of the value function was studied in [12], [13] (see also [6] for minimal time function).

Our main aim is to characterize the value function V_g^K as the unique solution to the corresponding PDE for an arbitrary discontinuous (bounded) g without any controllability assumption.

Of course, in the fully discontinuous case the characterization is based on a suitable definition of solution we introduce below. The definition of solution we propose is strongly related with Frankowska and Barron–Jensen semicontinuous solutions and Subbotin minimax solutions [17] (called bilateral solutions in [4]).

The key point is an observation of Frankowska that some invariance property of the epigraph and/or hypograph of the value-function for control can be used to define a notion of semicontinuous solution to some Hamilton–Jacobi equation [10] using Viability Theory [1].

From now on we make the following assumption:

(H1) Function g is bounded by $M > 0$ and F is a Lipschitz continuous set-valued map with compact convex values and linear growth.

2. Mayer problem without constraints

Throughout this section we suppose $K = \mathbb{R}^n$ and we note V_g the value function (4).

2.1. Generalized solutions and basic results. We shall need the following concept of solutions related to Subbotin minimax solution ([17]):

Definition 1. Let $H : [0,T] \times \mathbb{R}^{2n} \to \mathbb{R}$ be an Hamiltonian. The function $(t,x) \mapsto u(t,x)$ is a solution to (5) if and only if

$$\text{(7)} \qquad \begin{cases} \text{(i)} & u \text{ is the supremum on the set of viscosity subsolutions} \\ & \phi \text{ such that } \phi(T,x) \le g(x), \ \text{for all } x \in \mathbb{R}^n, \\ \text{(ii)} & u \text{ is the infimum on the set of viscosity supersolutions} \\ & \psi \text{ such that } \psi(T,x) \ge g(x), \ \text{for all } x \in \mathbb{R}^n. \end{cases}$$

Let us call supersolution any lower semicontinuous function $\psi : (0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall (t,x) \in (0,T) \times \mathbb{R}^n, \ \forall (p_t, p_x) \in \partial_- \psi(t,x), \ p_t + H(t,x,p_x) \le 0,$$

and we call subsolution any upper semicontinuous function $\phi : (0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\forall (t,x) \in (0,T) \times \mathbb{R}^n, \ \forall (p_t, p_x) \in \partial_+ \phi(t,x), \ p_t + H(t,x,p_x) \ge 0.$$

Here $\partial_- \psi(t,x)$ denotes the subdifferential of ψ and $\partial_+ \phi(t,x)$ the superdifferential of ϕ (see the definition below).

Let us recall some notions and facts from nonsmooth analysis. Let $D \subset \mathbb{R}^n$ be a nonempty subset and $x_0 \in D$. The contingent cone to D at x_0, $T_D(x_0)$, is defined by

$$v \in T_K(x_0) \leftrightarrow \liminf_{h \to 0^+} \frac{\operatorname{dist}(x_0 + hv, K)}{h} = 0.$$

A polar cone T^- to a subset $T \subset \mathbb{R}^n$ is defined by

$$T^- := \{p \in \mathbb{R}^n : \ \text{for all } v \in T, \ \langle p, v \rangle \le 0\}.$$

Let $\Omega \subset \mathbb{R}^n$ be an open subset and $w : \Omega \to \mathbb{R}$ be a lower semicontinuous function. The subdifferential of w at $x_0 \in \Omega$ is given by

$$\partial_- w(x_0) = \left\{ p \in \mathbb{R}^n : \liminf_{x \to x_0} \frac{w(x) - w(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \right\}.$$

It is well known (cf. [11] for instance) that

$$p \in \partial_- w(x_0) \leftrightarrow (p, -1) \in [T_{\mathrm{Epi}(w)}(x_0, w(x_0))]^-$$

where Epi stands for the epigraph. For an upper semicontinuous function w we define a superdifferential by

$$\partial_+ w(x_0) = \left\{ p \in \mathbb{R}^n : \limsup_{x \to x_0} \frac{w(x) - w(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \leq 0 \right\}$$

and we have

$$p \in \partial_+ w(x_0) \leftrightarrow (-p, 1) \in \left[T_{\mathrm{Hypo}(w)}(x_0, w(x_0))\right]^-$$

where Hypo stands for the hypograph. When normals to Epi or hypograph are of the form $(0, p)$ the following Rockafellar result (see [11]) is of great use.

Lemma 2. *Consider a lower semicontinuous function $w : \Omega \to \mathbb{R}$ and $x_0 \in \Omega$. If $(p, 0) \in [T_{\mathrm{Epi}(w)}(x_0, w(x_0))]^-$ then there exist $x_n \to x_0$, $p_n \to p$, $q_n \to 0$, $q_n < 0$ such that*

$$(p_n, q_n) \in [T_{\mathrm{Epi}(w)}(x_n, w(x_n))]^-.$$

2.2. Discontinuous optimal control without constraints. Let us state our main result.

Theorem 3. *Let assumption* (H1) *holds. Suppose that $K = \mathbb{R}^n$. Then the value-function $V_g : (0, T] \times \mathbb{R}^n \to \mathbb{R}$ given by* (4) *is the unique generalized solution, in the meaning of Definition* 1*, to the Hamilton–Jacobi–Bellmann equation* (5)*.*

For the proof we shall use the following classical viability theorem

Proposition 4 (Viability Theorem, [1, Theorem 3.2.4]). *Assume that F satisfies* (H1) *and let $D \subset \mathbb{R}^n$ be closed. If for every $z \in D$ we have*

$$\forall p \in [T_D(z)]^-, \quad \min_{y \in F(z)} \langle y, p \rangle \leq 0 \tag{8}$$

then for every $x_0 \in D$, $t_0 < T$, there exists a solution $x(\,\cdot\,)$ to (2) *such that*

$$x(t) \in D \quad \textit{for all } t \in [t_0, T].$$

Let us state the following technical

Lemma 5. *Assume that* (H1) *hold true. Suppose that* $\psi : (0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is a lower semicontinuous supersolution to*

$$\frac{\partial u}{\partial t} + H\left(x, \frac{\partial u}{\partial x}\right) = 0 \tag{9}$$

on $(0,T) \times \mathbb{R}^n$. *Then for every* $(t_0, x_0) \in (0,T) \times \mathbb{R}^n$ *there exists a solution* $x(\cdot)$ *to* (2) *such that*

$$\psi(t_0, x_0) \geq \psi(t, x(t)), \quad \textit{for all } t \in [t_0, T]. \tag{10}$$

Proof. Fix $t_0 \in (0,T)$. We set

$$D_\psi := \mathrm{cl}(\{(t,x,r) : \ t \in (0,T], \ x \in \mathbb{R}^n, \ r \geq \psi(t,x)\}) \cup [T, \infty) \times \mathbb{R}^n,$$

$$\widetilde{F}(t,x,r) = \begin{cases} 0 & \text{if } t < 0, \\ (t/t_0)(1, F(x), 0) & \text{if } t \in [0, t_0], \\ (1, F(x), 0) & \text{if } t \in (t_0, T], \\ (1, F(x), 0) & \text{if } t > T, \end{cases}$$

where cl denoted the closure. We show that (8) holds true for F and D replaced by $\widetilde{F}$ and D_ψ.

Let $z_0 = (s_0, x_0, r_0 := \psi(t_0, x_0)) \in D_\psi$. If $s_0 = 0$ then $\widetilde{F} = 0$. Obviously, (8) holds true.

If $s_0 \geq T$ and $(p_s, p_x, p_r) \in [T_{D_\psi}(s_0, x_0, r_0)]^-$, then $p_s \leq 0$, $p_x = 0$, $p_r = 0$. Hence, (8) holds true.

It remains to consider the case $s_0 \in (0,T)$. We have $[T_{D_\psi}(s_0, x_0, r_0)]^- \subset [T_{D_\psi}(s_0, x_0, \psi(s_0, x_0))]^-$.

Let $(p_s, p_x, p_r) \in [T_{D_\psi}(s_0, x_0, \psi(s_0, x_0))]^-$. If $p_r < 0$ then $(p_s/-p_r, p_x/-p_r) \in \partial_- \psi(s_0, x_0)$. Since ψ is a supersolution to (9) we have

$$\frac{p_s}{-p_r} + \min_{y \in (F(x_0)} \left\langle y, \frac{p_x}{-p_r} \right\rangle \leq 0.$$

Hence

$$\min_{\widetilde{y} \in \widetilde{F}(t_0, x_0, r_0)} \langle \widetilde{y}, (p_s, p_x, p_r) \rangle \leq 0.$$

Now, we consider the case $p_r = 0$. By Lemma 2, there exist $s_n \to s_0$, $x_n \to x_0$, $p_{sn} \to p_s$, $p_{xn} \to p_x$, $p_{rn} \to 0$, $p_{rn} < 0$ such that

$$(p_{sn}, p_{xn}, p_{rn}) \in [T_{\mathrm{Epi}(\psi)}(s_n, x_n, \psi(t_n, x_n))]^-.$$

Since $p_{rn} < 0$, from the previous case we obtain

$$\min_{\widetilde{y_n} \in \widetilde{F}(t_n, x_n, r_n)} \langle \widetilde{y}_n, (p_{sn}, p_{xn}, p_{rn}) \rangle \leq 0.$$

Since $\widetilde{F}$ is upper semicontinuous compact valued, we have

$$\min_{\widetilde{y}\in\widetilde{F}(t_0,x_0,r_0)} \langle \widetilde{y}, (p_s,p_x,p_r\rangle \leq 0.$$

In view of Proposition 4, there exists a solution $z(\,\cdot\,)$ to the Cauchy problem

$$z'(s) = \widetilde{F}(z(s)), \quad z(t_0) = z_0$$

such that $z(s)$ belongs to D_ψ for every $s \in [t_0,T]$. Let $z(s) = (t(s), x(s), r(s))$. By the definition of $\widetilde{F}$, we have $t(s) = s$, $r(s) = r_0 = \psi(t_0,x_0)$. It yields (10) for $t \in [t_0,T)$. Since ψ is lower semicontinuous, we obtain (10) for $t = T$. □

Using an Invariance Theorem ([1, Theorem 5.2.1]), one can prove using the same method the following technical

Lemma 6. *Assume that* (H1) *hold true. Suppose that* $\phi : (0,T] \times \mathbb{R}^n \to \mathbb{R}$ *is upper semicontinuous and is a subsolution to* (9) *on* $(0,T)\times\mathbb{R}^n$. *Then for every* $(t_0,x_0) \in (0,T)\times\mathbb{R}^n$ *for all solution* $x(\,\cdot\,)$ *to* (2) *such that*

$$\phi(t_0,x_0) \leq \phi(t,x(t)), \quad \textit{for all } t \in [t_0,T]. \tag{11}$$

Proof of Theorem 3. Let ψ be a lower semicontinuous supersolution such that $\psi(T,x) \geq g(x)$. In view of Proposition 5

$$\psi(t_0,x_0) \geq \psi(T,x(T)) \geq g(x(T)) \geq V(t_0,x_0).$$

In a similar way using Lemma 6, if ϕ is a upper semicontinuous subsolution such that $\phi(T,x) \leq g(x)$ then for all solutions $x(\,\cdot\,)$ to (2)

$$\phi(t_0,x_0) \leq \phi(T,x(T)) \leq g(x(T)).$$

Taking the infimum over all solutions $x(\,\cdot\,)$ to (2) yields $\phi(t_0,x_0) \leq V(t_0,x_0)$. Hence

$$\phi(t_0,x_0) \leq V(t_0,x_0) \leq \psi(t_0,x_0). \tag{12}$$

It remains to prove that $V(t_0,x_0)$ is the supremum of such $\phi(t_0,x_0)$ and the infimum of such $\psi(t_0,x_0)$. The end of the proof is divided into 2 steps.

Step 1. Suppose that g is upper semicontinuous. We define a sequence of functions $g_n : \mathbb{R}^n \to \mathbb{R}$ by

$$g_n(x) := \sup_{y\in\mathbb{R}^n} g(y) - n\|x-y\|.$$

Recall that functions g_n are lipschitz continuous, $g_n(x) \geq g_{n+1}(x)$ and $\lim_n g_n(x) = g(x)$ for every $x \in \mathbb{R}^n$. Consider value functions V_{g_n} which form clearly a decreasing sequence which limit is denoted by $W := \lim_n V_{g_n}$. From one hand, $g_n \geq g$ yields

$$W(t_0,x_0) \geq V_g(t_0,x_0). \tag{13}$$

From the other hand, V_{g_n} is the unique (Lipschitz) viscosity solution ([9]) to (9) with Lipschitz boundary condition g_n. So W appears to be a decreasing limit of subsolutions. In view of Theorem 4.1 in [3], W is a subsolution to (9). Because $W(T, \cdot) = g(\cdot)$,

$$W(t_0, x_0) = V_g(t_0, x_0)$$

follows from inequality (13) and Lemma 6. This proves 7(i).

Step 2. Suppose that g is an arbitrary function bounded by M. Fix $(t_0, x_0) \in (0, T] \times \mathbb{R}^n$. Let $\varepsilon > 0$. There exists x_ε solution to (2) such that

$$g(x_\varepsilon(T)) < V_g(t_0, x_0) + \varepsilon.$$

We define $h_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ by

$$h_\varepsilon(x) := \begin{cases} g(x) & \text{for } x = x_\varepsilon(T), \\ M & \text{for } x \neq x_\varepsilon(T). \end{cases}$$

Obviously, h is lower semicontinuous. By [11], V_{h_ε} is a supersolution to (9). Hence $V_{h_\varepsilon}(t_0, x_0) < V_g(t_0, x_0) + \varepsilon$. From (12), it may be concluded that

$$V_g(t_0, x_0) = \inf\{\psi(t_0, x_0) : \ \psi \text{ is a supersolution to (9)}, \ \psi(T, \cdot) \geq g(\cdot)\}.$$

Define the reachable set

$$R_F(t_0, x_0; T) := \{x(T) \, : \, x(\cdot) \text{ solution to (2)}\}$$

and the function $l : \mathbb{R}^n \to \mathbb{R}$ by

$$l(x) := \begin{cases} V_g(t_0, x_0) & \text{if } x \in R_F(t_0, x_0; T), \\ -M & \text{if } x \notin R_F(t_0, x_0; T). \end{cases}$$

By (H1), the reachable set $R_F(t_0, x_0; T)$ is closed. Therefore l is upper semicontinuous. Obviously, we have $V_g(t_0, x_0) = V_l(t_0, x_0)$. By step 1, the proof is complete. □

3. Discontinuous optimal control with state-constraint

We are interested in the PDE characterization of the value function V_g^K where K is an arbitrary closed subset of $\mathbb{R}^n$. The minimal requirement guaranteeing the function V_g^K to be well defined by formula (4) is

$$\text{(14)} \qquad \begin{cases} \text{for any initial condition } (t_0, x_0) \in [0, T] \times K \\ \text{there exists a solution } x(\cdot) \text{ to (2)} \\ \text{remaining in the set of constraints } K \text{ for every } t \in [t_0, T]. \end{cases}$$

Above property (14) called the viability property can be characterized by a geometrical condition in the way of Proposition 4.

Proposition 7. *Let $K \subset \mathbb{R}^n$ be closed and $g : \mathbb{R}^n \mapsto \mathbb{R}$ be a function bounded by $M > 0$. Assume that* (14) *and* (H1) *hold true. Then*

$$V_g^K(t,x) = U(t,x,0) \quad \text{for } x \in K$$

where $U : [0,T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is the unique solution to

$$\begin{cases} \dfrac{\partial U}{\partial t} + \widetilde{H}\left(x, y, \dfrac{\partial U}{\partial x}, \dfrac{\partial U}{\partial y}\right) = 0, \\ U(T,x,y) = g(x) + (2M+1)\chi_{(0,\infty)}(y), \end{cases} \tag{15}$$

where $\widetilde{H}(x,y,p_x,p_y) := H(x,p_x) + d_K(x)p_y$ and $\chi_{(0,\infty)}$ denotes the characteristic function of the open interval $(0,\infty)$.

Proof. The proof is based on a penalization of the cost for an augmented differential inclusion. We consider the following extended differential inclusion

$$\begin{cases} x'(t) \in F(x(t)), \\ y'(t) = d_K(x(t)), \end{cases}$$

where $d_K(x)$ denotes the distance from x to K. Obviously this differential inclusion satisfies (H1). By Theorem 3, we obtain that the function

$$U(t_0,x_0,y_0) = \inf_{\substack{\begin{cases} x'(t) \in F(x(t)), \\ x(t_0) = x_0, \end{cases}}} g(x(T)) + (2M+1)\chi_{(0,\infty)}\left(y_0 + \int_{t_0}^T d_K(x(t))\,dt\right)$$

is the unique generalized solution to (15). From the very definition, one can easily check that for every $x_0 \in K$ we have

$$V_g^K(t_0,x_0) = U(t_0,x_0,0). \qquad \square$$

Remarks. When g is lower semicontinuous, the new cost

$$g(x) + (2M+1)\chi_{(0,\infty)}(y)$$

is also lower semicontinuous, so Proposition 7 is still valid using Frankowska's concept of solutions instead of Definition 7. Results of this paper can be also obtained for nonconvex Hamiltonian [14].

Example 1. Let $g : \mathbb{R} \to \mathbb{R}$ be the characteristic function of rationals. The dynamics is given by an ODE $x' = f(x)$ where f is a lipschitz function. In this case the value $V(t_0,x_0) = g(x(T;\, t_0,x_0))$ is discontinuous at every point. Despite of this, by Theorem 3, V is the unique solution (in the sense of Definition 1) of the corresponding problem (9). Let us remark, that the concepts of solution from [1] and [17] do not apply to this example.

Example 2. We consider a Mayer problem for the control system

$$x'(t) \in [-1, 1]$$

and a terminal cost function $g : R \to R$ given by $g(x) = 1$ if $x \neq 0$ and $g(0) = 0$. One can check that $V(t, x) = 1$ if $t \in (0, T)$ and $V(T, x) = g(x)$. The Bellman equation corresponding to the problem is

$$V_t + |V_x| = 0 \quad \text{in } R \times (0, T).$$

In [BJ] the approach to the function is "from the one side only". So to avoid "a jump" at terminal time one have to assume that

$$g(x) = \limsup_{y \to x, t \to T^-} V(t, x).$$

Let us remark that Barron–Jensen approach can be used for a convex Hamiltonian and lower semicontinuous terminal cost (or concave Hamiltonian and upper semicontinuous terminal cost); here this approach is not applicable because H is convex and g is upper semicontinuous.

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