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USE FINITE FAMILY OF MULTIVALUED MAPS FOR CONSTRUCTING STABLE ABSORPTION OPERATOR

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Dedicated to the memory of Juliusz P. Schauder

Abstract. The differential game of pursuit-evasion over a fixed time segment is considered. the problem of construction of the stable absorption operator of control system is investigated. The attainability sets is appointed with the help of the stable absorption operator. The partition of the conjugate space on the finite regions of convexity of Hamiltonian is used for constructing stable absorption operator.

1. Problem set

Let us consider a conflict controlled system which dynamics over a time segment $[0, \vartheta]$ is described by the equation

$$dx/dt = f(t, x, u, v), \quad x[0] = x_0, \quad t \in [0, \vartheta], \ x \in \mathbb{R}^m \tag{1.1}$$

is the phase vector of the system, u and v are control vectors of 1st and 2nd players, $u \in P \subset \mathbb{R}^p$, $v \in Q \subset \mathbb{R}^q$, P and Q are compacts.

Let the following conditions are satisfied:

(A) The game takes place in bounded closed region D of variables $(t, x) \in [t_0, \vartheta] \times \mathbb{R}^m$.

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(B) $\|f(t, x^{(2)}, u, v) - f(t, x^{(1)}, u, v)\| \leq L\|x^{(2)} - x^{(1)}\|$ for all $(t, x^{(i)}, u, v) \in D \times P \times Q,\ i = 1, 2$.

(C) Motions $x(t)$ of the system (1.1) are continued on the segment $[t_0, \vartheta]$.

For the system (1.1) the pursuit-evasion problem with a target set $M \in \mathbb{R}^m$ to the fixed time ϑ is regarded.

2. Stable absorption operator

The solution of above problem can be determind as strategies, which are extremal to the stable bridge [6]–[8]. So the constructing stable bridge is important element of solving the problem. Its definition is based on the stability property and can be done, for example, using unification scheme (see [5]). The unification scheme is useful not only in differential games theory. In [10]–[12] is applied for determining generalized solutions of Hamilton–Jacobi equations. On the base of unification scheme the numerical procedures for constructing stable bridges and value function of differential game can be done.

In a common case in the unification scheme the dynamics of controlled system is described by some collection of sets $F_l(t, x),\ l \in S$, where S is unit sphere, i.e. the set of parameters of the collection is infinite. While numerical solving differential games the unit sphere is replaced by finite ε-gride.

Besides, there is a set of problem, which assumes schemes, based on the idea of unification and having a finite number of parameters. For example, let the controlled system is described by equation

$$dx/dt = f(t, x) + A(t, x)u + B(t, x)v, \tag{2.1}$$

where $u \in P,\ v \in Q$, P and Q are polyhedrones in Euclidean spaces $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Then we can replacethe infinite collection of sets $F_l(t, x),\ l \in S$ by finite collection $G_{q_k}(t, x) = f(t, x) + A(t, x)P + B(t, x)q_k$, where $\{q_k\}$ is the set of vertexes of polyhedrone Q.

In [14], [4], [1] more generalized definition of stability was given. It use the notion of the stable absorption operator, which is determined on the base of family of maps $\{F_\psi :\ D \to 2^{\mathbb{R}^m}\}$ (where $D = \mathbb{R}^m \times [t_0, \vartheta]$ and $\psi \in \Psi$), satisfying some special conditions, namely:

(A.1) for all $(t, x, \psi) \in D \times \Psi$ the set $F_\psi(t, x)$ is compact and uniform bounded;

(A.2) for all $(t, x, l) \in D \times S$ the equality is valid

$$\min_{\psi \in \Psi} \max_{f \in F_\psi(t,x)} \langle f, l\rangle = H(t, x, l)$$

where $H(t, x, l) = \max_{u \in P}\ \min_{v \in Q}\ \langle l, f(t, x, u, v)\rangle$ is Hamiltonian of controlled system.

(A.3) There exists a function $\omega^*(\delta)$ ($\omega^*(\delta) \downarrow 0$ when $\delta \downarrow 0$) such that for all (t_*, x_*), $(t^*, x^*) \in D$, and all $\psi \in \Psi$ the inequality is valid

$$\operatorname{dist}(F_\psi(t^*, x^*), F_\psi(t_*, x_*)) \le \omega^*(|t^* - t_*| + \|x^* - x_*\|)$$

where $\operatorname{dist}(F, G)$ is the Hausdorf distanse between F and G sets.

Let $Z^* \subset \mathbb{R}^m$. Denote: $X_\psi(t^*; t_*, x_*)$ is the attainability set of differential inclusion

$$\frac{dx}{dt} \in F_\psi(t, x), \quad x[t_*] = x_*,$$

to the moment $t^* \in (t_*, \vartheta]$ (see Figure 1);

$$X_\psi^{-1}(t_*; t^*, Z^*) = \{x_* \in \mathbb{R}^m : \ X_\psi(t^*; t_*, x_*) \cap Z^* \neq \emptyset\}.$$

The set $X_\psi^{-1}(t_*; t^*, Z^*)$ consist of points x_*, such that the attainability sets of corresponding differential inclusion with initial conditions (x_*, t_*) to the moment t^* intersect with Z^* (see Figure 2).

DEFINITION 1. Call by stable absorption operator $\pi = \pi(t_*; t^*, Z^*)$, $(t_0 \le t_* < t^* \le \vartheta,\ Z^* \subset \mathbb{R}^m)$ in the problem of pursuit with the target M to the moment ϑ the map π, which is determined by equation

$$\pi(t_*; t^*, Z^*) = \bigcap_{\psi \in \Psi} X_\psi^{-1}(t_*; t^*, Z^*).$$

DEFINITION 2. Call closed set $W \subset D$ by u-stable bridge in the problem of pursuit with the target M to the moment ϑ, if

(1) $W_\vartheta \subset M$,

(2) $W_{t_*} \subset \pi(t_*, t^*, W_{t^*})$ for all $t_*, t^* \in [t_0, \vartheta]$, $t_* < t^*$.

Here $W_t = \{x \in \mathbb{R}^m : (t, x) \in W\}$.

The assertion is true [13], that the set W_0 of positional absorption in the problem of pursuit with the target M to the fixed time ϑ is the maximal u-stable bridge.

3. Constructing family of multivalued maps, having finite numbers of parameters

As there is said above, the definitions, described here, are applied while constructing numerical solutions of differential games. That is why we are interesting in reducing calculating by reducing cardinality of set Ψ. We are interesting in constructing a scheme with finite number of parameters.

The method, used in the paper, is the coagulation of unification scheme to some finite collection of sets through the partition of the conjugate space on such cones, that Hamiltonian, regarding as function of conjugate variable, is convex on every cone. The cones of convexity of Hamiltonian were used first by Patsko and his collaborates while studying linear differential games of second order [3], [9].

Let us construct, for every $(t,x) \in D$, the finite collection of closed subsets $L_\psi(t,x)$ ($\psi \in \Psi$) of a unit sphere S, such that for every $L_\psi(t,x)$ following conditions satisfy:

(B.1) For all $(t,x) \in D$ the cone $K(L_\psi(t,x))$ is a convex set.

(B.2) For all $(t,x) \in D$ Hamiltonian $H(t,x,l)$ of controlled system is a convex function on $K(L_\psi(t,x))$ by the variable l.

(B.3) There exists a function $\overline{\omega}(\delta)$ ($\overline{\omega}(\delta) \downarrow 0$ when $\delta \downarrow 0$) such that for all (t_*,x_*) and (t^*,x^*) from D and for all $\psi \in \Psi$ the following inequality takes place:

$$d(L_\psi(t_*,x_*), L_\psi(t^*,x^*)) \le \overline{\omega}(|t^* - t_*| + \|x^* - x_*\|).$$

Here $K(L_\psi(t,x)) = \{l' : l' = \lambda l,\ \lambda > 0,\ l \in L_\psi(t,x)\}$ is a cone pulled over the set $L_\psi(t,x)$. At that $\bigcup_{\psi\in\Psi} L_\psi(t,x) = S,\ (t,x) \in D$, and some of them can intersect.

Let's introduce some definitions (see Figure 1).

$$F(t,x) = \mathrm{co}\{f(t,x,u,v) : u \in P,\ v \in Q\},$$
$$\Pi_l(t,x) = \{f \in \mathbb{R}^m : \langle l, f\rangle \le H(t,x,l)\},$$
$$A_\psi(t,x) = \bigcap_{l\in L_\psi(t,x)} \Pi_l(t,x),$$
$$F_\psi(t,x) = A_\psi(t,x) \cap F(t,x). \tag{3.1}$$

Let also that the following condition takes place:

(B.4) $\operatorname{int} F_\psi(t,x) \ne \emptyset$ for all $(t,x,\psi) \in D \times \Psi$.
$\operatorname{int} F$ is the set of interior points of F.

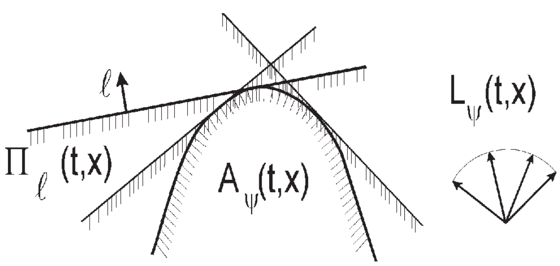

Figure 1

The following theorem was proved.

Theorem 1. *The family of maps* $\{F_\psi : D \to 2^{\mathbb{R}^m}\}$*, which is determined by equalities* (3.1) *satisfies conditions* (A.1)–(A.3)

Proof. Lets take into consideration (see Figure 2)

$$\Lambda^0_\psi(t,x) = \partial A_\psi(t,x) \cap \operatorname{int} F(t,x),$$
$$L^0_\psi(t,x) = \operatorname{cl}\{l \in S : \langle l, z\rangle = h_{A_\psi(t,x)}(l),\ z \in \Lambda^\psi_0(t,x)\}$$

Here ∂A and $\operatorname{cl} A$ are the boundary and the closure of the set A respectively, $h_A(l) = \max_{a\in A}\langle l, a\rangle$ is the supporting function of the set A.

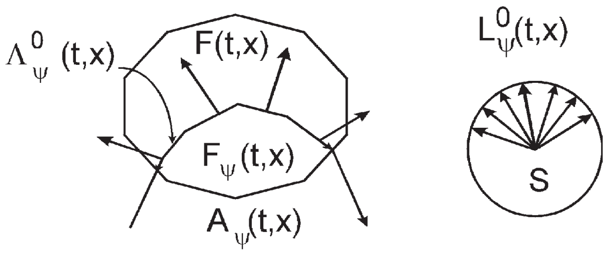

Figure 2

Lets appoint the basic facts, which are used while proving.

(U.1) For any point $(t,x) \in D$ and all vectors l, s ($(l,s) \in S\times S$) the inequality holds
$$h_{F_l(t,x)}(l) \le h_{F_s(t,x)}(l).$$
This fact is formulated in [6], and it can be called by extremal property of the system $\{F_l(t,x) : l \in S\}$. It means, that for every fixed $l \in S$ the minimum of the function $\xi(s) = h_{F_s(t,x)}$ is attained for $s = l$.

(U.2) Let together with the set $F_\psi(t,x)$ (for some fixed $\psi \in \Psi$), there exists a convex compact Φ in $F(t,x)$, such that $F_\psi(t,x) \cap \Phi = \emptyset$. Then there exists hyperplane $\Gamma^{\alpha^0}_{l_0} = \{z \in \mathbb{R}^m : \langle l^0, z\rangle = \alpha^0\}$, $l^0 \in L^0_\psi(t,x)$, $\alpha^0 \in (-\infty,\infty)$, which is supporting to the set $F_\psi(t,x)$ and is separating strictly sets $F_\psi(t,x)$ and Φ.

The essence of (U.2) is that the restriction on non-intersecting convex closed sets $F_\psi(t,x)$ and Φ allows to yield the addition information about position of some of hyperplanes, separating the sets.

Turn to the proving the theorem.

Proof of (A.1). Convex and closeness of the set $F_\psi(t,x)((t,x,\psi) \in D\times\Psi)$ follows from definition of $F_\psi(t,x)$ as intersection of convex and closed sets. Inclusion $F_\psi(t,x) \subset G$ follows from inclusions $F_\psi(t,x) \subset F(t,x)$ and $F(t,x) \subset G$. □

Proof of (A.2). Let (t,x,l) is arbitrary point in $D\times S$. We need to prove, that $\min_{\psi\in\Psi} h_{F_\psi(t,x)(l)=H(t,x,l)}$

(1) Lets prove the inequality

$$\min_{\psi\in\Psi} h_{F_\psi(t,x)}(l) \le H(t,x,l). \tag{3.2}$$

For some $\psi \in \Psi$ the inclusion $l \in L_\psi(t,x)$ holds. $A_\psi(t,x) \subset \Pi_l(t,x)$, and $F_\psi = F(t,x) \cap A_\psi(t,x)$ hence, $F_\psi(t,x) \subset F(t,x) \cap \Pi_l(t,x) = F_l(t,x)$. Then $h_{F_\psi(t,x)}(l) \le h_{F_l(t,x)}(l) = H(t,x,l)$. The last inequality implies (3.2).

(2) Lets prove, that

$$\min_{\psi\in\Psi} h_{F_\psi(t,x)}(l) \ge H(t,x,l). \tag{3.3}$$

Assuming contrary to the (3.3), namely,

$$\min_{\psi\in\Psi} h_{F_\psi(t,x)}(l) < H(t,x,l), \tag{3.4}$$

and taking into account, that the set Ψ is finite, we conclude, that there exists $\psi^* \in \Psi$ such, that

$$h_{F_{\psi^*}(t,x)}(l) < H(t,x,l). \tag{3.5}$$

Then, supposing that $\Phi = \{z \in F(t,x) : \langle l,z\rangle \ge H(t,x,l)\}$, we derive $\Phi \cap F_{\psi^*}(t,x) = \emptyset$.

In accordance with (U.2), there exist $l^0 \in L^0_{\psi^*}(t,x)$ and $\alpha^0 \in (-\infty,\infty)$ such that hyperplane $\Gamma^{\alpha^0}_{l^0}$, supporting to $F_\psi * (t,x)$, separates strictly sets $F_{\psi^*}(t,x)$ and Φ.

We need to prove, that $l^0 \in L_{\psi^*}(t,x)$ $l^0 \in L^0_\psi(t,x)$, then there exists the point $z^0 \in \operatorname{cl}\Lambda^0_{\psi^*}(t,x)$, such that the hyperplane $\Gamma^{\alpha^0}_{l^0}$ is supporting for $F_{\psi^*}(t,x)$, hence

$$\langle l^0, z^0\rangle = h_{F_{\psi^*}(t,x)}(l^0). \tag{3.6}$$

Let $z^0 \in \Lambda^0_{\psi^*}(t,x) = \partial A_{\psi^*}(t,x) \cap \operatorname{int} F(t,x)$. Then the equality is true

$$\langle l^0, z^0\rangle = h_{A_{\psi^*}(t,x)}(l^0). \tag{3.7}$$

Otherwise we would find the point $a \in A_{\psi^*}(t,x)$, such that (see Figure 3)

$$\langle l^0, z^0\rangle < \langle l^0, a\rangle \tag{3.8}$$

and, furthemore, for all points $a(\lambda) = z^0 + \lambda(a - z^0)$, $\lambda \in (0,1]$, were λ is small enough, the relations would be true

$$h_{F_{\psi^*}(t,x)}(l^0) = \langle l^0, z^0\rangle < \langle l^0, a(\lambda)\rangle,$$
$$a(\lambda) \in \operatorname{int} F(t,x) \cap A_{\psi^*}(t,x) \subset F_{\psi^*}(t,x).$$

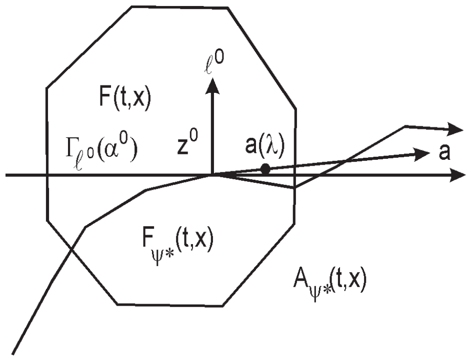

Figure 3

These relations contradict one to other. Thus, if $z^0 \in \Lambda^0_{\psi^*}(t,x)$, then (3.7) is true.

Let $z^0 \notin \partial\Lambda^0_{\psi^*}(t,x)$. Then, accordinly the definition of $\Lambda^0_{\psi^*}(t,x)$, $z^0 = \lim_{k\to\infty} z^0_k$ and $l^0 = \lim_{k\to\infty} l^0_k$, where $\{z^0_k\}$ and $\{l^0_k\}$ — are some consequences, such that $z^0_k \in \Lambda^0_{\psi^*}(t,x)$ and $l^0_k \in L^0_{\psi^*}(t,x)$, at that $\langle l^0_k, z^0_k\rangle = h_{F_{\psi^*}(t,x)}(l^0_k)$. Passing to the limits $z^0 = \lim_{k\to\infty} z^0_k$, $l^0 = \lim_{k\to\infty} l^0_k$ and taking to account $\langle l^0_k, z^0_k\rangle = h_{A_{\psi^*}(t,x)}(l^0_k)$ and the continuity property of supporting function $h_{A_{\psi^*}(t,x)}(l)$, we conclude, that in the case $z^0 \notin \partial\Lambda^0_{\psi^*}(t,x)$ the (3.7) also is true.

Thus, if vector $l^0 \in L_{\psi^*}(t,x)$ and point $z^0 \in \Lambda^0_{\psi^*}(t,x)$ satisfy (3.6), then (3.7) holds.

Let's denote by $K^{\perp}(L_{\psi^*}(t,x))$ the polar cone of $\widetilde{K}(L_{\psi^*}(t,x))$, that is $K^{\perp}(L_{\psi^*}(t,x)) = \{z : \langle z, z^*\rangle \le 0,\ z^* \in \widetilde{K}(L_{\psi^*}(t,x))\}$. And prove, that

$$z^0 + K^{\perp}(L_{\psi^*}(t,x)) = \{z^0 + z : z \in K^{\perp}(L_{\psi^*}(t,x))\} \subset A_{\psi^*}(t,x). \tag{3.9}$$

We need to show, that any point $z^* = z^0 + z,\ z \in K^{\perp}(L_{\psi^*}(t,x))$ is contained in $A_{\psi^*}(t,x)$. Since for all $l \in L_{\psi^*}(t,x)$ following relations are true $\langle l, z^*\rangle = \langle l, z^0 + z\rangle = \langle l, z^0\rangle + \langle l, z\rangle \le \langle l, z^0\rangle \le H(t,x,l)$ (that is $\langle l, z^*\rangle \le H(t,x,l)$), and $A_{\psi^*}(t,x) = \bigcap_{l\in L_{\psi^*}(t,x)} \Pi_l(t,x)$, hence (3.9) is valid.

Turn to the provinf inclusion $l^0 \in L_{\psi^*}(t,x)$. Assuming contrary: $l^0 \notin L_{\psi^*}(t,x)$. Since $K^{\perp}(L_{\psi^*}(t,x))$ is polar cone for $\widetilde{K}(L_{\psi^*}(t,x))$, then $\widetilde{K}(L_{\psi^*}(t,x))$ is polar cone for $K^{\perp}(L_{\psi^*}(t,x))$, and relation $l^0 \notin L_{\psi^*}(t,x)$ means, that there exists point $z^* \in K^{\perp}(L_{\psi^*}(t,x))$, which satisfy inequality $\langle z^*, l^0\rangle > 0$.

Further, taking into account (3.9), for points $z(\lambda) = z^0 + \lambda z^*$, $\lambda > 0$,

$$z(\lambda) \in z^0 + K^{\perp}(L_{\psi^*}(t,x)) \subset A_{\psi^*}(t,x) \tag{3.10}$$

is valid. In other side, for these points the folowing relation holds

$$\langle l^0, z(\lambda)\rangle = \langle l^0, z^0\rangle + \lambda\langle l^0, z^*\rangle > \langle l^0, z^0\rangle = h_{A_{\psi^*}(t,x)}(l^0),$$

which means, that

$$z(\lambda) \notin A_{\psi^*}(t,x). \tag{3.11}$$

(3.10) and (3.11) contradict one to other, hence, assumption $l^0 \notin L_{\psi^*}(t,x)$ is not true. The inclusion $l^0 \in L_{\psi^*}(t,x)$ is shown.

Considering the last inclusion, and taking into account coincidence on the cone $\widetilde{K}(L_{\psi^*}(t,x))$ values of functions $h_{F_{\psi^*}(t,x)}(l)$ and $H(t,x,l)$, we derive $\Gamma^{\alpha^0}_{l^0} = \{z \in \mathbb{R}^m : \langle l^0, z\rangle = H(t,x,l^0)\}$. It means, that $\min_{z\in\Phi}\langle l^0, z\rangle > H(t,x,l^0) = h_{F_{l^0}(t,x)}(l^0)$. That is $\Phi \cap F_{l^0}(t,x) = \emptyset$. Then $h_{F_{l^0}(t,x)}(l) < H(t,x,l)$, that is $h_{F_{l^0}(t,x)}(l) < h_{F_l(t,x)}(l)$ for some $(t,x) \in D$, $l \in S$, $l^0 \in S$. The last inequality contradicts (U.1). Hence, (3.4) can't takes place, and (3.3) is proved. (3.2) and (3.3) together prove (A.2). □

Proof of (A.3). Lets take into account function

$$\omega^*(\delta) = \sup_{(\psi,t_*,x_*,t^*,x^*)\in Y(\delta)} d(F_\psi(t^*,x^*), F_\psi(t_*,x_*)),$$

where $Y(\delta) = \{(\psi,t_*,x_*,t^*,x^*) : \ \psi \in \Psi,\ (t_*,x_*) \in D,\ (t^*,x^*) \in D,\ |t^* - t_*| + \|x^* - x_*\| \le \delta\}$, $\delta > 0$. The function $\omega^*(\delta)$ satisfies (A.3). We need to show, that $\omega^*(\delta) \downarrow 0$ while $\delta \downarrow 0$. Really, supposing contradiction and taking into account finiteness of the set Ψ, we derive, that there exist such $\varepsilon > 0$, $\widehat{\psi} \in \Psi$ and sequences $\{(t^*_k, x^*_k)\}$, $\{(t^0_k, x^0_k)\}$ from D, that $|t^*_k - t^0_k| + \|x^*_k - x^0_k\| \downarrow 0$ while $k \to \infty$, and

$$d(F_{\widehat{\psi}}(t^*_k, x^*_k), F_{\widehat{\psi}}(t^0_k, x^0_k)) \ge \varepsilon > 0. \tag{3.12}$$

Without less of community, will consider that there exists

$$\lim_{k\to\infty}(t^*_k, x^*_k) = \lim_{k\to\infty}(t^0_k, x^0_k) = (\widetilde{t}, \widetilde{x}) \in D. \tag{3.13}$$

Show, that equalities are true

$$\lim_{k\to\infty} F_{\widehat{\psi}}(t^*_k, x^*_k) = F_{\widehat{\psi}}(\widetilde{t}, \widetilde{x}), \quad \lim_{k\to\infty} F_{\widehat{\psi}}(t^0_k, x^0_k) = F_{\widehat{\psi}}(\widetilde{t}, \widetilde{x}). \tag{3.14}$$

Here convergence of the sets means the convergence in the Hausdorff metric.

Prove the first equality from (3.14). Let $z^* \in \lim_{k\to\infty} F_{\widehat{\psi}}(t^*_k, x^*_k)$. For some sequence $\{z_k \in F_{\widehat{\psi}}(t^*_k, x^*_k)\}$ the equality $z^* = \lim_{k\to\infty} z_k$ holds.

Vectors z_k satisfy relations

(1) $\langle l, z_k\rangle \le H(t^*_k, x^*_k, l)$ for any $l \in L_{\widehat{\psi}}(t^*_k, x^*_k)$,
(2) $z_k \in F(t^*_k, x^*_k)$.

Take arbitrary vector $\widetilde{l} \in L_{\widehat{\psi}}(\widetilde{t}, \widetilde{x})$ and such sequence $\{l_k \in L_{\widehat{\psi}}(t^*_k, x^*_k)\}$, that $\lim_{k\to\infty} l_k = \widetilde{l}$. Because of the supposition (B.3) such sequence exists. The inequality $\langle l_k, z_k\rangle \leq H(t_k^*, x_k^*, l_k)$ is true. Taking into account continuity of $H(t,x,l)$ by (t,x,l) and $F(t,x)$ on (t,x), derive

(1) $\langle \widetilde{l}, z^*\rangle \leq H(\widetilde{t}, \widetilde{x}, \widetilde{l})$,

(2) $z^* \in F(\widetilde{t}, \widetilde{x})$,

that is $z^* \in F_{\widehat{\psi}}(\widetilde{t}, \widetilde{x})$. Hence

$$\lim_{k\to\infty} F_{\widehat{\psi}}(t_k^*, x_k^*) \subset F_{\widehat{\psi}}(\widetilde{t}, \widetilde{x}). \tag{3.15}$$

Let's prove the opposite inclusion

$$\lim_{k\to\infty} F_{\widehat{\psi}}(t_k^*, x_k^*) \supset F_{\widehat{\psi}}(\widetilde{t}, \widetilde{x}). \tag{3.16}$$

Assume contrary to the (3.16), namely,

$$\operatorname{int} F_{\widehat{\psi}}(\widetilde{t}, \widetilde{x}) \setminus \lim_{k\to\infty} F_{\widehat{\psi}}(t_k^*, x_k^*) \neq \emptyset. \tag{3.17}$$

Let $z^* \in \operatorname{int} F_{\widehat{\psi}}(\widetilde{t}, \widetilde{x}) \setminus \lim_{k\to\infty} F_{\widehat{\psi}}(t_k^*, x_k^*)$. Then for some small enough $\delta > 0$ there is valid $O_\delta(z^*) \subset F_{\widehat{\psi}}(\widetilde{t}, \widetilde{x}) \setminus \lim_{k\to\infty} F_{\widehat{\psi}}(t_k^*, x_k^*)$, therefore

$$\langle l, z^*\rangle \leq H(\widetilde{t}, \widetilde{x}, l) - \delta \tag{3.18}$$

for all $l \in L_{\widehat{\psi}}(\widetilde{t}, \widetilde{x})$. The set $O_\delta(z^*)$ satisfies

$$O_\delta(z^*) \subset F(\widetilde{t}, \widetilde{x}). \tag{3.19}$$

Moreover, since $H(t,x,l)$ is uniform continuous on compact $D \times S$ and (3.13) holds, then k_0 and $\sigma > 0$ exist, such that for all $k \geq k_0$, $l \in S$, $l^* \in S$, $\|l - l^*\| \leq \sigma$ the inequality holds

$$H(\widetilde{t}, \widetilde{x}, l) - \delta/2 \leq H(t_k^*, x_k^*, l^*). \tag{3.20}$$

Then (3.18) and (3.20) gather, that k_0 and $\sigma > 0$ exist, such that for all $k \geq k_0$, $l \in L_{\widehat{\psi}}(\widetilde{t}, \widetilde{x})$, $l^* \in S$, $\|l - l^*\| \leq \sigma$, the inequality holds

$$\langle l, z^*\rangle \leq H(t_k^*, x_k^*, l^*) - \delta/2. \tag{3.21}$$

Then, since (B.3), k_1 exists, such that for all $k \geq k_1$ the inequality holds

$$d(L_{\widehat{\psi}}(t_k^*, x_k^*),\ L_{\widehat{\psi}}(\widetilde{t}, \widetilde{x})) \leq \min(\delta/2K, \sigma)$$

where $K = \max_{(t,x,u,v)\in D\times P\times Q} \|f(t,x,u,v)\|$. Hence, for any $k \geq k_1$, $l_k^* \in L_{\widehat{\psi}}(t_k^*, x_k^*)$ vector $l_k \in L_{\widehat{\psi}}(\widetilde{t}, \widetilde{x})$ exists, such that

$$\langle l_k^*, z^*\rangle - \langle l_k, z^*\rangle \leq \|l_k^* - l_k\| \|z^*\| \leq \delta/2, \quad \|l_k^* - l_k\| \leq \sigma. \tag{3.22}$$

Taking into account (3.21) and (3.22), we can conclude, that for any $k \geq \max\{k_0, k_1\}$, $l_k^* \in L_{\widehat{\psi}}(t_k^*, x_k^*)$ l_k exists, such that $l_k \in L_{\widehat{\psi}}(\widetilde{t}, \widetilde{x})$, $(\|l_k - l_k^*\| \leq \sigma)$ and

$$\langle l_k, z^*\rangle + (\langle l_k^*, z^*\rangle - \langle l_k, z^*\rangle) \leq H(t_k^*, x_k^*, l_k^*) - \delta/2 + \delta/2.$$

That is, for all $k \geq \max\{k_0, k_1\}$, $l_k^* \in L_{\widehat{\psi}}(t_k^*, x_k^*)$ the inequality is valid $\langle l_k^*, z^* \rangle \leq H(t_k^*, x_k^*, l_k^*)$, which means, that $z^* \in A_{\widehat{\psi}}(t_k^*, x_k^*)$ for all $k \geq \max\{k_0, k_1\}$. And, if we remind, that the set $F(t,x)$ depends continuously on (t,x) and that (3.19) is valid, then we can conclude, that $z^* \in F(t_k^*, x_k^*)$ for some k_2 and all $k \geq k_2$.

In the conclusion, for all $k \geq \max\{k_0, k_1, k_2\}$ the inclusion

$$z^* \in F_{\widehat{\psi}}(t_k^*, x_k^*)$$

takes place. The last inclusion contradicts supposition (3.17). Then the inclusion (3.16) is proved, and the first equality from (3.14) is true. The second equality can be proved by similar way. (3.14) contradicts (3.13), hence the assumption (3.13) is not true, and $\omega^*(\delta) \downarrow 0$ while $\delta \downarrow 0$. The (A.3) is proved. And the teorem is proved. □

4. Examples

Example 1. Let the controlled system has the dynamics

$$dx/dt = f(t,x) + u + v,\ x[t_0] = x_0, \quad u \in P,\ v \in Q, \tag{4.1}$$

$x \in \mathbb{R}^2$, P is rectangle with vertexes $(-0.5, 1.5)$, $(0.5, 1.5)$, $(0.5, -1.5)$ and $(-0.5, -1.5)$, Q is segment $(-1.0, 0.0)$, $(1.0, 0)$ as shown in the Figure 4.

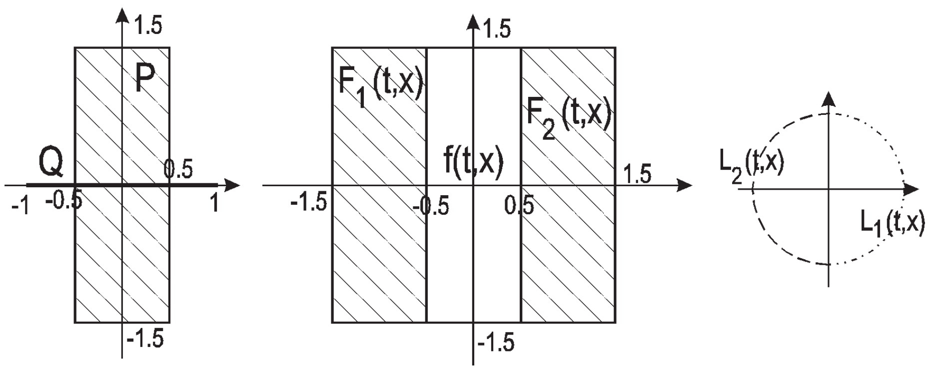

Figure 4

In the example there are two cones of convexity of Hamiltonian, which are two half-plane of variable l: $L_1 = \{l = (l_1, l_2) : l_1 \geq 0\}$ and $L_2 = \{l = (l_1, l_2) : l_1 \leq 0\}$. The corresponding sets $F_1(x)$ and $F_2(x)$ are constructed in Figure 4.

Example 2. Let the controlled system has the dynamics

$$dx/dt = f(t,x) + a(t,x)u + b(t,x)v, \quad x[0] = x0, \tag{4.2}$$

$x[t_0] = x_0$, $x \in \mathbb{R}^2$, $u \in P$, $v \in Q$, P and Q are quadrangle with vertexes $(1,0)$, $(-1,0)$, $(0,\varepsilon)$, $(0,-\varepsilon)$, $\varepsilon > 0$.

Let

$$\det\begin{pmatrix} a_{1,1}(t,x), & a_{1,2}(t,x) \\ a_{2,1}(t,x), & a_{2,2}(t,x) \end{pmatrix} \neq 0, \quad \det\begin{pmatrix} b_{1,1}(t,x), & b_{1,2}(t,x) \\ b_{2,1}(t,x), & b_{2,2}(t,x) \end{pmatrix} \neq 0$$

for all $(t,x) \in D$ and $a_{i,j}(t,x)$, and $b_{i,j}(t,x)$, $i,j = 1,2$ are continuous by (t,x). Denote $b_i(t,x)$, $i = 1,2$ ith column of matrix $b(t,x)$ and

$$\beta_1(t,x) = \arccos(-(b_{2,1}(t,x) + \varepsilon b_{2,2})/\|b_1 + \varepsilon b_2\|),$$
$$\beta_2(t,x) = \arccos(-(b_{2,1}(t,x) - \varepsilon b_{2,2})/\|b_1 - \varepsilon b_2\|)$$

(see Figure 5). Let, for example, that for some (t,x) the inequality $\beta_1 < \beta_2$ fulfils, and let $l = (r\cos\phi,\ r\sin\phi)$. We can write Hamiltonian of controlled system by following way

$$H(t,x,l) = f(t,x) + \max_{u\in P}\langle a(t,x)u, l\rangle + \min_{v\in Q}\langle b(t,x)v, l\rangle,$$

$$\min_{v\in Q}\langle b(t,x)v, l\rangle = \begin{cases} -|\langle b_1(t,x), l\rangle|, \\ l:\ \beta_1(t,x) \le \phi \le \beta_2(t,x) \vee \beta_1(t,x) + \pi \le \phi \le \beta_2(t,x) + \pi \\ -|\langle \varepsilon b_2(t,x), l\rangle|, \\ l:\ \beta_2(t,x) < \phi < \beta_1(t,x) + \pi \vee \beta_2(t,x) - \pi < \phi < \beta_1(t,x). \end{cases}$$

Then, if we denote

$$\delta(t,x,l) = \begin{cases} 1 & \beta_1(t,x) \le \phi \le \beta_2(t,x) \vee \beta_1(t,x) + \pi \le \phi \le \beta_2(t,x) + \pi, \\ 0 & \beta_2(t,x) < \phi < \beta_1(t,x) + \pi \vee \beta_2(t,x) - \pi < \phi < \beta_1(t,x), \end{cases}$$

we can gather

$$H(t,x,l) = f(t,x) + \max_{u\in P}\langle a(t,x)u, l\rangle - |\langle b_1(t,x), l\rangle| \cdot \delta(t,x,l) - |\langle \varepsilon b_2(t,x), l\rangle| \cdot (1 - \delta(t,x,l)).$$

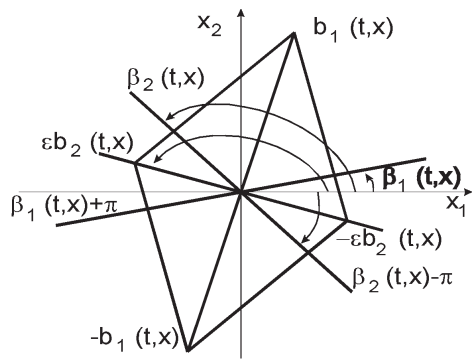

Figure 5

Part the space of variable $l = (r\cos\phi, r\sin\phi)$ onto 4 subsets:

$$L_1^*(t,x) = \{(r\cos\phi,\ r\sin\phi) : 0 < r < \infty,\ \beta_1(t,x) \le \phi \le \beta_2(t,x)\},$$
$$L_2^*(t,x) = \{(r\cos\phi,\ r\sin\phi) : 0 < r < \infty,\ \beta_2(t,x) \le \phi \le \beta_1(t,x) + \pi\},$$
$$L_3^*(t,x) = \{(r\cos\phi,\ r\sin\phi) : 0 < r < \infty,\ \beta_1(t,x) + \pi \le \phi \le \beta_2(t,x) + \pi\},$$
$$L_4^*(t,x) = \{(r\cos\phi,\ r\sin\phi) : 0 < r < \infty,\ \beta_2(t,x) - \pi \le \phi \le \beta_1(t,x)\},$$

then $\Psi = \{1,2,3,4\}$, and

$$L_\psi(t,x) = L_\psi^*(t,x) \cap S. \tag{4.4}$$

Then it can be proved that the partition, determined by equality (4.4) satisfies conditions (B.1)–(B.4)

(B.1) is satisfies accordinly by construction.

(B.2) Note, that the addendum $\max_{u\in P}\langle a(t,x)u, l\rangle$ is the supporting function of a convex set, hence, is convex function by the variable l. The addendum $-|\langle b_1(t,x), l\rangle| \cdot \delta_2(t,x,l)$ is linear function by the variable l, hence, if convex function by l on each of the cones $\{(r\cos\phi,\ r\sin\phi) : 0 < r < \infty,\ \beta_1(t,x) \le \phi \le \beta_2(t,x)\}$ and $\{(r\cos\phi,\ r\sin\phi) : 0 < r < \infty,\ \beta_1(t,x) + \pi \le \phi \le \beta_2(t,x) + \pi\}$, and the addendum $-|\langle \varepsilon b_2(t,x), l\rangle| \cdot (1 - \delta_2(t,x,l))$ is linear (that is, convex) function by l on each of the cones $\{(r\cos\phi,\ r\sin\phi) : 0 < r < \infty,\ \beta_2(t,x) \le \phi \le \beta_1(t,x) + \pi\}$ and $\{(r\cos\phi,\ r\sin\phi) : 0 < r < \infty,\ \beta_2(t,x) - \pi \le \phi \le \beta_1(t,x)\}$. Then Hamiltonian, as it is the sum of convex functions, is convex function on each of the cones $K(L_\psi(t,x))$.

(B.3) Let denote

$$\widetilde{\omega}(\delta) = \sup_{(\psi,t_*,x_*,t^*,x^*)\in Y(\delta)} d(L_\psi(t_*,x_*), L_\psi(t^*,x^*)),$$

where $Y(\delta) = \{(\psi,t_*,x_*,t^*,x^*) : \psi \in \Psi,\ (t_*,x_*) \in D,\ (t^*,x^*) \in D,\ |t^* - t_*| + \|x^* - x_*\| \le \delta\}$, $\delta > 0$. The function satisfies condition (B.3). We need to prove, that $\widetilde{\omega}(\delta) \downarrow 0$ while $\delta \downarrow 0$.

Take arbitrary $\psi \in \Psi$ and prove, that for $(\psi,t_*,x_*,t^*,x^*) \in Y(\delta)$

$$d(L_\psi(t_*,x_*), L_\psi(t^*,x^*)) \downarrow 0 \tag{4.5}$$

is true while $\delta \downarrow 0$. The set $L_\psi(t,x)$ is the arc of the circle, wich is determined by angles $\beta_1(t,x)$ and $\beta_2(t,x)$. To prove (4.5) we need to prove, that functions $\beta_1(t,x)$ and $\beta_2(t,x)$ are uniform continuous. Let prove, that for all $(t,x) \in D$.

$$\|b_1(t,x) + \varepsilon b_2(t,x)\| \ne 0. \tag{4.6}$$

Propose the opposite: let for some pair $(t,x) \in D$

$$\|b_1(t,x) + \varepsilon b_2(t,x)| = (b_{1,1}(t,x) + \varepsilon b_{1,2}(t,x))^2 + (b_{2,1}(t,x) + \varepsilon b_{2,2}(t,x))^2 = 0. \tag{4.7}$$

Then

$$b_{1,1}(t,x) + \varepsilon b_{1,2}(t,x) = 0, \qquad b_{2,1}(t,x) + \varepsilon b_{2,2}(t,x) = 0,$$

or

$$b_{1,1}(t,x) = -\varepsilon b_{1,2}(t,x), \qquad b_{2,1}(t,x) = -\varepsilon b_{2,2}(t,x).$$

Hence

$$\det b(t,x) = -\varepsilon \cdot \det \begin{pmatrix} b_{1,2}(t,x) & b_{1,2}(t,x) \\ b_{2,2}(t,x) & b_{2,2}(t,x) \end{pmatrix} = 0.$$

Last equality contradicts condition (4.3). Hence, proposition (4.7) is false, and (4.6) is true. Also we can prove by the similar way, that for all $(t,x) \in D$ the inequality is true $\|b_1(t,x) - \varepsilon b_2(t,x)\| \neq 0$. That is the functions $\beta_1(t,x)$ and $\beta_2(t,x)$ are continuous on compact, hence, uniform continuous. That is why (4.5) fulfils for any $\psi \in \Psi$. And (B.3) satisfyes because of Ψ is finite set.

(B.4) satisfies because of (4.3).

Example 3. Let the controlled system has the dynamics

$$dx/dt = f(t,x) + u + v, \quad x[0] = x0, \quad x \in \mathbb{R}^3.$$

P is cube with vertexes (−1, −1, −1), (−1, −1, 1), (−1, 1, −1), (−1, 1, 1), (1, −1, −1), (1, −1, 1), (1, 1, −1), (1, 1, 1); Q isooctahedron with vertexes $(-\sqrt{2},0,0)$, $(0,-\sqrt{2},0)$, $(0,0,-\sqrt{2})$, $(0,0,\sqrt{2})$, $(0,\sqrt{2},0)$, $(\sqrt{2},0,0)$ as shown in the Figure 6.

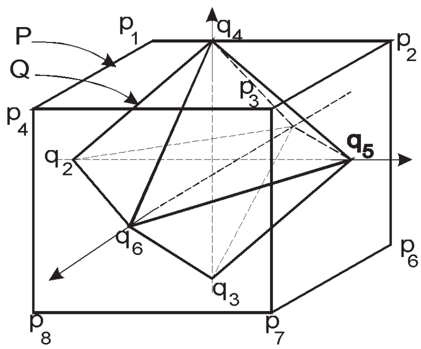

Figure 6

There are 6 cones $K(L_\psi)$ of convexity of Hamiltonian in that example. They are cones of normales to Q in its vertexes. The cone $K(L_{\psi^*})$, shown on the Figure 7, consist of 4 cones of linearity of Hamiltonian: S_1, S_2, S_3, S_4, where cone S_i is intersection $K(L_{\psi^*})$ with octant of $\mathbb{R}^3$, which contains vertex v_i.

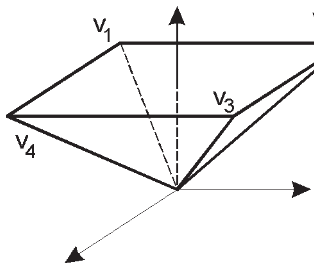

Figure 7

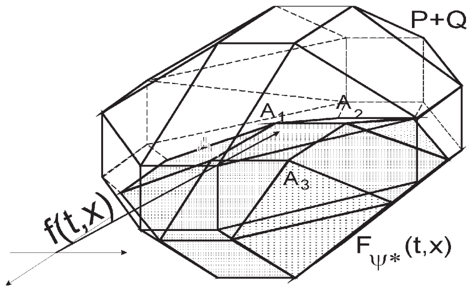

Figure 8

If we write Hamiltonian on $K(L_{\psi^*})$

$$h(t,x,l) = \begin{cases} \langle l, p_1\rangle + \langle l, q_5\rangle = \left\langle l, \begin{pmatrix} -1 \\ -1 \\ -0.4 \end{pmatrix}\right\rangle, & l \in S_1, \\ \langle l, p_2\rangle + \langle l, q_5\rangle = \left\langle l, \begin{pmatrix} -1 \\ 1 \\ -0.4 \end{pmatrix}\right\rangle, & l \in S_2, \\ \langle l, p_3\rangle + \langle l, q_5\rangle = \left\langle l, \begin{pmatrix} 1 \\ 1 \\ -0.4 \end{pmatrix}\right\rangle, & l \in S_3, \\ \langle l, p_4\rangle + \langle l, q_5\rangle = \left\langle l, \begin{pmatrix} 1 \\ -1 \\ -0.4 \end{pmatrix}\right\rangle, & l \in S_4, \end{cases}$$

then we can easy prove, that Hamiltonian is convex function on $K(L_{\psi^*})$. The set $F_{\psi^*}(t,x)$, corresponding to L_{ψ^*}, is constructed in Figure 8.

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